Commun. Math. Phys. 84, 257–295 (1982)

Communications in
Mathematical
Physics

Analytic Interpolation and Borel Summability of the $(\frac{\lambda}{N}|\Phi_N|^{:4})_2$ Models

I. Finite Volume Approximation

C. Billionnet and P. Renouard

Centre de Physique Théorique, Ecole Polytechnique, F-91128 Palaiseau Cedex, France

Abstract. Analytic interpolation in the variable $1/N$ of $\left(\frac{\lambda}{N}|\Phi_N|^{:4}\right)_2$ models is constructed at finite volume approximation. We prove Borel summability of the Taylor series at $1/N=0$ of their Schwinger functions. We also give an extension of the domain of analyticity in the coupling constant.

Introduction

We study an analytic interpolation and the asymptotic behaviour of a family of vector quantum fields, self-coupled with a quartic interaction, in a two dimensional space-time. So we carry on the study of the "$\frac{1}{N}$ expansion" for the family of $(\frac{\lambda}{N}\ |\Phi_N|^{:4})_2$ models, initiated by Kupiainen [2].

More precisely, for each integer N, we start with the Schwinger functions of a vector field Φ_N, with N components, submitted to the $\frac{\lambda}{N}|\Phi_N|^{:4}$ interaction; their (momentum and volume cut-off) approximations have a representation which allows us to "complexify" the parameter N.

In this paper, we obtain, as limits of these, analytic functions of two complex variables λ, z, which continue (in λ) and interpolate (in $z \sim \frac{1}{N}$) the given Schwinger functions without ultra-violet cut-off. (The removal of the volume cut-off using the Glimm-Jaffe-Spencer cluster expansion if $|\lambda|$ is sufficiently small does not seem to entail any essential difficulty.) We show that these analytic functions have an indefinitely derivable (in an angle) continuation to points of the form $(\lambda, z=0)$, if $|\lambda|$ is sufficiently small, and that their Taylor series at these points are Borel summable.

This property improves the relation between the "$\frac{1}{N}$ expansion" (known to be asymptotic [2]) and the function itself. It allows the construction of convergent approximations which depends only on the beginning of the series; these are "explicit" (as sums of Feynman graphs). Moreover it allows us to characterize the constructed interpolation among all analytic functions which coincide at $z=\frac{1}{N}$, $(N\in\mathbb{N})$ with the given Schwinger functions.

0010-3616/82/0084/0257/$07.80

Besides, we obtain an extension of the previously known analyticity domain in the coupling constant λ of the Schwinger functions themselves. In particular, for each θ arbitrarily close to π, this domain contains a sector $\{\lambda \in \mathbb{C};\ |\mathrm{Arg}\,\lambda| < \theta, |\lambda| < r_\theta\}$ to which extends the Borel summability of the Taylor series at zero (i.e. the "perturbation series"). However we note that, certainly, the constructed analyticity domain is unnecessarily restricted by technicalities.

1. Analytic Interpolation

In this chapter, we first recall, in order to fix notations, the definition of the generating functionals of Schwinger functions of the "finite volume" approximation of $\frac{\lambda}{N}|\Phi_N|^{:4}$ models in two dimensional space-time (1.1). Then we introduce, for their "ultraviolet cut-off" regularizations (1.2), an integral representation (1.3) which allows the construction of an analytic interpolation (1.4), the limit of which (Theorem I, Sect. 1.5) gives the expected continuation. Sections 1.6–1.10 are devoted to the proof of Theorem I.

1.1. Description of the Model

Let E be a two dimensional euclidean space, $\mathscr{S}$ the real topological vector space $\mathscr{S}(E, \mathbb{R})$ of indefinitely differentiable, fast decreasing functions, $\mathscr{S}'$ its dual space, $\mathfrak{A}$ the σ-algebra over $\mathscr{S}'$ generated by the linear functions

$$\omega \mapsto \langle \omega, f \rangle, \quad (\omega \in \mathscr{S}'), \quad f \in \mathscr{S},$$

and $m > 0$ being fixed, $\mu_m \in \mathscr{M}^1(\mathscr{S}', \mathfrak{A})$ the gaussian measure whose Fourier transform is

$$\hat{\mu}_m(f) \equiv \int_{\mathscr{S}'} e^{i\langle \omega, f \rangle} \mu_m(d\omega) = \exp(-\tfrac{1}{2}(f, \Sigma_m^{-1} f)_2), \quad f \in \mathscr{S}, \tag{1.1.1}$$

where $\Sigma_m = -\Delta + m^2$, and $(\cdot,\cdot)_2$ is the scalar product in $L^2(E)$.

For each integer $N \geqq 1$, $\Phi_N : \mathscr{S}^N \to L^1(\mathscr{S}'^N, \mathfrak{A}^{\otimes N}, \mu_m^{\otimes N})$ denotes the canonical process, [that is, if $\mathbf{f} = (f_j)_{1 \leqq j \leqq N} \in \mathscr{S}^N$, $\Phi_N(\mathbf{f})$ is the (class modulo $\mu_m^{\otimes N}$ of the) function

$$\boldsymbol{\omega} \mapsto \langle \boldsymbol{\omega}, \mathbf{f} \rangle = \sum_{j=1}^{N} \langle \omega_j, f_j \rangle, \quad (\boldsymbol{\omega} = (\omega_j)_{1 \leqq j \leqq N} \in \mathscr{S}'^N)].$$

If g is a real function, defined on E, such that[1]

$$M_4(g)^2 = (2\pi)^{-8} \int_{E^4} \left| \hat{g}\left(\sum_{i=1}^{4} p_i \right) \right| \prod_{j=1}^{4} \frac{dp_j}{|p_j|^2 + 1} < +\infty, \tag{1.1.2}$$

$|\Phi_N|^{:4}(g) \in L^1(\mathscr{S}'^N, \mu_m^{\otimes N})$ is the function defined by

$$E_{\mu_m^{\otimes N}}[|\Phi_N|^{:4}(g) \exp(i\Phi_N(\mathbf{f}))] = \left(g, \left[\sum_{j=1}^{N} (\Sigma_m^{-1} f_j)^2 \right]^2 \right)_2 \prod_{j=1}^{N} \hat{\mu}_m(f_j),$$
$$\forall \mathbf{f} = (f_j)_{1 \leqq j \leqq N} \in \mathscr{S}^N. \tag{1.1.3}$$

1 $\hat{g}$ is the Fourier transform of g, specified by $\hat{g}(p) = \int_E e^{-i(p,x)} g(x)\,dx$, $(p \in E)$

It is well known[2] that, if $g \geqq 0$,

$$\exp(-|\Phi_N|^{:4}(g)) \in \bigcap_{1 \leqq p < \infty} L^p(\mathscr{S}'^N, \mu_m^{\otimes N}). \tag{1.1.4}$$

One supposes g satisfies (1.1.2), $0 \leqq g \leqq 1$ and $\operatorname{Re}\lambda \geqq 0$, and one defines

$$Z^{\star}_{\frac{1}{N},\lambda,g}(\mathbf{f}) = E_{\mu_m^{\otimes N}}[\exp(i\Phi_N(\mathbf{f}))\exp(-\tfrac{\lambda}{N}|\Phi_N|^{:4}(g))], \quad \mathbf{f} \in \mathscr{S}^N, \tag{1.1.5}$$

and if[3]

$$\begin{aligned} Z^{0}_{\frac{1}{N},\lambda,g}(\mathbf{f}) &\equiv Z^{\star}_{\frac{1}{N},\lambda,g}(\mathbf{0}) \neq 0, \\ S^{\star}_{\frac{1}{N},\lambda,g}(\mathbf{f}) &= \frac{Z^{\star}_{\frac{1}{N},\lambda,g}(\mathbf{f})}{Z^{0}_{\frac{1}{N},\lambda,g}}. \end{aligned} \tag{1.1.6}$$

Then, for $\lambda \geqq 0$, $S^{\star}_{\frac{1}{N},\lambda,g}$ is the generating functional of the Schwinger functions of the $\frac{\lambda}{N}|\Phi_N|^{:4}$ model with "volume cut-off" g.

1.2. Ultraviolet Cut-off Regularizations

If $1 \leqq j \leqq N$, let $\Phi_N^{(j)} : \mathscr{S} \to L^1(\mathscr{S}'^N, \mu_m^{\otimes N})$ be the "j^{th} component of Φ_N," that is, for $f \in \mathscr{S}$, $\Phi_N^{(j)}(f) = \Phi_N(\mathbf{I}_j f)$, where $\mathbf{I}_j : \mathscr{S} \to \mathscr{S}^N$ is defined by

$$(\mathbf{I}_j f)_k = \begin{cases} f, & \text{if} \quad k = j, \\ 0, & \text{if} \quad k \neq j, \end{cases} \qquad f \in \mathscr{S},$$

$\left[\text{so that } \Phi_N(\mathbf{f}) = \sum_{j=1}^{N} \Phi_N^{(j)}(f_j), \text{ if } \mathbf{f} = (f_j)_{1 \leqq j \leqq N} \in \mathscr{S}^N\right]$.

Given $\chi \in \mathscr{S}$, one sets

$$\Phi_{N,\chi}^{(j)}(x) = \Phi_N^{(j)}(\check{\chi}(\cdot - x)), \qquad x \in E, \tag{1.2.1}$$

[thus $\Phi_{N,\chi}^{(j)}(x)[\boldsymbol{\omega}] = \omega_j * \chi(x)$], and

$$\mathbf{c}_\chi = E_{\mu_m}^{\otimes N}[\Phi_{N,\chi}^{(j)}(x)^2] = (\chi, \Sigma_m^{-1}\chi)_2, \tag{1.2.2}$$

next,

$$\Phi_{N,\chi}^{(j)}(x)^{:2} = \Phi_{N,\chi}^{(j)}(x)^2 - \mathbf{c}_\chi, \tag{1.2.3}$$

and,

$$\Phi_{N,\chi}^{(j)}(x)^{:4} = \Phi_{N,\chi}^{(j)}(x)^4 - 6\mathbf{c}_\chi \Phi_{N,\chi}^{(j)}(x)^2 + 3\mathbf{c}_\chi^2, \tag{1.2.4}$$

last,

$$|\Phi_{N,\chi}(x)|^{:4} = \sum_{j=1}^{N} \Phi_{N,\chi}^{(j)}(x)^{:4} + 2 \sum_{1 \leqq i < j \leqq N} \Phi_{N,\chi}^{(i)}(x)^{:2} \cdot \Phi_{N,\chi}^{(j)}(x)^{:2}, \tag{1.2.5}$$

---

2 It follows from Nelson's theorem [3]; see also [1, Theorem 2.1.4]

3 This is true for $\lambda \geqq 0$ (from Jensen's inequality, because $E_{\mu_m^{\otimes N}}[|\Phi_N|^{:4}(g)] = 0$), and, by continuity, in a neighbourhood of the positive real axis

so that

$$|\Phi_{N,\chi}(x)|^{:4}=\left(\sum_{j=1}^{N}\Phi_{N,\chi}^{(j)}(x)^{:2}-2\mathbf{c}_{\chi}\right)^{2}-2(N+2)\mathbf{c}_{\chi}^{2}. \tag{1.2.6}$$

Then if [g satisfying (1.1.2)] one sets

$$|\Phi_{N,\chi}|^{:4}(g)=\int_{E}|\Phi_{N,\chi}(x)|^{:4}g(x)dx, \tag{1.2.7}$$

and, if one replaces χ by an approximate unit $(\chi_n)_{n\in\mathbb{N}}$, one knows[4] that

$$|\Phi_N|^{:4}(g)=\lim_{n\to\infty}|\Phi_{N,\chi_n}|^{:4}(g),\quad \text{in}\quad L^p(\mathscr{S}'^N,\mu_m^{\otimes N}),\quad 1\leqq p<+\infty, \tag{1.2.8}$$

and, if moreover $g\geqq 0$, that[5]

$$\exp(-|\Phi_N|^{:4}(g))=\lim_{n\to\infty}\exp(-|\Phi_{N,\chi_n}|^{:4}(g)),$$
$$\text{in}\quad L^p(\mathscr{S}'^N,\mu_m^{\otimes N}),\quad 1\leqq p<+\infty. \tag{1.2.9}$$

Thus, if one sets

$$\overset{\star}{Z}_{\frac{1}{N},\lambda,g,\chi}(\mathbf{f})=E_{\mu_m^{\otimes N}}[e^{i\Phi(\mathbf{f})}\exp(-\tfrac{\lambda}{N}|\Phi_{N,\chi}|^{:4}(g))],\quad \mathbf{f}\in\mathscr{S}^N, \tag{1.2.10}$$

one has

$$\overset{\star}{Z}_{\frac{1}{N},\lambda,g}(\mathbf{f})=\lim_{n\to\infty}\overset{\star}{Z}_{\frac{1}{N},\lambda,g,\chi_n}(\mathbf{f}). \tag{1.2.11}$$

1.3. An Integral Representation

Now let $\nu\in\mathscr{M}^1(\mathscr{S}',\mathfrak{A})$ be the gaussian measure with Fourier transform

$$\hat{\nu}(f)=\exp(-\tfrac{1}{2}\|f\|_2^2),\qquad f\in\mathscr{S}; \tag{1.3.1}$$

let also ϱ be a real number, g a C^∞-function[6] on E with compact support, such that $0\leqq g\leqq 1$, and $\boldsymbol{\omega}=(\omega_j)_{1\leqq j\leqq N}\in\mathscr{S}'^N$. From (1.2.6) and (1.2.7), one has[7]

$$\exp(-\tfrac{\varrho^2}{8N}|\Phi_{N,\chi}|^{:4}(g^2))[\boldsymbol{\omega}]$$
$$=\exp\left(\frac{N+2}{4}\frac{\varrho^2}{N}\mathbf{c}_\chi^2\|g\|_2^2\right)\int_{\mathscr{S}'}\exp\left(-\frac{i}{2}\frac{\varrho}{\sqrt{N}}\left\langle\sigma,\left(\sum_{j=1}^{N}\Phi_{N,\chi}^{(j)}(\cdot)^{:2}[\boldsymbol{\omega}]-2\mathbf{c}_\chi\right)g\right\rangle\right)\nu(d\sigma) \tag{1.3.2}$$

4 See [1, Lemma 2.1.6]

5 This follows from (1.1.4) and (1.2.8), using Duhammel's formula

$$\exp(-|\Phi_N|^{:4}(g))-\exp(-|\Phi_{N,\chi}|^{:4}(g))$$
$$=(|\Phi_{N,\chi}|^{:4}(g)-|\Phi_N|^{:4}(g))\int_0^1\exp(-u|\Phi_N|^{:4}(g))\exp(-(1-u)|\Phi_{N,\chi}|^{:4}(g))du$$

6 This regularity condition is only needed to avoid inessential complications: with slight modifications one can assume, in all that follows, that g is only measurable, bounded, and vanishes outside a compact set

7 A representation of this form is introduced in [2, Sect. 4]

Then, substituting (1.3.2) in (1.2.10), one obtains from Fubini's theorem (the integrant is a bounded function),

$$Z^{\star}_{\frac{1}{N},\frac{\varrho^2}{8},g^2,\chi}(\mathbf{f}) = \int_{\mathscr{S}'} \exp\left(\frac{1}{2}\frac{\varrho^2}{N}\mathbf{c}_\chi^2\|g\|_2^2\right)\exp\left(i\frac{\varrho}{\sqrt{N}}\mathbf{c}_\chi\langle\sigma,g\rangle\right)$$
$$\cdot\prod_{j=1}^{N}\left(\exp\left(\frac{1}{4}\frac{\varrho^2}{N}\mathbf{c}_\chi^2\|g\|_2^2\right)\int_{\mathscr{S}'}\exp(i\langle\omega_j,f_j\rangle)\exp\left(-\frac{i}{2}\frac{\varrho}{\sqrt{N}}\langle\sigma,[(\omega_j*\chi)^2-\mathbf{c}_\chi]g\rangle\right)\right.$$
$$\left.\cdot\mu_m(d\omega_j)\right)\nu(d\sigma). \tag{1.3.3}$$

For $\sigma\in\mathscr{S}'$, let $\mathbf{A}_{g,\chi}(\sigma)\in\mathscr{T}_2(\mathscr{H}^1)$[8] be the real, self-adjoint operator, defined by

$$\mathbf{A}_{g,\chi}(\sigma)\psi=\Sigma_m^{-1}(\check{\chi}*(\sigma g\cdot(\chi*\psi))),\qquad \psi\in\mathscr{H}^1, \tag{1.3.4}$$

one has,

$$\int_{\mathscr{S}'}\langle\sigma,[(\omega*\chi)^2-\mathbf{c}_\chi]g\rangle e^{i\langle\omega,f\rangle}\mu_m(d\omega)$$
$$=-\langle\sigma,(\chi*\Sigma_m^{-1}f)^2g\rangle\hat{\mu}_m(f)$$
$$=-(\Sigma_m^{-1}f,\mathbf{A}_{g,\chi}(\sigma)\Sigma_m^{-1}f)_{\mathscr{H}^1}\hat{\mu}_m(f),\qquad\forall f\in\mathscr{S}. \tag{1.3.5}$$

Thus, for any real t such that $(I+t\mathbf{A}_{g,\chi}(\sigma))>0$,

$$\int_{\mathscr{S}'}\exp\left(-\frac{t}{2}\langle\sigma,[(\omega*\chi)^2-\mathbf{c}_\chi]g\rangle\right)e^{i\langle\omega,f\rangle}\mu_m(d\omega)$$
$$=[\det_2(I+t\mathbf{A}_{g,\chi}(\sigma))]^{-1/2}\exp(-\tfrac{1}{2}(\Sigma_m^{-1}f,[I+t\mathbf{A}_{g,\chi}(\sigma)]^{-1}\Sigma_m^{-1}f)_{\mathscr{H}^1}). \tag{1.3.6}$$

By analytic continuation, (1.3.6) also holds for any complex t such that $(I+\mathrm{Re}\,t\cdot\mathbf{A}_{g,\chi}(\sigma))>0$, provided, in this case, the square root is specified by

$$[\det_2(I+tA)]^\alpha\equiv\exp\left(\alpha\sum_{k\in\mathbb{N}}[\log(1+t\xi_k)-t\xi_k]\right),\quad(\mathrm{Im}\,t\neq0,\alpha\in\mathbb{C}),$$

with

$$|\mathrm{Im}\log(1+t\xi)|<\pi,\qquad\forall\xi\in\mathbb{R}, \tag{1.3.7}$$

where $(\xi_k)_{k\in\mathbb{N}}$ is the sequence of proper values (each repeated according to its own multiplicity) of $A=A^*\in\mathscr{T}_2$.
Then, inserting (1.3.6) in (1.3.3), one obtains

$$Z^{\star}_{\frac{1}{N},\frac{\varrho^2}{8},g^2,\chi}(\mathbf{f}) = \int_{\mathscr{S}'} \exp\left(\frac{1}{2}\frac{\varrho^2}{N}\mathbf{c}_\chi^2\|g\|_2^2\right)\exp\left(i\frac{\varrho}{\sqrt{N}}\mathbf{c}_\chi\langle\sigma,g\rangle\right)$$
$$\cdot\left[\exp\left(-\frac{1}{2}\frac{\varrho^2}{N}\mathbf{c}_\chi^2\|g\|_2^2\right)\det_2\left(I+i\frac{\varrho}{\sqrt{N}}\mathbf{A}_{g,\chi}(\sigma)\right)\right]^{-\frac{N}{2}}$$
$$\cdot\prod_{j=1}^{N}\exp\left(-\frac{1}{2}\left(\Sigma_m^{-1}f_j,\left[I+i\frac{\varrho}{\sqrt{N}}\mathbf{A}_{g,\chi}(\sigma)\right]^{-1}\Sigma_m^{-1}f_j\right)_{\mathscr{H}^1}\right)\cdot\nu(d\sigma). \tag{1.3.8}$$

8 $\mathscr{H}^1$ stands for the Sobolev space $\mathscr{H}^1(E)$ with norm $\|\psi\|_{\mathscr{H}^1}=\|\Sigma_m^{-1/2}\psi\|_2$, and $\mathscr{T}_p$, $(p\geq1)$, is the ideal of operators A such that $|A|^p$ is trace class

1.4. Construction of an Analytic Interpolation

Now one constructs an analytic continuation of (1.3.8) with the help of the following lemma:

For $a>0$, let $\mathbf{D}_a$ be the domain[9] of all $(\varrho,\zeta)\in\mathbb{C}^2$ such that

$$\varrho\zeta\neq 0 \quad\text{and}\quad |\operatorname{Arg}\varrho\zeta|\neq\frac{\pi}{2}, \tag{1.4.1}$$

and

$$\left.\begin{array}{l} |\varrho|^2\left\{\dfrac{1-|\cos\operatorname{Arg}\varrho\zeta|}{4|\cos\operatorname{Arg}\varrho\zeta|}|\cos\operatorname{Arg}\zeta^2|+\operatorname{Arc}\cos(-\sin|\operatorname{Arg}\varrho\zeta|)\cdot|\sin\operatorname{Arg}\zeta^2|\right\}<a,\\ \quad\text{if}\quad \operatorname{Re}\zeta^2\geqq 0,\\ |\varrho|^2\{\tfrac{1}{2}|\cos\operatorname{Arg}\zeta^2|+\operatorname{Arc}\cos(-\sin|\operatorname{Arg}\varrho\zeta|)\cdot|\sin\operatorname{Arg}\zeta^2|\}<a,\\ \quad\text{if}\quad \operatorname{Re}\zeta^2\leqq 0,\end{array}\right\} \tag{1.4.2}$$

then

Lemma. *Suppose* $\|g\|_\infty\leqq 1$, $\|\hat{\chi}\|_\infty\leqq 1$, *and let* $a<4\pi m^2$; *if* $p\in\left[1,\dfrac{4\pi m^2}{a}\right[$, *then* $\left[\det_2(I+i\varrho\zeta\mathbf{A}_{g,\chi})\right]^{-\frac{1}{2\zeta^2}}\in L^p(\mathscr{S}',\nu)$, *and*

$$\|[\det_2(I+i\varrho\zeta\mathbf{A}_{g,\chi})]^{-\frac{1}{2\zeta^2}}\|_p<\exp\left(\frac{a}{2}\left(1-\frac{ap}{4\pi m^2}\right)^{-1}\mathbf{c}_\chi^2\|g\|_2^2\right),\quad \forall(\varrho,\zeta)\in\mathbf{D}_a \tag{1.4.3}$$

(the left hand side of (1.4.3) *is defined by* (1.3.7)*).*

Proof. First, for $(\varrho,\zeta)\in\mathbf{D}_a$, one has

$$\operatorname{Re}\left\{-\frac{1}{2\zeta^2}[\log(1+i\varrho\zeta\xi)-i\varrho\zeta\xi]\right\}<\frac{a}{2}\xi^2,\quad\forall\xi\in\mathbb{R}. \tag{1.4.4}$$

Indeed, one shows elementarily, that if $|\operatorname{Arg}u|\neq\pi$, then

$$\operatorname{Re}\{\log(1+u)-u\}\leqq\tfrac{1}{2}|u|^2, \tag{1.4.5}$$

$$-\operatorname{Re}\{\log(1+u)-u\}\leqq\frac{1-|\sin\operatorname{Arg}u|}{4|\sin\operatorname{Arg}u|}|u|^2, \tag{1.4.6}$$

and if moreover $|\operatorname{Im}\log(1+u)|<\pi$,

$$|\operatorname{Im}\{\log(1+u)-u\}|\leqq|\operatorname{Arg}u|\cdot|u|^2,\quad[|\operatorname{Arg}u|<\pi], \tag{1.4.7}$$

which entails (1.4.4), according to (1.4.2). Therefore

$$|[\det_2(I+i\varrho\zeta\mathbf{A}_{g,\chi})]^{-\frac{1}{2\zeta^2}}|<\exp\left(\frac{a}{2}\|\mathbf{A}_{g,\chi}\|_{\mathscr{T}_2}^2\right), \tag{1.4.8}$$

(where $\|\cdot\|_{\mathscr{T}_2}$ is the Hilbert-Schmidt norm). But

$$E_\nu[\|\mathbf{A}_{g,\chi}\|_{\mathscr{T}_2}^2 e^{i\langle\cdot,f\rangle}]=\{\operatorname{Tr}B_{g,\chi}-(f,B_{g,\chi}f)_2\}\hat{\nu}(f),\quad\forall f\in\mathscr{S}, \tag{1.4.9}$$

9 Some graphic representations of this domain are shown in appendix

where $B_{g,\chi} \in \mathcal{T}_1(L^2)$ is the real, positive, self adjoint operator defined by

$$B_{g,\chi}\psi = g\cdot((\Sigma_m^{-1}\check{\chi}*\chi)^2*(g\cdot\psi)), \quad \psi \in L^2(E), \tag{1.4.10}$$

$\Big[B_{g,\chi}$ is trace class because if $C_{g,\chi}: L^2(E) \to L^2(E^2)$ is given by $C_{g,\chi}\psi(x_1,x_2) = \int_E \Sigma_m^{-1/2}\check{\chi}(x_1-y)\Sigma_m^{-1/2}\chi(x_2-y)g(y)\psi(y)dy$, $C_{g,\chi}$ is obviously of Hilbert-Schmidt class and $B_{g,\chi} = C^*_{g,\chi}C_{g,\chi} = |C_{g,\chi}|^2\Big]$.

Then, if G_m denotes the kernel of Σ_m^{-1} (namely $\Sigma_m^{-1}f = G_m * f$), one has

$$\begin{aligned}\|B_{g,\chi}\| &\leqq \|g\|_\infty^2 \|(\Sigma_m^{-1}\check{\chi}*\chi)^2\|_1 = \|g\|_\infty^2 \|G_m*\check{\chi}*\chi\|_2^2 \\ &\leqq \|g\|_\infty^2 \|\hat{\chi}\|_\infty^4 \|G_m\|_2^2 \leqq \frac{1}{4\pi m^2} \quad (\text{if } \|g\|_\infty \leqq 1, \|\hat{\chi}\|_\infty \leqq 1).\end{aligned} \tag{1.4.11}$$

Thus if $a < 4\pi m^2$ and $1 \leqq p < \dfrac{4\pi m^2}{a}$, then $(I - paB_{g,\chi}) > 0$, thus from (1.4.9), $\exp(\frac{1}{2}pa\|\mathbf{A}_{g,\chi}\|^2_{\mathcal{T}_2}) \in L^1(\mathcal{S}', \nu)$, and moreover,

$$\begin{aligned}E_\nu\left[\exp\left(p\frac{a}{2}\|\mathbf{A}_{g,\chi}\|^2_{\mathcal{T}_2}\right)\right] &= [\det(I - paB_{g,\chi})]^{-1/2} \\ &= [\det(I + pa[I - paB_{g,\chi}]^{-1}B_{g,\chi})]^{1/2} \\ &\leqq \exp(\tfrac{1}{2}pa\|[I - paB_{g,\chi}]^{-1}\|\,\|B_{g,\chi}\|_{\mathcal{T}_1}).\end{aligned} \tag{1.4.12}$$

But, from (1.4.11),

$$\|[I - paB_{g,\chi}]^{-1}\| \leqq (1 - pa\|B_{g,\chi}\|)^{-1} \leqq \left(1 - \frac{pa}{4\pi m^2}\right)^{-1}, \tag{1.4.13}$$

and, as $B_{g,\chi} \geqq 0$,

$$\|B_{g,\chi}\|_{\mathcal{T}_1} = \operatorname{Tr} B_{g,\chi} = \mathbf{c}_\chi^2 \|g\|_2^2, \tag{1.4.14}$$

then (1.4.3) follows from (1.4.8), (1.4.12), (1.4.13) and (1.4.14). □

Then, for $(\varrho,\zeta) \in \mathbf{D}_{(4\pi m^2)}$, $K \in \mathbb{N}$, and $\mathbf{f} = (f_j)_{1 \leqq j \leqq K} \in \mathcal{S}^K$, one sets

$$\begin{aligned}Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, \chi}(\mathbf{f}) = \int_{\mathcal{S}'} &\exp(\tfrac{1}{2}\varrho^2\zeta^2\mathbf{c}_\chi^2\|g\|_2^2)\exp(i\varrho\zeta\mathbf{c}_\chi\langle\sigma, g\rangle) \\ &\cdot[\exp(-\tfrac{1}{2}\varrho^2\zeta^2\mathbf{c}_\chi^2\|g\|_2^2)\det{}_2(I + i\varrho\zeta\mathbf{A}_{g,\chi}(\sigma))]^{-\frac{1}{2\zeta^2}} \\ &\cdot\prod_{j=1}^K \exp(-\tfrac{1}{2}(\Sigma_m^{-1}f_j, [I + i\varrho\zeta\mathbf{A}_{g,\chi}(\sigma)]^{-1}\Sigma_m^{-1}f_j)_{\mathcal{H}^1})\cdot\nu(d\sigma).\end{aligned} \tag{1.4.15}$$

[The integral in the right hand side of (1.4.15) is well defined from Lemma 1.4 and Hölder's inequality, because on the one hand, if $(\varrho,\zeta) \in \mathbf{D}_{(4\pi m^2)}$, there exists $a < 4\pi m^2$ such that $(\varrho,\zeta) \in \mathbf{D}_a$ and, on the other hand,

$$e^{i\varrho\zeta\langle\cdot, g\rangle} \in \bigcap_{1 \leqq q < \infty} L^q(\mathcal{S}', \nu)$$

and

$$\exp(-\tfrac{1}{2}(\Sigma_m^{-1}f, [I+i\varrho\zeta\mathbf{A}_{g,\chi}]^{-1}\Sigma_m^{-1}f)_{\mathscr{H}^1}) \in L^\infty(\mathscr{S}', \nu)$$

because, as $\mathbf{A}_{g,\chi}(\sigma)$ is self-adjoint, one has $\|[I+i\varrho\zeta\mathbf{A}_{g,\chi}(\sigma)]^{-1}\| \leqq |\cos \operatorname{Arg}\varrho\zeta|^{-1}$; besides, this integral defines actually a function of ϱ^2 and ζ^2, because $\nu=\check{\nu}$.]

Functions (1.4.15) are obviously analytic in $(\varrho,\zeta)\in\mathbf{D}_{(4\pi m^2)}$ and, for $\varrho\in\mathbb{R}$, $\zeta=\dfrac{1}{\sqrt{N}}$, and $K\leqq N$, $(N\in\mathbb{N}_+)$, they coincide with functions (1.3.8) in which the f_j are nonvanishing only for K values of the index j.

If, besides, $\mathscr{U}$ denotes the family of domains

$$U_{a,\varepsilon}=\left\{(\varrho,\zeta)\in\mathbf{D}_a;\left|\frac{\pi}{2}-|\operatorname{Arg}\varrho\zeta|\right|>\varepsilon\right\},\quad (a<4\pi m^2, \varepsilon>0), \tag{1.4.16}$$

then, from (1.4.8) and Lebesgue's theorem, for any $U\in\mathscr{U}$, each function (1.4.15) has a continuous continuation to the closure $\bar{U}$ of U; we use the same symbol to denote the function (1.4.15) and its continuation to $\mathbf{D}^+_{(4\pi m^2)} \equiv \bigcup_{U\in\mathscr{U}} \bar{U}$. For all $\zeta\in\mathbb{C}$, one has $(0,\zeta)\in\mathbf{D}^+_{(4\pi m^2)}$ and

$$Z^{\star}_{\zeta^2,0,g^2,\chi}(\mathbf{f})=\prod_{j=1}^{K} e^{-\frac{1}{2}(f_j,\Sigma_m^{-1}f_j)_2}, \tag{1.4.17}$$

and, if $\varrho\in\mathbb{C}$ is such that [10] $(\varrho,0)\in\mathbf{D}^+_{(4\pi m^2)}$,

$$\begin{aligned} Z^{\star}_{0,\frac{\varrho^2}{8},g^2,\chi}(\mathbf{f}) &= \int_{\mathscr{S}'} \exp(\tfrac{1}{4}\varrho^2\mathbf{c}_\chi^2\|g\|_2^2)\exp(-\tfrac{1}{4}\varrho^2\|\mathbf{A}_{g,\chi}(\sigma)\|^2_{\mathscr{T}_2}) \\ &\quad \cdot \prod_{j=1}^{K}\exp(-\tfrac{1}{2}(f_j,\Sigma_m^{-1}f_j)_2)\nu(d\sigma) \\ &= \left[\det{}_2\left(I+\frac{\varrho^2}{2}B_{g,\chi}\right)\right]^{-1/2}\prod_{j=1}^{K}\exp(-\tfrac{1}{2}(f_j,\Sigma_m^{-1}f_j)_2). \end{aligned} \tag{1.4.18}$$

The function

$$Z^0_{\zeta^2,\frac{\varrho^2}{8},g^2,\chi} \equiv Z^{\star}_{\zeta^2,\frac{\varrho^2}{8},g^2,\chi}(\mathbf{0})$$

is thus nonvanishing on $\mathbf{D}^+_{(4\pi m^2)}\backslash\mathbf{D}_{(4\pi m^2)}$ and, consequently, the meromorphic functions defined on $\mathbf{D}_{(4\pi m^2)}$ by

$$S^{\star}_{\zeta^2,\frac{\varrho^2}{8},g^2,\chi}(\mathbf{f})=\frac{Z^{\star}_{\zeta^2,\frac{\varrho^2}{8},g^2,\chi}(\mathbf{f})}{Z^0_{\zeta^2,\frac{\varrho^2}{8},g^2,\chi}},\quad (\mathbf{f}\in\mathscr{S}^K; K\in\mathbb{N}), \tag{1.4.19}$$

have, under the above conditions, a continuous continuation to $\mathbf{D}^+_{(4\pi m^2)}$, which one is constant (and independent of g and χ) over $\mathbf{D}^+_{(4\pi m^2)}\backslash\mathbf{D}_{(4\pi m^2)}$; especially, if

10 This condition is particularly true for all $\varrho\in\mathbb{R}$, and also if $|\varrho|^2<8\pi m^2$. Under this assumption, the integral in Eq. (1.4.18) is well defined (from Lebesgue's theorem); this, according to (1.4.9), implies that $\left(I+\operatorname{Re}\frac{\varrho^2}{2}\cdot B_{g,\chi}\right)>0$, and $\left[\det_2\left(I+\frac{\varrho^2}{2}B_{g,\chi}\right)\right]^{-1/2}$ is then defined following (1.3.7)

$(\varrho, 0) \in \mathbf{D}^{+}_{(4\pi m^2)}$,

$$S^{\star}_{0,\frac{\varrho^2}{8},g^2,\chi}(\mathbf{f}) = \prod_{j=1}^{K} \exp(-\tfrac{1}{2}(f_j, \Sigma_m^{-1} f_j)_2) = \prod_{j=1}^{K} \hat{\mu}_m(f_j) \tag{1.4.20}$$

is the generating functional of a free field.

1.5. Removing the Ultraviolet Cut-off

One now obtains the expected analytic continuation of functions (1.1.5) and (1.1.6) as limit of functions (1.4.15) and (1.4.19) where one substitutes a suitable approximate unit to χ; more precisely, we want to show

Theorem I. *Let $\chi_q \in \mathscr{S}$, $(q \in \mathbb{N})$ be an approximate unit such that*[11], *for any $q \in \mathbb{N}$, $\hat{\chi}_q$ has compact support and $\|\hat{\chi}_q\|_\infty \leqq 1$; for any compact set $K \subset E$, there exists $q_K \in \mathbb{N}$ such that $\hat{\chi}_{q|K} = 1$, $\forall q \geqq q_K$; then, for any $(\varrho, \zeta) \in \mathbf{D}^{+}_{(4\pi m^2)}$, the sequence*

$$(Z^{\star}_{\zeta^2,\frac{\varrho^2}{8},g^2,\chi_q}(\mathbf{f}))_{q\in\mathbb{N}}$$

converges and its limit

$$Z^{\star}_{\zeta^2,\frac{\varrho^2}{8},g^2}(\mathbf{f}) \equiv \lim_{q\to\infty} Z^{\star}_{\zeta^2,\frac{\varrho^2}{8},g^2,\chi_q}(\mathbf{f}) \tag{1.5.1}$$

is independent of the chosen approximate unit; moreover, for any $U \in \mathscr{U}$, (defined by (1.4.16)), the convergence is uniform for (ϱ, ζ) in any bounded subset of $\bar{U}$.

As convergence is, in particular, uniform on any compact set of $\mathbf{D}_{(4\pi m^2)}$, functions

$$(\varrho, \zeta) \mapsto Z^{\star}_{\zeta^2,\frac{\varrho^2}{8},g^2}(\mathbf{f})$$

are holomorphic; for any $U \in \mathscr{U}$ they are continuous on $\bar{U}$ and, from (1.2.11), they continue functions (1.1.5).

One notes that, for $\zeta^2 = \dfrac{1}{N}$, $(N \in \mathbb{N}_+)$ one so obtains an extension of the previously known analyticity domain[12] of functions

$$\lambda \mapsto Z^{\star}_{\frac{1}{N},\lambda,g^2}(\mathbf{f}).$$

This extended domain contains especially the region[13]

$$\bigcup_{-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}} e^{i\theta} \Delta_{(\pi m^2)},$$

where $\Delta_r = \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda^{-1} > r^{-1}\}$; on the other hand, if $|\operatorname{Arg}\lambda| \leqq \pi - \varepsilon$, one has from (1.4.17),

$$\lim_{|\lambda|\to 0} Z^{0}_{\frac{1}{N},\lambda,g^2} = 1,$$

11 A little more work allows less restrictive assumptions

12 One already knows that these functions are analytic in the half-plane $\operatorname{Re}\lambda > 0$, see (1.1.5)

13 Nevanlinna-Sokal's theorem [5] allows us to obtain Borel summability in this region (see Theorem II below), however we can obtain uniform bounds only for $|\theta| < \pi - \varepsilon$

thus the "normalized" functions

$$\lambda \mapsto S^{\star}_{\frac{1}{N},\lambda,g^2}(\mathbf{f}) \equiv Z^{\star}_{\frac{1}{N},\lambda,g^2}(\mathbf{f})/Z^{0}_{\frac{1}{N},\lambda,g^2}$$

are holomorphic in a region of the form

$$\bigcup_{0<\varepsilon<\pi} \{\lambda \in \mathbb{C}; |\operatorname{Arg}\lambda| < \pi - \varepsilon, 0 < |\lambda| < r_\varepsilon\}.$$

Theorem I is an easy consequence of the following lemma: let

$$\mathscr{W} = \{h \in \mathscr{S}'; \hat{h} \in C^\infty(E), \|\hat{h}\|_\infty \leqq 1\},$$

let also $k \in \mathscr{S}$ be a function such that $0 \leqq \hat{k}(p) \leqq 1$, $\forall p \in E$; $\hat{k}(p) = 1$ if $|p| \leqq 1$; $\hat{k}(p) = 0$ if $|p| \geqq 2$; for $\kappa > 0$ let $k_\kappa(x) = \kappa^2 k(\kappa x)$, $(x \in E)$, then one has

Lemma. *Let $\kappa_0 > 0$ and $\kappa_n = \kappa_0 e^n$, $(n \in \mathbb{N})$, then for any $(\varrho, \zeta) \in \mathbf{D}^+_{(4\pi m^2)}$, the sequence*

$$(Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, h_* k_{\kappa_n}}(\mathbf{f}))_{n \in \mathbb{N}}$$

converges, uniformly with respect to $h \in \mathscr{W}$ and, for any $U \in \mathscr{U}$, uniformly on any bounded subset of $\bar{U}$.

First we deduce Theorem I from this lemma as follows: one notes that the Dirac measure δ belongs to $\mathscr{W}$ and that, if $(\chi_q)_{q \in \mathbb{N}}$ is an approximate unit satisfying the assumptions of the theorem, one has $\chi_q \in \mathscr{W}$, $\forall q \in \mathbb{N}$, and

$$\lim_{n \to \infty} Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, \chi_q * k_{\kappa_n}}(\mathbf{f}) = Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, \chi_q}(\mathbf{f}), \quad \forall q \in \mathbb{N}, \tag{1.5.2}$$

because $\chi_q * k_{\kappa_n} = \chi_q$ for sufficiently large n. Then, given some $\varepsilon > 0$, one first chooses $n_0 \in \mathbb{N}$ such that

$$\left| \lim_{n \to \infty} Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, h_* k_{\kappa_n}}(\mathbf{f}) - Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, h_* k_{\kappa_{n_0}}}(\mathbf{f}) \right| \leqq \frac{\varepsilon}{2}, \quad \forall h \in \mathscr{W}, \tag{1.5.3}$$

and then $q_0 \in \mathbb{N}$ such that $\chi_q * k_{\kappa_{n_0}} = k_{\kappa_{n_0}}$, $\forall q \geqq q_0$: then one has

$$\begin{aligned} &\left| \lim_{n \to \infty} Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, k_{\kappa_n}}(\mathbf{f}) - Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, \chi_q}(\mathbf{f}) \right| \\ &\leqq \left| \lim_{n \to \infty} Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, \delta_* k_{\kappa_n}}(\mathbf{f}) - Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, \delta_* k_{\kappa_{n_0}}}(\mathbf{f}) \right| \\ &+ \left| Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, \chi_q * k_{\kappa_{n_0}}}(\mathbf{f}) - \lim_{n \to \infty} Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, \chi_q * k_{\kappa_n}}(\mathbf{f}) \right| \leqq \varepsilon, \quad \forall q \geqq q_0. \end{aligned} \tag{1.5.4}$$

As the choice of n_0 and q_0 can be done independently of (ϱ, ζ) in each bounded subset of $\bar{U}$, $(U \in \mathscr{U})$, this proves the theorem. □

1.6. Introduction of Auxiliary Variables

Now one reduces the proof of Lemma 1.5 to that of Proposition 1.6 below, which is given in Sect. 1.7–1.10.

Suppose the function $k \in \mathscr{S}$ and the sequence $(\kappa_n)_{n \in \mathbb{N}}$ are chosen according to the assumptions of Lemma 1.5; for any sequence $\underline{t} = (t_j)_{1 \leqq j < +\infty} \in [0,1]^{\mathbb{N}_+}$, vanishing outside some finite set of values of the index j, one sets

$$k_{(\underline{t})}=k_{\kappa_0}+\sum_{j=1}^{+\infty} t_j(k_{\kappa_j}-k_{\kappa_{j-1}}), \tag{1.6.1}$$

(so that, $k_{\kappa_0}=k_{(0)}$ and, for $n\geqq 1$, $k_{\kappa_n}=k_{(1_n)}$, where 1_n is the characteristic function of $\{1,\dots,n\}\subset\mathbb{N}_+$).

On the other hand, let $(\ell_n)_{n\in\mathbb{N}}$, ($\ell_0=0$, $\ell_n>0$ if $n>0$), be a given sequence of integers (to be determined later); one sets

$$\mathscr{E}_j^n=\{\underline{q}=(q_i)_{j\leqq i\leqq n}\in\mathbb{N}^{n-j+1};\, q_j=\ell_j, 0\leqq q_i<\ell_i \text{ if } j<i\leqq n\},\ (0\leqq j\leqq n),$$

and, for $t\in[0,1]$,

$$(J_0(t))_i=0,\quad \forall i\in\mathbb{N}_+\,;\quad (J_j(t))_i=\begin{cases}0, & \text{if } i>j,\\ t, & \text{if } i=j,\\ 1, & \text{if } 1\leqq i<j,\end{cases}\qquad \text{if } j\in\mathbb{N}_+.$$

Last $D_j=\dfrac{\partial}{\partial t_j}$, then if one supposes provisionally that the function

$$\mathscr{Z}(\underline{t})\equiv Z^{\star}_{\zeta^2,\frac{\varrho^2}{8},g^2,h*k_{(\underline{t})}}(\mathbf{f}) \tag{1.6.2}$$

is sufficiently derivable, one has, from Taylor's formula[14]

$$\mathscr{Z}(\mathbf{1}_n)=\sum_{j=0}^{n}\sum_{\underline{q}\in\mathscr{E}_j^n}\frac{1}{\prod_{i=j}^{n}(q_i!)}\cdot\ell_j\int_0^1(1-t)^{\ell_j-1}\cdot\left[\prod_{i=j}^{n}D_i^{q_i}\right]\mathscr{Z}(J_j(t))\cdot dt. \tag{1.6.3}$$

If now one admits the

Proposition. *The function $\mathscr{Z}$ defined by* (1.6.2) *is indefinitely derivable. For any $\varepsilon>0$, $\eta<\frac{1}{4}$, there exist constants $b>0$, $c>0$, independent*[15] *of $h\in\mathscr{W}$ and of (ϱ,ζ) in any bounded set of $\bar{U}$, ($U\in\mathscr{U}$), such that for all $j\geqq 0$, $n\geqq j$, $q=(q_i)_{j\leqq i\leqq n}\in\mathbb{N}^{n-j+1}$, ($q_0=0$), and $t\in[0,1]$, one has*

$$\left|\left[\prod_{i=j}^{n}D_i^{q_i}\right]\mathscr{Z}(J_j(t))\right|\leqq\exp(b\kappa_j^{\varepsilon})\left[\left(\sum_{i=j}^{n}q_i\right)!\right]^2\prod_{i=j}^{n}(c\kappa_{i-1})^{-\eta q_i}, \tag{1.6.4}$$

the end of the proof is standard; moreover, the actual value [2 in the present case] of the exponent of the factor $\left[\left(\sum_{i=j}^{n}q_i\right)!\right]$ in the right hand side of (1.6.4) is irrelevant, so we want to conclude under the assumption

$$\left|\left[\prod_{i=j}^{n}D_i^{q_i}\right]\mathscr{Z}(J_j(t))\right|\leqq\exp(b\kappa_j^{\varepsilon})\left[\left(\sum_{i=j}^{n}q_i\right)!\right]^{s+1}\prod_{i=j}^{n}(c\kappa_{i-1})^{-\eta q_i}, \tag{1.6.5}$$

14 For $j=0$, one has $q_0=\ell_0=0$; one agrees that $D_0^{q_0}$ is the identity and that $\ell_0\int_0^1(1-t)^{\ell_0-1}dt=1$

15 These constants are also independent of $m\geqq m_0>0$

(where $s>0$ is arbitrary), which generalizes (1.6.4): first one chooses $\xi>0$, arbitrarily small, and sets $\alpha_i=(1-e^{-\xi})^{-1}\left(\frac{\kappa_{i-1}}{\kappa_0}\right)^{\xi}$, $(i\geqq 1)$, so that[16] $\sum_{i=1}^{\infty}\alpha_i^{-1}=1$, therefore

$$\left(\sum_{i=j}^{n} q_i\right)! \leqq \prod_{i=j}^{n} q_i!\alpha_i^{q_i}, \qquad (q_i\in\mathbb{N}, \quad q_0=0). \tag{1.6.6}$$

Inserting (1.6.6) in (1.6.5) one sees that for any $\eta_1<\eta$, there exists $c_1>0$ such that

$$\left\|\left[\prod_{i=j}^{n} D_i^{q_i}\right]\mathscr{Z}(J_j(t))\right| \leqq \exp(b\kappa_j^{\varepsilon})\prod_{i=j}^{n}(q_i!)^{s+1}(c_1\kappa_{i-1})^{-\eta_1 q_i}. \tag{1.6.7}$$

Then, from (1.6.3),

$$\left|\mathscr{Z}(\mathbf{1}_n)-\mathscr{Z}(\mathbf{1}_{n-1})\right| \leqq \sum_{j=0}^{n}\sum_{\substack{q\in\mathscr{E}_j^n\\ q_n\neq 0}} \exp(b\kappa_j^{\varepsilon})\prod_{i=j}^{n}(q_i!)^{s}(c_1\kappa_{i-1})^{-\eta_1 q_i}. \tag{1.6.8}$$

Now one chooses the sequence $(\ell_i)_{i\in\mathbb{N}}$: one sets

$$(c_1\kappa_{i-1})^{\eta_1}=\exp(2v_i), \qquad (i\geqq 1), \qquad v_0=-\infty,$$

and

$$\ell_i=\mathbf{E}[e^{(v_i/s)-1}], \qquad (i\geqq 0).^{17}$$

One checks that

$$(q!)^s\exp(-2v_iq) \leqq \begin{cases} \exp\left(-\frac{s}{e}e^{v_i/s}\right), & \text{if } \; q=\ell_i, \\ \exp(-v_iq), & \text{if } \; 0\leqq q<\ell_i, \quad i\geqq 0). \end{cases} \tag{1.6.9}$$

Therefore,

$$\begin{aligned} |\mathscr{Z}(\mathbf{1}_n)-\mathscr{Z}(\mathbf{1}_{n-1})| &\leqq \sum_{j=0}^{n}\sum_{\substack{q\in\mathscr{E}_j^n\\ q_n\neq 0}} \exp(b\kappa_j^{\varepsilon})\prod_{i=j}^{n}(q_i!)^s\exp(-2v_iq_i) \\ &\leqq \sum_{j=0}^{n}\exp\left(b\kappa_j^{\varepsilon}-\frac{s}{e}e^{v_j/s}\right)\cdot\prod_{i=j+1}^{n-1}\left(\sum_{q_i=0}^{\ell_i-1}\exp(-v_iq_i)\right)\cdot\sum_{q_n=1}^{\ell_n-1}\exp(-v_nq_n) \\ &\leqq \left(\sum_{j=0}^{\infty}\exp\left(b\kappa_j^{\varepsilon}-\frac{s}{e}e^{v_j/s}\right)\right)\cdot\prod_{i=1}^{\infty}\left(\sum_{r_i=0}^{\infty}\exp(-v_ir_i)\right)\cdot\exp(-v_n) \\ &\leqq M\exp\left(-\frac{\eta_1}{2}n\right), \quad \left(\text{if } \; \varepsilon<\frac{\eta_1}{2s} \; \text{ and } \; \kappa_n=\kappa_0e^n\right). \quad \square \end{aligned} \tag{1.6.10}$$

---

16 Recall that $\kappa_j=\kappa_0e^j$, $(j\in\mathbb{N})$

17 $\mathbf{E}$ denotes the entire part; one assumes, for simplicity, κ_0 is chosen sufficiently large, so that $v_i\geqq s$, $\forall i\geqq 1$; last, conventionaly, $+\infty\cdot 0=0$

1.7. Computation of the Derivatives of $\mathscr{Z}$

To compute the derivatives of $\mathscr{Z}$ one uses the following "integration by parts" formula:
Let F be an $\mathfrak{A}$-measurable function on $\mathscr{S}'$ and let $g\in\mathscr{S}$, one assumes there exists $p>1$ such that for any α in a real neighbourhood of 0 one has $F(\cdot+\alpha g)\in L^p(\mathscr{S}',\nu)$, then

$$\int_{\mathscr{S}'} F(\sigma+\alpha g)\exp(-\tfrac{1}{2}\alpha^2\|g\|_2^2-\alpha\langle\sigma,g\rangle)\nu(d\sigma)=\int_{\mathscr{S}'} F(\sigma)\nu(d\sigma). \tag{1.7.1}$$

One assumes moreover that, for ν-almost all $\sigma\in\mathscr{S}'$ the function $\alpha\mapsto F(\sigma+\alpha g)$ is derivable in some neighbourhood of 0, and that the functions $|F(\cdot+\alpha g)|$ and $\left|\frac{d}{d\alpha}F(\cdot+\alpha g)\right|$ are dominated by some fixed positive function in $L^p(\mathscr{S}',\nu)$. Then if one sets $\partial_g F(\sigma)=\left.\frac{d}{d\alpha}\right|_{\alpha=0}F(\sigma+\alpha g)$, one obtains

$$\int_{\mathscr{S}'}\langle\sigma,g\rangle F(\sigma)\nu(d\sigma)=\int_{\mathscr{S}'}\partial_g F(\sigma)\nu(d\sigma) \tag{1.7.2}$$

by derivation of (1.7.1) under the integral sign.

Coming back to function $\mathscr{Z}$, we shorten notations by writing, from (1.6.2) and (1.4.15) where one substitutes $h_* k_{(\underline{t})}$ for χ

$$\mathscr{Z}(\underline{t})=\int_{\mathscr{S}'}\exp(\tfrac{1}{2}\varrho^2\zeta^2\mathbf{c}(\underline{t})^2\|g\|_2^2+i\varrho\zeta\mathbf{c}(\underline{t})\langle\sigma,g\rangle)Y(\underline{t};\sigma)\nu(d\sigma), \tag{1.7.3}$$

and one checks recursively the formula

$$\left[\prod_{k=j}^{n}D_k^{q_k}\right]\mathscr{Z}(\underline{t})=\int_{\mathscr{S}'}\exp(\tfrac{1}{2}\varrho^2\zeta^2\mathbf{c}(\underline{t})^2\|g\|_2^2+i\varrho\zeta\mathbf{c}(\underline{t})\langle\sigma,g\rangle)\cdot\left[\prod_{k=j}^{n}(D_k+i\varrho\zeta(D_k\mathbf{c})\partial_g)^{q_k}\right]Y(\underline{t},\sigma)\cdot\nu(d\sigma), \tag{1.7.4}$$

(where the derivations $(D_k+i\varrho\zeta(D_k\mathbf{c})\partial_g)$ are mutually commutative). Indeed, if X denotes any of the functions $\left[\prod_k(D_k+i\varrho\zeta(D_k\mathbf{c})\partial_g)\right]Y$, one justifies easily derivations under the integral sign by estimates mainly coming from (1.4.8), and one obtains

$$\begin{aligned}
D_k&\int_{\mathscr{S}'}\exp(\tfrac{1}{2}\varrho^2\zeta^2\mathbf{c}(\underline{t})^2\|g\|_2^2+i\varrho\zeta\mathbf{c}(\underline{t})\langle\sigma,g\rangle)X(\underline{t};\sigma)\nu(d\sigma)\\
&=\int_{\mathscr{S}'}\exp(\tfrac{1}{2}\varrho^2\zeta^2\mathbf{c}(\underline{t})^2\|g\|_2^2+i\varrho\zeta\mathbf{c}(\underline{t})\langle\sigma,g\rangle)\\
&\quad\cdot\{(D_k\mathbf{c}(\underline{t}))[\varrho^2\zeta^2\mathbf{c}(\underline{t})\|g\|_2^2+i\varrho\zeta\langle\sigma,g\rangle]X(\underline{t};\sigma)+D_kX(\underline{t};\sigma)\}\nu(d\sigma)\\
&=\int_{\mathscr{S}'}\exp(\tfrac{1}{2}\varrho^2\zeta^2\mathbf{c}(\underline{t})^2\|g\|_2^2+i\varrho\zeta\mathbf{c}(\underline{t})\langle\sigma,g\rangle)\\
&\quad\cdot\{i\varrho\zeta(D_k\mathbf{c}(\underline{t}))\partial_gX(\underline{t};\sigma)+D_kX(\underline{t};\sigma)\}\nu(d\sigma),
\end{aligned} \tag{1.7.5}$$

where the second equality comes from (1.7.2) applied to the function

$$F(\sigma)=e^{i\varrho\zeta\alpha(t)\langle\sigma,g\rangle}X(\underline{t},\sigma). \quad \square$$

1.8. Explicit Expression of the Derivatives of $\mathscr{X}$

To compute the right hand side of (1.7.4), it seems useful to introduce $\tilde{\mathbf{c}}(\underline{t},\underline{u})\in\mathbb{C}$, $\tilde{\mathbf{A}}(\underline{t},\underline{u};\sigma)\in\mathscr{T}_2(\mathscr{H}^1)$, where $\underline{t},\underline{u},\in[0,1]^{\mathbb{N}_+}$ are sequences which vanish outside a finite set, by

$$\tilde{\mathbf{c}}(\underline{t},\underline{u})=(h*k_{(\underline{t})},\Sigma_m^{-1}h*k_{(\underline{u})})_2\,, \tag{1.8.1}$$

$$\tilde{\mathbf{A}}(\underline{t},\underline{u};\sigma)\psi=\Sigma_m^{-1}(\check{h}*\check{k}_{(\underline{t})}*(\sigma\cdot g\cdot(h*k_{(\underline{u})}*\psi))),\quad \forall\psi\in\mathscr{H}^1,\quad (\sigma\in\mathscr{S}'), \tag{1.8.2}$$

[one has $\tilde{\mathbf{c}}(\underline{t},\underline{u})=\tilde{\mathbf{c}}(\underline{u},\underline{t})$ and $\tilde{\mathbf{A}}(\underline{t},\underline{u};\sigma)=\tilde{\mathbf{A}}(\underline{u},\underline{t};\sigma)^*$], and to set[18]

$$\tilde{Y}_0(\underline{t},\underline{u};\sigma)=[\exp(-\tfrac{1}{2}\varrho^2\zeta^2\tilde{\mathbf{c}}(\underline{t},\underline{u})^2\|g\|_2^2)\cdot\det{}_2(I+i\varrho\zeta\tilde{\mathbf{A}}(\underline{t},\underline{u};\sigma))]^{-\frac{1}{2\zeta^2}}, \tag{1.8.3}$$

and,

$$\tilde{Y}(\underline{t},\underline{u};\sigma)=\tilde{Y}_0(\underline{t},\underline{u};\sigma)\cdot\prod_{j=1}^{K}\exp(-\tfrac{1}{2}(\Sigma_m^{-1}f_j,[I+i\varrho\zeta\tilde{\mathbf{A}}(\underline{t},\underline{u};\sigma)]^{-1}\Sigma_m^{-1}f_j)_{\mathscr{H}^1}), \tag{1.8.4}$$

so that, from (1.7.3), (1.6.2), (1.4.15), (1.2.2), and (1.3.4), one has

$$Y(\underline{t};\sigma)=\tilde{Y}(\underline{t},\underline{t};\sigma). \tag{1.8.5}$$

Besides, one notes $D'_j=\dfrac{\partial}{\partial t_j}$, $D''_j=\dfrac{\partial}{\partial u_j}$, $(j\in\mathbb{N}_+)$, and one sets[19]

$$\tilde{\mathbf{c}}^{(j,k)}=D'_jD''_k\tilde{\mathbf{c}},\qquad \tilde{\mathbf{A}}^{(j,k)}=D'_jD''_k\tilde{\mathbf{A}},\qquad (j,k\in\mathbb{N}), \tag{1.8.6}$$

(here D'_0 and D''_0 denotes conventionally the identity). Then, for $r_j\in\mathbb{N}_+$, $(1\leqq j\leqq n)$, one has from (1.8.5),

$$\begin{aligned}&\left[\prod_{j=1}^{n}(D_{r_j}+i\varrho\zeta(D_{r_j}\mathbf{c})\partial_g)\right]Y(\underline{t};\sigma)\\&=\left[\prod_{j=1}^{n}\{(D'_{r_j}+i\varrho\zeta\tilde{\mathbf{c}}^{(r_j,0)}\partial_g)+(D''_{r_j}+i\varrho\zeta\tilde{\mathbf{c}}^{(0,r_j)}\partial_g)\}\right]\tilde{Y}(\underline{t},\underline{t};\sigma),\end{aligned} \tag{1.8.7}$$

where one notes that the derivations $(D'_r+i\varrho\zeta\tilde{\mathbf{c}}^{(r,0)}\partial_g)$, $(r\geqq1)$, and $(D''_s+i\varrho\zeta\tilde{\mathbf{c}}^{(0,s)}\partial_g)$, $(s\geqq1)$, are mutually commutative.

To make explicit the right hand side of (1.8.7), one introduces the following notations:

$$\tilde{\mathbf{A}}^{(j,k)}_{[g]}(\underline{t},\underline{u})=\tilde{\mathbf{A}}^{(j,k)}(\underline{t},\underline{u};g)=\partial_g\tilde{\mathbf{A}}^{(j,k)}(\underline{t},\underline{u};\sigma),\qquad \forall\sigma\in\mathscr{S}'. \tag{1.8.8}$$

---

18 As $[I+i\varrho\zeta\tilde{\mathbf{A}}(\underline{t},\underline{u};\sigma)]$ is invertible if $\underline{t}=\underline{u}$, so it is in some neighbourhood (depending a priori of all parameters) of the diagonal

19 One notes that $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{A}}$ are affine functions of t_j and $u_j (j\geqq1)$, so that their only nonvashing derivatives are those given by (1.8.6)

Next,

$$
“\prod_{j=k}^{n}”\tilde{\mathbf{A}}^{(r_j,s_j)}=\begin{cases}1, & (k>n),\\ \tilde{\mathbf{A}}^{(r_k,s_k)}, & (k=n),\\ \tilde{\mathbf{A}}^{(r_k,s_k)}\cdot“\prod_{i=k+1}^{n}”\tilde{\mathbf{A}}^{(r_i,s_i)}-\tilde{\mathbf{c}}^{(r_{k+1},s_k)}\tilde{\mathbf{A}}_{[g]}^{(r_k,s_{k+1})}\cdot“\prod_{i=k+2}^{n}”\tilde{\mathbf{A}}^{(r_i,s_i)}, & (k<n),\end{cases}
\tag{1.8.9}
$$

and,

$$
“\boxed{\prod}_{j=k}^{n}”\tilde{\mathbf{A}}^{(r_j,s_j)}=\begin{cases}“\prod_{j=k}^{n}”\tilde{\mathbf{A}}^{(r_j,s_j)}, & (k=n-1),\\ “\prod_{j=k}^{n}”\tilde{\mathbf{A}}^{(r_j,s_j)}-\tilde{\mathbf{c}}^{(r_k,s_n)}\tilde{\mathbf{A}}_{[g]}^{(r_n,s_k)}\cdot“\prod_{i=k+1}^{n-1}”\tilde{\mathbf{A}}^{(r_i,s_i)}, & (k<n-1).\end{cases}
\tag{1.8.10}
$$

Last, $\underline{a}=\{a_j\in\mathbb{N}_+;1\leqq j\leqq n_1\}$ and $\underline{b}=\{b_j\in\mathbb{N}_+;1\leqq j\leqq n_2\}$ being given, one denotes by $\mathfrak{S}_{\bigcirc}^{k}(\underline{a},\underline{b})$, $(k\geqq 2)$, the set of $(\underline{r},\underline{s})=\{(r_i,s_i)\in\mathbb{N}\times\mathbb{N};1\leqq i\leqq k\}$, satisfying the conditions:

$(S_{\bigcirc},i)$ – the set of nonvanishing $(r_i)_{1\leqq i\leqq k}$, [respectively $(s_i)_{1\leqq i\leqq k}$], is a permutation of $(a_j)_{1\leqq j\leqq n_1}$, [respectively $(b_j)_{1\leqq j\leqq n_2}$].

$(S_{\bigcirc},ii)$ – $r_1\neq 0$ or $s_k\neq 0$ and for all $i\in\{1,\dots,k-1\}$, either $s_i\neq 0$ or $r_{i+1}\neq 0$;

$\mathfrak{S}_n^{(\underline{k})}(\underline{a},\underline{b})$, $(n\geqq 1,\underline{k}=\{k_q\geqq 2;1\leqq q\leqq n\})$, the set of $\{(\underline{r}^{(q)},\underline{s}^{(q)})=\{(r_i^{(q)},s_i^{(q)})\in\mathbb{N}\times\mathbb{N};1\leqq i\leqq k_q\};1\leqq q\leqq n\}$, satisfying

(S_n,i) – the set of nonvanishing $(r_i^{(q)})_{1\leqq i\leqq k_q}^{1\leqq q\leqq n}$, $\left[\text{respectively}\,(s_i^{(q)})_{1\leqq i\leqq k_q}^{1\leqq q\leqq n}\right.$ is a permutation of $(a_j)_{1\leqq j\leqq n_1}$, [respectively $(b_j)_{1\leqq j\leqq n_2}$].

(S_n,ii) – for each $q\in\{1,\dots,n\}$, $r_1^{(q)}=s_{k_q}^{(q)}=0$, and for each $i\in\{1,\dots,k_q-1\}$, either $s_i^{(q)}\neq 0$ or $r_{i+1}^{(q)}\neq 0$;

$\mathfrak{T}_n^{(\underline{k})}(\underline{a},\underline{b})$, $(n\geqq 0$, $\underline{k}=\{(k_q)_{0\leqq q\leqq n}$; $k_0\geqq 0$, $k_n\geqq 0$, $k_q\geqq 2$ if $1\leqq q\leqq n-1\})$, the set of $\{(\underline{r}^{(q)},\underline{s}^{(q)})=\{(r_i^{(q)},s_i^{(q)})\in\mathbb{N}\times\mathbb{N};1\leqq i\leqq k_q\};0\leqq q\leqq n\}$, satisfying

(T_n,i)– the set of nonvanishing $(r_i^{(q)})_{1\leqq i\leqq k_q}^{0\leqq q\leqq n}$, $\left[\text{respectively}\ (s_i^{(q)})_{1\leqq i\leqq k_q}^{0\leqq q\leqq n}\right.$ is a permutation of $(a_j)_{1\leqq j\leqq n_1}$, [respectively $(b_j)_{1\leqq j\leqq n_2}$].

(T_n,ii) – $r_1^{(0)}\neq 0$ and $r_1^{(q)}=0$, $(1\leqq q\leqq n)$; $s_{k_n}^{(n)}\neq 0$ and $s_{k_q}^{(q)}=0$, $0\leqq q\leqq n-1)$; for each $q\in\{0,\dots,n\}$ and each $i\in\{1,\dots,k_q-1\}$, either $s_i^{(q)}\neq 0$ or $r_{i+1}^{(q)}\neq 0$.

With the above notations, one has

Lemma, *Let* $\underline{a}=\{a_j\in\mathbb{N}_+;1\leqq j\leqq n_1\}$ *and* $\underline{b}=\{b_j\in\mathbb{N}_+;1\leqq j\leqq n_2\}$.

i) *One has*

$$
\left[\prod_{j=1}^{n_1}(D'_{a_j}+i\varrho\zeta\tilde{\mathbf{c}}^{(a_j,0)}\partial_g)\cdot\prod_{k=1}^{n_2}(D''_{b_k}+i\varrho\zeta\tilde{\mathbf{c}}^{(0,b_k)}\partial_g)\right]\log\tilde{Y}_0
=\frac{1}{2\zeta^2}\sum_{n=0}^{n_1+n_2}\mathfrak{U}_n(\underline{a},\underline{b}),
\tag{1.8.11}
$$

with

$$\mathfrak{U}_0(\underline{a},\underline{b}) = \sum_{k\geqq 2} \frac{(-i\varrho\zeta)^k}{k} \sum_{(\underline{r},\underline{s})\in\mathfrak{S}_0^k(\underline{a},\underline{b})} \mathrm{Tr}\left\{\text{“}\prod_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}\right\}, \tag{1.8.12}$$

and, for $n \geqq 1$,

$$\mathfrak{U}_n(\underline{a},\underline{b}) = \frac{1}{n} \sum_{(k_q\geqq 2)_{1\leqq q\leqq n}} (-i\varrho\zeta)^{\sum_{q=1}^{n} k_q}$$

$$\sum_{(\underline{r}^{(q)},\underline{s}^{(q)})_{1\leqq q\leqq n}\in\mathfrak{S}_n^{(\underline{k})}(\underline{a},\underline{b})} \mathrm{Tr}\left\{\prod_{q=1}^{n}\left([I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\cdot\text{“}\prod_{j_q=1}^{k_q}\text{”}\,\tilde{\mathbf{A}}^{(r_{j_q}^{(q)},s_{j_q}^{(q)})}\right)\right\}. \tag{1.8.13}$$

ii) *If* $\psi\in\mathscr{H}^1$, *one has*

$$\left[\prod_{j=1}^{n_1}(D'_{a_j}+i\varrho\zeta\tilde{\mathbf{c}}^{(a_j,0)}\partial_g)\cdot\prod_{k=1}^{n_2}(D''_{b_k}+i\varrho\zeta\tilde{\mathbf{c}}^{(0,b_k)}\partial_g)\right]\cdot(\psi,[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\psi)_{\mathscr{H}^1}$$

$$= \sum_{n=0}^{n_1+n_2+1} \mathfrak{B}_n(\underline{a},\underline{b}), \tag{1.8.14}$$

with

$$\mathfrak{B}_n(\underline{a},\underline{b}) = \sum_{\substack{k_0\geqq 0,\,k_n\geqq 0\\(k_q\geqq 2)_{1\leqq q\leqq n-1}}} (-i\varrho\zeta)^{\sum_{q=0}^{n} k_q}\cdot \sum_{(\underline{r}^{(q)},\underline{s}^{(q)})_{0\leqq q\leqq n}\in\mathfrak{T}_n^{(\underline{k})}(\underline{a},\underline{b})}$$

$$\cdot\left(\psi,\text{“}\prod_{j_0=1}^{k_0}\text{”}\,\tilde{\mathbf{A}}^{(r_{j_0}^{(0)},s_{j_0}^{(0)})}\cdot\prod_{q=1}^{n}\left([I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\cdot\text{“}\prod_{j_q=1}^{k_q}\text{”}\,\tilde{\mathbf{A}}^{(r_{j_q}^{(q)},s_{j_q}^{(q)})}\right)\cdot\psi\right)_{\mathscr{H}^1}. \tag{1.8.15}$$

Proof. One shows (1.8.11) and (1.8.14) recursively.

1. First (1.8.11) holds if $n_1+n_2=1$. Indeed from (1.8.3),

$$(D''_j+i\varrho\zeta\tilde{\mathbf{c}}^{(0,j)}\partial_g)\log\tilde{Y}_0$$

$$= -\frac{\varrho^2}{2}[-\tilde{\mathbf{c}}^{(0,j)}\tilde{\mathbf{c}}\|g\|_2^2+\mathrm{Tr}\{[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}(\tilde{\mathbf{A}}^{(0,j)}\tilde{\mathbf{A}}+i\varrho\zeta\tilde{\mathbf{c}}^{(0,j)}\tilde{\mathbf{A}}\cdot\partial_g\tilde{\mathbf{A}})\}], \tag{1.8.16}$$

but, from (1.8.2) and (1.8.8), one has $\partial_g\tilde{\mathbf{A}}(\underline{t},\underline{u};\sigma)=\tilde{\mathbf{A}}_{[g]}^{(0,0)}(\underline{t},\underline{u})\in\mathscr{T}_1(\mathscr{H}^1)$, [if $C(\underline{u})\psi=\Sigma_m^{-1/2}(g\cdot(h*k_{(\underline{u})}*\psi))$, $(\psi\in\mathscr{H}^1)$, one has $C(\underline{u})\in\mathscr{T}_2(\mathscr{H}^1)$ with $\|C(\underline{u})\|_{\mathscr{T}_2}=\|g\|_2\cdot\|\Sigma_m^{-1/2}h*k_{(\underline{u})}\|_2$, and $\tilde{\mathbf{A}}_{[g]}^{(0,0)}(\underline{t},\underline{u})=C(\underline{t})^*C(\underline{u})$], and,

$$\mathrm{Tr}\{\tilde{\mathbf{A}}_{[g]}^{(0,0)}(\underline{t},\underline{u})\}=\tilde{\mathbf{c}}(\underline{t},\underline{u})\|g\|_2^2. \tag{1.8.17}$$

Inserting (1.8.17) in (1.8.16), one obtains

$$(D''_j+i\varrho\zeta\tilde{\mathbf{c}}^{(0,j)}\partial_g)\log\tilde{Y}_0$$

$$= -\frac{\varrho^2}{2}\mathrm{Tr}\{[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\cdot(\tilde{\mathbf{A}}^{(0,j)}\tilde{\mathbf{A}}^{(0,0)}-\tilde{\mathbf{c}}^{(0,j)}\tilde{\mathbf{A}}_{[g]}^{(0,0)})\}, \tag{1.8.18}$$

which is exactly (1.8.11) for $n_1=0$, $n_2=1$.
For $n_1=1$, $n_2=0$, one shows in an analogous way that

$$(D'_j+i\varrho\zeta\tilde{\mathbf{c}}^{(j,0)}\partial_g)\,\mathrm{Log}\,\tilde{Y}_0$$

$$= -\frac{\varrho^2}{2}\mathrm{Tr}\{(\tilde{\mathbf{A}}^{(0,0)}\tilde{\mathbf{A}}^{(j,0)}-\tilde{\mathbf{c}}^{(j,0)}\tilde{\mathbf{A}}_{[g]}^{(0,0)})\cdot[1+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\}. \tag{1.8.19}$$

Next, if X is a derivable function with values in $\mathscr{T}_1(\mathscr{H}^1)$, from the obvious equality $[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}=I-i\varrho\zeta\tilde{\mathbf{A}}[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}$, one has

$$\begin{aligned}&(D'_j+i\varrho\zeta\tilde{\mathbf{c}}^{(j,0)}\partial_g)\{X[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\}\\&\quad=\{(D'_jX)[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\}-i\varrho\zeta\{(X\tilde{\mathbf{A}}^{(j,0)}-\mathbf{c}^{(j,0)}\partial_gX)[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\}\\&\qquad-\varrho^2\zeta^2\{X[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}(\tilde{\mathbf{A}}^{(0,0)}\tilde{\mathbf{A}}^{(j,0)})-\tilde{\mathbf{c}}^{(j,0)}\tilde{\mathbf{A}}^{(0,0)}_{[g]})[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\}\,.\end{aligned}\tag{1.8.20}$$

and,

$$\begin{aligned}&(D''_j+i\varrho\zeta\tilde{\mathbf{c}}^{(0,j)}\partial_g)\{[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}X\}\\&\quad=\{[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}(D''_jX)\}-i\varrho\zeta\{[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}(\tilde{\mathbf{A}}^{(0,j)}X-\tilde{\mathbf{c}}^{(0,j)}\partial_gX)\}\\&\qquad-\varrho^2\zeta^2\{[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}(\tilde{\mathbf{A}}^{(0,j)}\tilde{\mathbf{A}}^{(0,0)}-\tilde{\mathbf{c}}^{(0,j)}\tilde{\mathbf{A}}^{(0,0)}_{[g]})[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}X\}\,.\end{aligned}\tag{1.8.21}$$

from which one deduces immediately (1.8.14) in the cases $n_1=1$, $n_2=0$:

$$\begin{aligned}&(D'_j+i\varrho\zeta\tilde{\mathbf{c}}^{(j,0)}\partial_g)(\psi,[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\psi)_{\mathscr{H}^1}=-i\varrho\zeta(\psi,\tilde{\mathbf{A}}^{(j,0)}[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\psi)_{\mathscr{H}^1}\\&\quad-\varrho^2\zeta^2(\psi,[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}(\tilde{\mathbf{A}}^{(0,0)}\tilde{\mathbf{A}}^{(j,0)}-\tilde{\mathbf{c}}^{(j,0)}\tilde{\mathbf{A}}^{(0,0)}_{[g]})[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\psi)_{\mathscr{H}^1}\,.\end{aligned}\tag{1.8.22}$$

and $n_1=0$, $n_2=1$:

$$\begin{aligned}&(D''_j+i\varrho\zeta\tilde{\mathbf{c}}^{(0,j)}\partial_g)(\psi,[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\psi)_{\mathscr{H}^1}=-i\varrho\zeta(\psi,[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\tilde{\mathbf{A}}^{(0,j)}\psi)_{\mathscr{H}^1}\\&\quad-\varrho^2\zeta^2(\psi,[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}(\tilde{\mathbf{A}}^{(0,j)}\tilde{\mathbf{A}}^{(0,0)}-\tilde{\mathbf{c}}^{(0,j)}\tilde{\mathbf{A}}^{(0,0)}_{[g]})[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\psi)_{\mathscr{H}^1}\,.\end{aligned}\tag{1.8.23}$$

2. If now $\underline{r}=\{r_j\in\mathbb{N}\,;1\leqq j\leqq k\}$ and $a\in\mathbb{N}_+$, for $i\in\{1,\dots,k\}$ one denotes by $\underline{r}_i^{\downarrow}(a)$ the sequence defined by

$$r_i^{\downarrow}(a)_j=\begin{cases}a, & \text{if } j=i,\\ r_j, & \text{if } j\neq i.\end{cases}$$

Then one has obivously (recursively with respect to k),

$$D'_a\text{“}\prod_{j=1}^{k}\text{”}\tilde{\mathbf{A}}^{(r_j,s_j)}=\sum_{\{i;r_i=0\}}\text{“}\prod_{j=1}^{k}\text{”}\tilde{\mathbf{A}}^{(r_i^{\downarrow}(a)_j,\,s_j)},\tag{1.8.24}$$

and

$$D''_b\text{“}\prod_{j=1}^{k}\text{”}\tilde{\mathbf{A}}^{(r_j,s_j)}=\sum_{\{i;s_i=0\}}\text{“}\prod_{j=1}^{k}\text{”}\tilde{\mathbf{A}}^{(r_j,s_i^{\downarrow}(b)_j)},\tag{1.8.25}$$

as also the analogous formulas where one substitutes “$\boxed{\circ}$” to “$\prod$”. It follows particularly that

$$\begin{aligned}&[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\left(D'_a\text{“}\prod_{j=1}^{k}\text{”}\tilde{\mathbf{A}}^{(r_j,s_j)}\right)\\&=\text{“}\prod_{j=1}^{k}\text{”}\tilde{\mathbf{A}}^{(r_1^{\downarrow}(a)_j,s_j)}-i\varrho\zeta[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\tilde{\mathbf{A}}^{(0,0)}\cdot\text{“}\prod_{j=1}^{k}\text{”}\tilde{\mathbf{A}}^{(r_1^{\downarrow}(a)_j,s_j)}\\&\quad+\sum_{\{2\leqq i\leqq k;r_i=0\}}[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\cdot\text{“}\prod_{j=1}^{k}\text{”}\tilde{\mathbf{A}}^{(r_i^{\downarrow}(a)_j,s_j)},\quad\text{if}\quad r_1=0,\end{aligned}\tag{1.8.26}$$

likewise

$$\left(D_b'' \text{“}\prod_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}\right)[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}$$
$$=\text{“}\prod_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,\overset{\downarrow}{s_k}(b)_j)}-i\varrho\zeta\text{“}\prod_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,\overset{\downarrow}{s_k}(b)_j)}\cdot\tilde{\mathbf{A}}^{(0,0)}[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}$$
$$+\sum_{\{1\leqq i\leqq k-1;\,s_i=0\}}\text{“}\prod_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,\overset{\downarrow}{s_i}(b)_j)}[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1},\quad \mathit{if}\quad s_k=0. \tag{1.8.27}$$

Then, if $b_0\in\mathbb{N}_+$, one computes $(D''_{b_0}+i\varrho\zeta\tilde{\mathbf{c}}^{(0,b_0)}\partial_g)\mathfrak{U}_n(\underline{a},\underline{b})$, [see (1.8.12) and (1.8.13)], by systematic use of (1.8.21) and then of (1.8.27), using the cyclicity property of the trace: one writes[20]

$$(D''_{b_0}+i\varrho\zeta\tilde{\mathbf{c}}^{(0,b_0)}\partial_g)\left[\sum_{n\geqq 0}\mathfrak{U}_n(\underline{a},\underline{b})\right]=\sum_{n\geqq 0}\mathfrak{U}''_n(\underline{a},\underline{b};b_0), \tag{1.8.28}$$

where $\mathfrak{U}''_n(\underline{a},\underline{b};b_0)$ is defined as the sum of all terms of the obtained expansion in which the resolvant $[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}$ appears exactly n times. In the same way one defines $\mathfrak{B}''_n(\underline{a},\underline{b};b_0)$ substituting $\mathfrak{B}_n$ [see (1.8.15)] to $\mathfrak{U}_n$, next $\mathfrak{U}'_n(\underline{a},\underline{b};a_0)$, $\mathfrak{B}'_n(\underline{a},\underline{b};a_0)$ substituting D'_{a_0} to D''_{b_0}, $\tilde{\mathbf{c}}^{(a_0,0)}$ for $\tilde{\mathbf{c}}^{(0,b_0)}$, and using (1.8.20) instead of (1.8.26).

3. From the equality

$$\text{“}\prod_{j=1}^{n}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}=\text{“}\prod_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}\cdot\text{“}\prod_{i=k+1}^{n}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}$$
$$-\text{“}\prod_{j=1}^{k-1}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}\cdot\tilde{\mathbf{c}}^{(r_{k+1},s_k)}\tilde{\mathbf{A}}_{[g]}^{(r_k,s_{k+1})}\cdot\text{“}\prod_{i=k+2}^{n}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)},\qquad(1\leqq k\leqq n-1), \tag{1.8.29}$$

established recursively from (1.8.9), one deduces recursively

$$\partial_g\text{“}\prod_{j=1}^{n}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}=\sum_{k=1}^{n}\text{“}\prod_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}\cdot\tilde{\mathbf{A}}_{[g]}^{(r_k,s_k)}\cdot\text{“}\prod_{i=k+1}^{n}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}, \tag{1.8.30}$$

and, for $n\geqq 2$,

$$\partial_g\,\mathrm{Tr}\left\{\text{“}\mathop{\boxed{\circ}}_{j=1}^{n}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}\right\}=\sum_{k=1}^{n}\mathrm{Tr}\left\{\tilde{\mathbf{A}}_{[g]}^{(r_k,s_k)}\cdot\text{“}\prod_{j=1}^{n-1}\text{”}\,\tilde{\mathbf{A}}^{(r_{(k+j)|n},\,s_{(k+j)|n})}\right\}, \tag{1.8.31}$$

where one sets $j|n=\begin{cases} j, & \text{if}\quad 1<j\leqq n\\ j-n, & \text{if}\quad n<j\leqq 2n-1\end{cases}.$

Next, if $(\underline{r},\underline{s})\in\mathfrak{S}^k_{\bigcirc}(\underline{a},\underline{b})$, $(k\geqq 2)$, one defines on the one hand $({}^{i\leftarrow}\underline{r},{}^{i\leftarrow}\underline{s})\in\mathfrak{S}^k_{\bigcirc}(\underline{a},\underline{b})$ by ${}^{i\leftarrow}r_j=r_{(i+j)|k}$, ${}^{i\leftarrow}s_j=s_{(i+j)|k}$, $(1\leqq j\leqq k)$, and one notes that

$$\mathrm{Tr}\left\{\text{“}\mathop{\boxed{\circ}}_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}\right\}=\mathrm{Tr}\left\{\text{“}\mathop{\boxed{\circ}}_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{({}^{i\leftarrow}r_j,\,{}^{i\leftarrow}s_j)}\right\},\qquad(1\leqq i\leqq k), \tag{1.8.32}$$

20 We do not write here explicit expressions for $\mathfrak{U}''_n(\underline{a},\underline{b};b_0)$ and the analogous terms, because of their excessive length

and, on the other hand, $(\underline{r}^+, \underline{s}^+) \in \mathfrak{S}_1^{(k+1)}(\underline{a}, \underline{b})$, by $r_1^+ = s_{k+1}^+ = 0$, $r_j^+ = r_{j|k}$, $(2 \leqq j \leqq k+1)$, and $s_j^+ = s_j$, $(1 \leqq j \leqq k)$.
Then, from (1.8.31), one has

$$-\tilde{\mathbf{c}}^{(0,b_0)}\partial_g \operatorname{Tr}\left\{\text{``}\mathop{\boxed{\circ}}_{j=1}^{k}\text{''}\,\tilde{\mathbf{A}}^{(r_j,s_j)}\right\} + \sum_{i=1}^{k} \operatorname{Tr}\left\{\text{``}\prod_{j=1}^{k+1}\text{''}\,\tilde{\mathbf{A}}^{(({}^{i}\overleftarrow{r})_j^+,\, ({}^{i}\overleftarrow{s})^+{}_{(k\overset{\downarrow}{+}1)}(b_0)_j)}\right\}$$

$$= \sum_{i=1}^{k} \operatorname{Tr}\left\{\text{``}\mathop{\boxed{\circ}}_{j=1}^{k+1}\text{''}\,\tilde{\mathbf{A}}^{(({}^{i}\overleftarrow{r})^+,\, ({}^{i}\overleftarrow{s})_j^+{}_{(k\overset{\downarrow}{+}1)}(b_0)_j)}\right\}, \tag{1.8.33}$$

and, from (1.8.32) and (1.8.33) one deduces that

$$\mathfrak{U}_0''(\underline{a}, \underline{b}; b_0) = \mathfrak{U}_0(\underline{a}, \underline{b} \cup \{b_0\}). \tag{1.8.34}$$

After, if $(\underline{r}, \underline{s}) = \{(r_j, s_j) \in \mathbb{N} \times \mathbb{N};\ 1 \leqq j \leqq k\}$, and $i \in \{1, \ldots, k\}$, one defines $(\overset{i}{\underline{\check{r}}}, \overset{i}{\underline{\check{s}}})$ by

$$\overset{i}{\check{r}}_j = \begin{cases} r_j, & 1 \leqq j \leqq i \\ 0, & j = i+1 \\ r_{j-1}, & i+2 \leqq j \leqq k+1 \end{cases}, \quad \overset{i}{\check{s}}_j = \begin{cases} s_j, & 1 \leqq j < i \\ 0, & j = i \\ s_{j-1}, & i+1 \leqq j \leqq k+1 \end{cases},$$

and $(\overset{i}{\underline{\check{r}}}{}^{+(q)}, \overset{i}{\underline{\check{s}}}{}^{+(q)})_{q=1,2}$ by

$$(\overset{i}{\check{r}}{}^{+(1)}_{j_1}, \overset{i}{\check{s}}{}^{+(1)}_{j_1}) = (\overset{i}{\check{r}}_{j_1}, \overset{i}{\check{s}}_{j_1}), \quad 1 \leqq j_1 \leqq i, \quad (\overset{i}{\check{r}}{}^{+(2)}_{j_2}, \overset{i}{\check{s}}{}^{+(2)}_{j_2}) = (\overset{i}{\check{r}}_{i+j_2}, \overset{i}{\check{s}}_{i+j_2}), \quad 1 \leqq j_2 \leqq k-i+1.$$

Then, if $b_0 \in \mathbb{N}_+$, from (1.8.29) and (1.8.30), one has

$$\begin{aligned} &-\tilde{\mathbf{c}}^{(0,b_0)}\partial_g \text{``}\prod_{j=1}^{k}\text{''}\,\tilde{\mathbf{A}}^{(r_j,s_j)} \\ &\quad + \sum_{i=1}^{k} \text{``}\prod_{j_1=1}^{i}\text{''}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}{}^{+(1)}_{j_1},\, \overset{i}{\check{s}}{}^{+(1)}\underset{i}{\downarrow}(b_0)_{j_1})} \,.\, \text{``}\prod_{j_2=1}^{k-i+1}\text{''}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}{}^{+(2)}_{j_2},\, \overset{i}{\check{s}}{}^{+(2)}_{j_2})} \\ &= \sum_{i=1}^{k} \text{``}\prod_{j=1}^{k+1}\text{''}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}_j,\, \overset{i}{\check{s}}\underset{i}{\downarrow}(b_0)_j)}. \end{aligned} \tag{1.8.35}$$

One notes that if $(\underline{r}, \underline{s}) \in \mathfrak{S}_1^{(k)}(\underline{a}', \underline{b}')$, $(k \geqq 2)$, one has

$$\begin{aligned} (\overset{i}{\underline{\check{r}}}{}^{+(q)}, \overset{i}{\underline{\check{s}}}{}^{+(q)})_{q=1,2} &\in \mathfrak{S}_2^{(i,k-i+1)}(\underline{a}', \underline{b}'), \quad (2 \leqq i \leqq k-1), \\ (\overset{i}{\underline{\check{r}}}, \overset{i}{\underline{\check{s}}}\underset{i}{\downarrow}(b_0)) &\in \mathfrak{S}_1^{(k+1)}(\underline{a}', \underline{b}' \cup \{b_0\}), \quad (1 \leqq i \leqq k), \end{aligned}$$

and from (1.8.35),

$$\begin{aligned} &\text{``}\prod_{j=1}^{k}\text{''}\,\tilde{\mathbf{A}}^{(r_j,\, s\underset{k}{\downarrow}(b_0)_j)} \,.\, \tilde{\mathbf{A}}^{(0,0)} + (\tilde{\mathbf{A}}^{(0,b_0)} - \tilde{\mathbf{c}}^{(0,b_0)}\partial_g)\text{``}\prod_{j=1}^{k}\text{''}\,\tilde{\mathbf{A}}^{(r_j,s_j)} \\ &\quad + \sum_{i=2}^{k-1} \text{``}\prod_{j_1=1}^{i}\text{''}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}{}^{+(1)}_{j_1},\, \overset{i}{\check{s}}{}^{+(1)}\underset{i}{\downarrow}(b_0)_{j_1})} \,.\, \text{``}\prod_{j_2=1}^{k-i+1}\text{''}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}{}^{+(2)}_{j_2},\, \overset{i}{\check{s}}{}^{+(2)}_{j_2})} \\ &= \sum_{i=1}^{k} \text{``}\prod_{j=1}^{k+1}\text{''}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}_j,\, \overset{i}{\check{s}}\underset{i}{\downarrow}(b_0)_j)}, \end{aligned} \tag{1.8.36}$$

from which one deduces

$$\mathfrak{U}_n''(\underline{a},\underline{b}\,;b_0)=\mathfrak{U}_n(\underline{a},\underline{b}\cup\{b_0\}),\qquad (n\geqq 1). \tag{1.8.37}$$

[For $n=1$, one uses directly (1.8.36), with $\underline{a}'=\underline{a}$, $\underline{b}'=\underline{b}$. For $n>1$, one applies successively (1.8.36) to each factor of each term in (1.8.13), using that for each $(\underline{r}^{(q)},\underline{s}^{(q)})_{1\leqq q\leqq n}\in\mathfrak{S}_n^{(k)}(\underline{a},\underline{b})$ there exists a partition $\underline{a}=\bigcup_{q=1}^{n}\underline{a}^{(q)}$, $\underline{b}=\bigcup_{q=1}^{n}\underline{b}^{(q)}$, such that $(\underline{r}^{(q)},\underline{s}^{(q)})\in\mathfrak{S}_1^{(k_q)}(\underline{a}^{(q)},\underline{b}^{(q)})$, $1\leqq q\leqq n$.]

In the same way, if $(\underline{r},\underline{s})\in\mathfrak{T}_0^{(k)}(\underline{a},\underline{b})$, $(k\geqq 1)$, one has

$$(\underline{\overset{i}{\check{r}}}{}^{+(q)},\underline{\overset{i}{\check{s}}}{}^{+(q)})_{q=1,2}\in\mathfrak{T}_1^{(i,k+i-1)}(\underline{a},\underline{b}),\qquad (1\leqq i\leqq k),$$
$$(\underline{\overset{i}{\check{r}}},\underline{\overset{i}{\check{s}}}{}_i^{\downarrow}(b_0))\in\mathfrak{T}_0^{(k+1)}(\underline{a},\underline{b}\cup\{b_0\}),\qquad (1\leqq i\leqq k),$$

and one deduces from (1.8.35) that

$$\mathfrak{B}_0''(\underline{a},\underline{b}\,;b_0)=\mathfrak{B}_0(\underline{a},\underline{b}\cup\{b_0\}). \tag{1.8.38}$$

Next, if $(\underline{r},\underline{s})\in\mathfrak{T}_1^{(0,k)}(\underline{a}',\underline{b}')$, $(k\geqq 1)$, one has

$$(\underline{\overset{i}{\check{r}}}{}^{+(q)},\underline{\overset{i}{\check{s}}}{}^{+(q)})_{q=1,2}\in\mathfrak{T}_2^{(0,i,k-i+1)}(\underline{a}',\underline{b}'),\qquad (2\leqq i\leqq k),$$
$$(\underline{\overset{i}{\check{r}}},\underline{\overset{i}{\check{s}}}{}_i^{\downarrow}(b_0))\in\mathfrak{T}_1^{(0,k+1)}(\underline{a}',\underline{b}'\cup\{b_0\}),\qquad (1\leqq i\leqq k),$$

and, from (1.8.35),

$$\begin{aligned}
&(\tilde{\mathbf{A}}^{(0,b_0)}-\tilde{\mathbf{c}}^{(0,b_0)}\partial_g)\text{“}\prod_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}\\
&\quad+\sum_{i=2}^{k}\text{“}\prod_{j_1=1}^{i}\text{”}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}{}^{+(1)}_{j_1},\,\overset{i}{\check{s}}{}^{+(1)}\underset{i}{\downarrow}(b_0)_{j_1})}\cdot\text{“}\prod_{j_2=1}^{k-i+1}\text{”}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}{}^{+(2)}_{j_2},\,\overset{i}{\check{s}}{}^{+(2)}_{j_2})}\\
&=\sum_{i=1}^{k}\text{“}\prod_{j=1}^{k+1}\text{”}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}_j,\,\overset{i}{\check{s}}\underset{i}{\downarrow}(b_0)_j)}\,.
\end{aligned} \tag{1.8.39}$$

Last, if $(\underline{r},\underline{s})\in\mathfrak{T}_1^{(k,0)}(\underline{a}',\underline{b}')$, $(k\geqq 1)$, one has

$$(\underline{\overset{i}{\check{r}}}{}^{+(q)},\underline{\overset{i}{\check{s}}}{}^{+(q)})_{q=1,2}\in\mathfrak{T}_2^{(i,k-i+1,0)}(\underline{a}',\underline{b}'),\qquad (1\leqq i\leqq k-1),$$
$$(\underline{\overset{i}{\check{r}}},\underline{\overset{i}{\check{s}}}{}_i^{\downarrow}(b_0))\in\mathfrak{T}_1^{(k+1,0)}(\underline{a}',\underline{b}'\cup\{b_0\}),\qquad (1\leqq i\leqq k),$$

and, from (1.8.35),

$$\begin{aligned}
&\text{“}\prod_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,\,s\underset{k}{\downarrow}(b_0)_j)}\cdot\tilde{\mathbf{A}}^{(0,0)}-\tilde{\mathbf{c}}^{(0,b_0)}\partial_g\text{“}\prod_{j=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_j,s_j)}\\
&\quad+\sum_{i=1}^{k-1}\text{“}\prod_{j_1=1}^{i}\text{”}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}{}^{+(1)}_{j_1},\,\overset{i}{\check{s}}{}^{+(1)}\underset{i}{\downarrow}(b_0)_{j_1})}\cdot\text{“}\prod_{j_2=1}^{k-i+1}\text{”}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}{}^{+(2)}_{j_2},\,\overset{i}{\check{s}}{}^{+(2)}_{j_2})}\\
&=\sum_{i=1}^{k}\text{“}\prod_{j=1}^{k+1}\text{”}\,\tilde{\mathbf{A}}^{(\overset{i}{\check{r}}_j,\,\overset{i}{\check{s}}\underset{i}{\downarrow}(b_0)_j)}\,,
\end{aligned} \tag{1.8.40}$$

and from (1.8.36), (1.8.39) and (1.8.40) one obtains

$$\mathfrak{V}''_n(\underline{a},\underline{b}\,;b_0)=\mathfrak{V}_n(\underline{a},\underline{b}\cup\{b_0\}),\qquad (n\geqq 1). \tag{1.8.41}$$

A similar computation of $\mathfrak{U}'_n(\underline{a},\underline{b}\,;a_0)$ and $\mathfrak{V}'_n(\underline{a},\underline{b}\,;\mathrm{a}_0)$ achieves the proof. □

1.9. Estimates

Estimation of the right hand side of (1.8.11) and (1.8.14) relies on the

Lemma. *For each $\eta<1/4$, there exists a constant $c>0$ such that, if $\underline{a}=\{a_i\in\mathbb{N}_+\,;\ 1\leqq i\leqq n_1\}$, $\underline{b}=\{b_i\in\mathbb{N}_+\,;\ 1\leqq i\leqq n_2\}$; $j\in\mathbb{N}$, $j\leqq\min\{\underline{a},\underline{b}\}$; $t\in[0,1]$ and $p\in[1,+\infty[$, one has*[21]

i) *for any $(\underline{r},\underline{s})\in\mathfrak{S}^k_{\bigcirc}(\underline{a},\underline{b})$, $(k\geqq 2)$,*

$$\left\|\mathrm{Tr}\left\{\text{“}\boxed{\prod_{i=1}^{k}}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}(J_j(t),J_j(t))\right\}\right\|_p\leqq p^{k/2}(k!)^{1/2}\prod_{i\in\underline{a}\cup\underline{b}}(c\kappa_{i-1})^{-\eta}\,; \tag{1.9.1}$$

ii) *for any $(\underline{r},\underline{s})\in\mathfrak{S}^{(k)}_1(\underline{a},\underline{b})$, $(k\geqq 2)$,*

$$\left\|\mathrm{Tr}\left\{\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}(J_j(t),J_j(t))\right\}\right\|_p\leqq p^{k/2}(k!)^{1/2}\prod_{i\in\underline{a}\cup\underline{b}}(c\kappa_{i-1})^{-\eta}, \tag{1.9.2}$$

and

$$\|\ \|\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}(J_j(t),J_j(t))\|_{\mathscr{T}_2}\|_p\leqq p^{k/2}(k!)^{1/2}\prod_{i\in\underline{a}\cup\underline{b}}(c\kappa_{i-1})^{-\eta}, \tag{1.9.3}$$

iii) *for any $(\underline{r},\underline{s})\in\mathfrak{T}^{(k)}_0(\underline{a},\underline{b})$, $(k\geqq 1)$, and $\psi\in\mathscr{H}^1$,*

$$\left\|\left(\psi,\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}(J_j(t),J_j(t))\psi\right)_{\mathscr{H}^1}\right\|_p\leqq\|\psi\|^2_{\mathscr{H}^1}p^{k/2}(k!)^{1/2}\prod_{i\in\underline{a}\cup\underline{b}}(c\kappa_{i-1})^{-\eta}\,; \tag{1.9.4}$$

iv) *for any $(\underline{r},\underline{s})\in\mathfrak{T}^{(0,k)}_1(\underline{a},\underline{b})$, $(k\geqq 1)$, and $\psi\in\mathscr{H}^1$,*

$$\|\ \|\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}(J_j(t),J_j(t))\psi\|_{\mathscr{H}^1}\|_p\leqq\|\psi\|_{\mathscr{H}^1}p^{k/2}(k!)^{1/2}\prod_{i\in\underline{a}\cup\underline{b}}(c\kappa_{i-1})^{-\eta}. \tag{1.9.5}$$

Proof. For each $n\in\mathbb{N}$, one denotes by $\mathscr{F}_n\subset L^2(\mathscr{S}',\nu)$ the “n-particles space,” one recalls that $\mathscr{F}_n\subset\bigcap_{1\leqq p<+\infty}L^p(\mathscr{S}',\nu)$, and that, from Nelson's “hypercontractivity” estimate

$$\|W\|_p\leqq p^{k/2}\|W\|_2,\qquad\forall W\in\bigoplus_{n=0}^{k}\mathscr{F}_n,\qquad(1\leqq p<+\infty). \tag{1.9.6}$$

One notes that functions $\mathrm{Tr}\left\{\text{“}\boxed{\prod_{i=1}^{k}}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right\}$, $\mathrm{Tr}\left\{\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right\}$ and $\left(\psi,\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\psi\right)_{\mathscr{H}^1}$ belong to $\bigoplus_{n=0}^{k}\mathscr{F}_n$, and that $\left\|\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right\|^2_{\mathscr{T}_2}$ and $\left\|\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\cdot\psi\right\|^2_{\mathscr{H}^1}$ belong to $\bigoplus_{n=0}^{2k}\mathscr{F}_n$, so one needs only estimate the L^2-norm of each of them.

21 See definition of $J_j(t)$, above (1.6.3), and definitions preceding Lemma 1.8. On the other hand, $\underline{a}\cup\underline{b}$ denotes the disjoint union of the two families of indices

Now, for $\underline{t}\in[0,1]^{\mathbb{N}_+}$, vanishing outside a finite set, one sets $F^{(0)}_{(\underline{t})}=\Sigma_m^{-1/2}h*k_{(\underline{t})}$, and, $F^{(i)}_{(\underline{t})}=D_iF^{(0)}_{(\underline{t})}$, $(i\in\mathbb{N}_+)$.[22] Next $K^{(i,j)}_{(\underline{t})}(y_1,x,y_2)=\check{F}^{(i)}_{(\underline{t})}(y_1-x)g(x)F^{(j)}_{(\underline{t})}(x-y_2)$, $(x,y_1,y_2\in E,\ i,j\in\mathbb{N})$; and, on the other hand, if $P\in\mathfrak{P}(e_k)$ is a partition of $e_k=\{1,\dots,k\}$, one denotes by $\mathfrak{v}_P:e_k\to P$ the mapping defined by $i\in\mathfrak{v}_P(i)$, $\forall i\in e_k$: then one has[23]

$$E_\nu\left[\left|\mathrm{Tr}\left\{\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}(\underline{t},\underline{t})\right\}\right|^2\right]=\sum_{P\in\mathfrak{Q}_1(e_{2k})}I^{(\underline{r},\underline{s})}_{1,P}(\underline{t}),\tag{1.9.7}$$

$$E_\nu\left[\left|\left(\psi,\text{“}\sum_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}(\underline{t},\underline{t})\psi\right)_{\mathscr{H}_1}\right|^2\right]=\sum_{P\in\mathfrak{Q}_1(e_{2k})}I^{(\underline{r},\underline{s})}_{2,P}(\underline{t}),\tag{1.9.8}$$

and

$$E_\nu\left[\left|\mathrm{Tr}\left\{\text{“}\boxed{\prod_{i=1}^{k}}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}(\underline{t},\underline{t})\right\}\right|^2\right]=\sum_{P\in\mathfrak{Q}_2(e_{2k})}I^{(\underline{r},\underline{s})}_{1,P}(\underline{t}),\tag{1.9.9}$$

where $\mathfrak{Q}_1(e_{2k})\subset\mathfrak{P}(e_{2k})$ is the set of all partitions into pairs including no one of the pairs $\{i,i+1\}$, $(1\leqq i\leqq k-1$ and $k+1\leqq i\leqq 2k-1)$, $\mathfrak{Q}_2(e_{2k})$ the subset of elements of $\mathfrak{Q}_1(e_{2k})$ which moreover includes neither $\{1,k\}$ nor $\{k+1,2k\}$, and where

$$I^{(\underline{r},\underline{s})}_{1,P}=\int_{E^P}\left[\int_E H^{(\underline{r},\underline{s})}_1(y_1;x_{\mathfrak{v}_P(1)},\dots,x_{\mathfrak{v}_P(k)};y_1)dy_1\right.$$
$$\left.\cdot\int_E\tilde{H}^{(\underline{r},\underline{s})}_1(y_2;x_{\mathfrak{v}_P(k+1)},\dots,x_{\mathfrak{v}_P(2k)};y_2)dy_2\right]\cdot\prod_{u\in P}dx_u,\tag{1.9.10}$$

$$I^{(\underline{r},\underline{s})}_{2,P}=\int_{E^P}\left[\int_{E^2}(\Sigma_m^{1/2}\bar{\psi})(y_1)\cdot H^{(\underline{r},\underline{s})}_1(y_1;x_{\mathfrak{v}_P(1)},\dots,x_{\mathfrak{v}_P(k)};y_2)\cdot(\Sigma_m^{1/2}\psi)(y_2)dy_1dy_2\right.$$
$$\left.\cdot\int_{E^2}(\Sigma_m^{1/2}\bar{\psi})(y_3)\cdot\tilde{H}^{(\underline{r},\underline{s})}_1(y_3;x_{\mathfrak{v}_P(k+1)},\dots,x_{\mathfrak{v}_P(2k)};y_4)\cdot(\Sigma_m^{1/2}\psi)(y_4)dy_3dy_4\right]\prod_{u\in P}dx_u,\tag{1.9.11}$$

with

$$H^{(\underline{r},\underline{s})}_1(y_1;x_1,\dots,x_k;y_{k+1})=\int_{E^{k-1}}\prod_{i=1}^{k}K^{(r_i,s_i)}(y_i,x_i,y_{i+1})\cdot\prod_{j=2}^{k}dy_j,\tag{1.9.12}$$

and

$$\tilde{H}^{(\underline{r},\underline{s})}_1(y_1;x_1,\dots,x_k;y_{k+1})=\int_{E^{k-1}}\prod_{i=1}^{k}K^{(s_{k-i+1},r_{k-i+1})}(y_i,x_i,y_{i+1})\cdot\prod_{j=2}^{k}dy_j\tag{1.9.13}$$

22 See (1.6.1) and (1.6.2)

23 To obtain equalities (1.9.7), (1.9.8), and (1.9.14), [respectively (1.9.9) and (1.9.15)], one computes, recursively with respect to k, the integrals $E_\nu\left[\mathrm{Tr}\left\{B\cdot\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right\}\cdot W\right]$, $\Big[$respectively $E_\nu\left[\left(\psi,B\cdot\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\psi\right)_{\mathscr{H}^1}\cdot W\right]\Big]$, where B is an arbitrary monomial in $\tilde{\mathbf{A}}^{(a_j,b_j)}$ and $\tilde{\mathbf{A}}^{(a_j,b_j)}_{[g]}$, and W belongs to some $\mathscr{F}_n$

Next,

$$E_\nu\left[\left\|``\prod_{i=1}^{k}"\,\tilde{\mathbf{A}}^{(r_i,s_i)}(\underline{t},\underline{t})\right\|^4_{\mathscr{T}_2}\right]=\sum_{P\in\mathfrak{Q}_3(e_{4k})} I^{(\underline{r},\underline{s})}_{3,P}(\underline{t}),\tag{1.9.14}$$

and

$$E_\nu\left[\left\|``\prod_{i=1}^{k}"\,\tilde{\mathbf{A}}^{(r_i,s_i)}(\underline{t},\underline{t})\cdot\psi\right\|^4_{\mathscr{H}^1}\right]=\sum_{P\in\mathfrak{Q}_3(e_{4k})} I^{(\underline{r},\underline{s})}_{4,P}(\underline{t}),\tag{1.9.15}$$

where $\mathfrak{Q}_3(e_{4k})\subset\mathfrak{P}(e_{4k})$ is the set of partitions into pairs which do not include any one of the pairs $\{i,i+1\}$, $(1\leqq i\leqq k-1,\ k+1\leqq i\leqq 2k-1,\ 2k+1\leqq i\leqq 3k-1,\ 3k+1\leqq i\leqq 4k-1)$, and where

$$\begin{aligned}I^{(\underline{r},\underline{s})}_{3,P}=&\int_{E^P}\Big[\int_E H^{(\underline{r},\underline{s})}_2(y_1;x_{\upsilon_P(1)},\dots,x_{\upsilon_P(2k)};y_1)dy_1\\&\cdot\int_E H^{(\underline{r},\underline{s})}_2(y_2;x_{\upsilon_P(2k+1)},\dots,x_{\upsilon_P(4k)};y_2)dy_2\Big]\cdot\prod_{u\in P}dx_u,\end{aligned}\tag{1.9.16}$$

$$\begin{aligned}I^{(\underline{r},\underline{s})}_{4,P}=&\int_{E^P}\Big[\int_{E^2}(\Sigma_m^{1/2}\bar{\psi})(y_1)\cdot H^{(\underline{r},\underline{s})}_2(y_1;x_{\upsilon_P(1)},\dots,x_{\upsilon_P(2k)};y_2)\cdot(\Sigma_m^{1/2}\psi)(y_2)dy_1dy_2\\&\cdot\int_{E^2}(\Sigma_m^{1/2}\bar{\psi})(y_3)\cdot H^{(\underline{r},\underline{s})}_2(y_3;x_{\upsilon_P(2k+1)},\dots,x_{\upsilon_P(4k)};y_4)\cdot(\Sigma_m^{1/2}\psi)(y_4)dy_3dy_4\Big]\prod_{u\in P}dx_u,\end{aligned}\tag{1.9.17}$$

with

$$\begin{aligned}&H^{(\underline{r},\underline{s})}_2(y_1;x_1,\dots,x_{2k};y_2)\\&\quad=\int_E\tilde{H}^{(\underline{r},\underline{s})}_1(y_1;x_1,\dots,x_k;y)H^{(\underline{r},\underline{s})}_1(y;x_{k+1},\dots,x_{2k};y_2)dy.\end{aligned}\tag{1.9.18}$$

One first integrates (1.9.10), (1.9.11), (1.9.16), and (1.9.17), with respect to the measure $\prod_{u\in P}dx_u$ [Fubini's theorem applies obviously because $g\in\mathscr{S}$ and $F^{(i)}_{(\underline{t})}\in\mathscr{S}$]; if $u\in P$ is not a pair of "consecutive" indices, that is, if u is not one of the following pairs:

in the computation of (1.9.10): $\{1,k\}$ or $\{k+1,2k\}$,

in that of (1.9.16): $\{1,2k\}$, $\{k,k+1\}$, $\{2k+1,4k\}$ or $\{3k,3k+1\}$,

and for (1.9.17): $\{k,k+1\}$ or $\{3k,3k+1\}$,

integration with respect to dx_u produces a function of four distinct variables $(y^{(u)}_{i_n})_{1\leqq n\leqq 4}$, namely

$$G_u((y_n)_{1\leqq n\leqq 4})=\int_E g(x)^2\cdot\prod_{n=1}^{4}F^{(i^{(u)}_n)}(x-y_n)\cdot dx,\qquad((y_n)_{1\leqq n\leqq 4}\in E^4).\tag{1.9.19}$$

If on the contrary u is one of the pairs of "consecutive" indices, integration with respect to dx_u gives a function of only three distinct variables, which, according to the assumptions on the sequence $(\underline{r},\underline{s})$, writes

$$\tilde{G}_u(y;y_1,y_2)=\int_E g(x)^2\cdot F^{(0)}(x-y)^2\cdot\prod_{n=1}^{2}F^{(i^{(u)}_n)}(x-y_n)\cdot dx.\tag{1.9.20}$$

Then integration with respect to dy gives[24]

$$G_u(y_1, y_2) \equiv \int_E \tilde{G}_u(y; y_1, y_2) dy = \mathbf{c} \int_E g(x)^2 \prod_{n=1}^{2} F^{(i_n^{(u)})}(x - y_n) \cdot dx, \quad ((y_1, y_2) \in E^2). \tag{1.9.21}$$

Once the partial integrations described above have been performed, the obtained integral is estimated in a standard way[25] which gives

$$|I_{1,P}^{(r,s)}| \leqq \prod_{u \in P} \|G_u\|_2, \quad |I_{2,P}^{(r,s)}| \leqq \|\psi\|_{\mathscr{H}^1}^4 \prod_{u \in P} \|G_u\|_2, \quad (P \in \mathfrak{Q}_1(e_{2k})),$$
$$|I_{3,P}^{(r,s)}| \leqq \prod_{u \in P} \|G_u\|_2, \quad |I_{4,P}^{(r,s)}| \leqq \|\psi\|_{\mathscr{H}^1}^4 \prod_{u \in P} \|G_u\|_2, \quad (P \in \mathfrak{Q}_3(e_{4k})). \tag{1.9.22}$$

But, (from Young's and Hölder's inequalities), one has

$$\begin{aligned}
&\int_{E^l} \left| \int_E g(x)^2 \prod_{n=1}^{l} F^{(i_n)}(x - y_n) \cdot dx \right|^2 \prod_{n=1}^{l} dy_n \\
&= \int_{E^{l+2}} g(x_1)^2 g(x_2)^2 \prod_{n=1}^{l} F^{(i_n)}(x_1 - y_n) \check{F}^{(i_n)}(y_n - x_2) \cdot dx_1 dx_2 \prod_{n=1}^{l} dy_n \\
&\leqq \|g^2\|_2^2 \left\| \widehat{\prod_{n=1}^{l} F^{(i_n)} * \check{F}^{(i_n)}} \right\|_\infty \leqq (2\pi)^{-2(l-1)} \|g\|_4^4 \prod_{n=1}^{l} \|\widehat{F^{(i_n)}}\|_{\frac{2l}{l-1}}^2 .
\end{aligned} \tag{1.9.23}$$

Next, from (1.6.1) and the assumptions made on k_{κ_n}, $(n \in \mathbb{N})$, one has $\|\hat{k}_{(\underline{t})}\|_\infty \leqq 1$, and, for any $i \in \mathbb{N}_+$, $\|\hat{k}_{\kappa_i} - \hat{k}_{\kappa_{i-1}}\|_\infty \leqq 1$ and $\hat{k}_{\kappa_i}(p) - \hat{k}_{\kappa_{i-1}}(p) = 0$ if $|p| \leqq \kappa_{i-1}$, therefore, as $\|\hat{h}\|_\infty \leqq 1$, one has

$$\|\hat{F}_{(\underline{t})}^{(0)}\|_{\frac{2l}{l-1}} \leqq \left(\int_E \frac{dp}{(|p|^2 + m^2)^{l/(l-1)}} \right)^{\frac{l-1}{2l}} = (\pi(l-1))^{(l-1)/2l} \cdot m^{-1/l}, \tag{1.9.24}$$

and, for $i \in \mathbb{N}_+$,

$$\|\hat{F}_{(\underline{t})}^{(i)}\|_{\frac{2l}{l-1}} \leqq \left(\int_{|p| \geqq \kappa_{i-1}} \frac{dp}{(|p|^2 + m^2)^{l/(l-1)}} \right)^{\frac{l-1}{2l}} \leqq (\pi(l-1))^{(l-1)/2l} \cdot (\kappa_{i-1})^{-1/l}. \tag{1.9.25}$$

on the other hand, for $j \in \mathbb{N}$ and $t \in [0, 1]$,

$$0 \leqq \mathbf{c}(J_j(t)) \leqq (2\pi)^{-2} \int_{|p| \leqq 2\kappa_j} \frac{dp}{(|p|^2 + m^2)} = \frac{1}{4\pi} \log(1 + 4\kappa_j^2/m^2), \tag{1.9.26}$$

since $\hat{k}_{(J_j(t))}(p) = 0$, if $|p| \geqq 2\kappa_j$.

---

24 One recalls that $\mathbf{c}$ denotes the function $\mathbf{c}(\underline{t}) \equiv \mathbf{c}_{h * k_{(\underline{t})}}$

25 Each function G_u has to be considered as the kernel of some Hilbert-Schmidt operator

To achieve the proof, one collects the above estimates, using the inequalities[26] $|\mathfrak{Q}_2(e_{2k})|\leqq|\mathfrak{Q}_1(e_{2k})|\leqq 2^k k!$, $|\mathfrak{Q}_3(e_{4k})|\leqq 2^{4k}(k!)^2$, $k\leqq|\underline{a}|+|\underline{b}|+1$, [according to any one of the assumptions on $(\underline{r},\underline{s})$], and one dominates the right hand side of (1.9.26) by an arbitrarily small power of one of the κ_{i-1}, $(i\in\underline{a}\cup\underline{b})$, $[\kappa_j\leqq e\kappa_{i-1}$, $\forall i\in\underline{a}\cup\underline{b}$, since, by assumption, $j\leqq i]$. □

1.10. Proof of Proposition 1.6

One deduces Proposition 1.6 from Lemmas 1.8 and 1.9 as follows: First one estimates the right hand side of (1.8.11) and (1.8.14) [with the assumption that $|\varrho|\leqq c_1$, $|\zeta|\leqq c_2$, and, according to (1.4.16) $|\cos\operatorname{Arg}\varrho\zeta|^{-1}\leqq c_3$, and, on the other hand, that $\underline{t}=\underline{u}=J_j(t)$, with $j\leqq i$, $\forall i\in\underline{a}\cup\underline{b}$ as in the hypothesis of Lemma 1.9]:

i) One notes that, if $\mathfrak{S}^k_{\bigcirc}(\underline{a},\underline{b})\neq\emptyset$, one has $\max\cdot\{|\underline{a}|,|\underline{b}|\}\leqq k\leqq|\underline{a}|+|\underline{b}|$, and $|\mathfrak{S}^k_{\bigcirc}(\underline{a},\underline{b})|\leqq\dfrac{k!}{(k-|\underline{a}|)!}\cdot\dfrac{k!}{(k-|\underline{b}|)!}\leqq 4^{(|\underline{a}|+|\underline{b}|)}\cdot(|\underline{a}|+|\underline{b}|)!$.

Then, from (1.8.12) and (1.9.1), one sees easily that, (if $\eta<1/4$), there exists $c'>0$ such that

$$\|\mathfrak{U}_0(\underline{a},\underline{b})\|_p\leqq p^{(|\underline{a}|+|\underline{b}|)/2}[(|\underline{a}|+|\underline{b}|)!]^{3/2}\prod_{i\in\underline{a}\cup\underline{b}}(c'\kappa_{i-1})^{-\eta}. \tag{1.10.1}$$

ii) For $(\underline{r},\underline{s})\in\mathfrak{S}^{(k)}_1(\underline{a},\underline{b})$, one has

$$\begin{aligned}&\operatorname{Tr}\left\{[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right\}\\&=\operatorname{Tr}\left\{\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right\}-i\varrho\zeta\operatorname{Tr}\left\{\tilde{\mathbf{A}}[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right\},\end{aligned} \tag{1.10.2}$$

and

$$\left|\operatorname{Tr}\left\{\tilde{\mathbf{A}}[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right\}\right|\leqq\|[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\|\,\|\tilde{\mathbf{A}}\|_{\mathcal{T}_2}\left\|\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right\|_{\mathcal{T}_2}, \tag{1.10.3}$$

but $\tilde{\mathbf{A}}(\underline{t},\underline{t};\sigma)$ is self-adjoint. Thus

$$\|[I+i\varrho\zeta\tilde{\mathbf{A}}(\underline{t},\underline{t};\sigma)]^{-1}\|\leqq|\cos\operatorname{Arg}\varrho\zeta|^{-1},\qquad\forall\sigma\in\mathcal{S}', \tag{1.10.4}$$

and therefore

$$\begin{aligned}&\left\|\operatorname{Tr}\left\{\tilde{\mathbf{A}}(\underline{t},\underline{t})[I+i\varrho\zeta\tilde{\mathbf{A}}(\underline{t},\underline{t})]^{-1}\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}(\underline{t},\underline{t})\right\}\right\|_p\\&\leqq|\cos\operatorname{Arg}\varrho\zeta|^{-1}\cdot\|\,\|\tilde{\mathbf{A}}(\underline{t},\underline{t})\|_{\mathcal{T}_2}\|_{(k+1)p}\cdot\left\|\left\|\text{“}\prod_{i=1}^{k}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}(\underline{t},\underline{t})\right\|_{\mathcal{T}_2}\right\|_{\frac{k+1}{k}p}.\end{aligned} \tag{1.10.5}$$

Now $\|\tilde{\mathbf{A}}\|^2_{\mathcal{T}_2}\in\bigoplus_{n=0}^{2}\mathcal{F}_n$, thus from (1.9.6),

$$\|\,\|\tilde{\mathbf{A}}(\underline{t},\underline{t})\|_{\mathcal{T}_2}\|_{(k+1)p}\leqq\left(\frac{k+1}{2}p\right)^{1/2}E_\nu[\|\tilde{\mathbf{A}}(\underline{t},\underline{t})\|^4_{\mathcal{T}_2}]^{1/4}, \tag{1.10.6}$$

26 If X is a finite set, $|X|$ denotes the number of elements of X

and, from (1.9.23),

$$E_\nu[\|\tilde{\mathbf{A}}(\underline{t},\underline{t})\|^4_{\mathcal{T}_2}] \leqq \mathbf{c}(\underline{t})^4 \|g\|^4_2 + 2\cdot(2\pi)^{-6}\|g\|^4_4 \|\hat{F}^{(0)}_{(\underline{t})}\|^8_{8/3}\,. \tag{1.10.7}$$

One estimates the right hand side of (1.10.7) by (1.9.24) and (1.9.26), one inserts this result as also (1.9.3) in (1.10.5), and one dominates $\log(1+4\kappa_j^2/m^2)$ by an arbitrarily small power of κ_{i-1}, $(i\in\underline{a}\cup\underline{b})$; then from (1.8.13), (1.10.2), (1.9.2) and (1.10.5), one sees that, if $\eta<1/4$, there exists $c'>0$ such that

$$\|\mathfrak{U}_1(\underline{a},\underline{b})\|_p \leqq p^{(|\underline{a}|+|\underline{b}|)/2+1}[(|\underline{a}|+|\underline{b}|)!]^{3/2} \prod_{i\in\underline{a}\cup\underline{b}} (c'\kappa_{i-1})^{-\eta}, \tag{1.10.8}$$

[one noted that $\max\{|\underline{a}|,|\underline{b}|\}\leqq k-1\leqq|\underline{a}|+|\underline{b}|$, if $\mathfrak{S}_1^{(k)}(\underline{a},\underline{b})\neq\emptyset$, and that $|\mathfrak{S}_1^{(k)}(\underline{a},\underline{b})| \leqq 4^{(|\underline{a}|+|\underline{b}|)}(|\underline{a}|+|\underline{b}|)!$.]

iii) If $n>2$, one has

$$\begin{aligned}
&\left|\mathrm{Tr}\left\{\prod_{q=1}^{n}\left([I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\cdot\text{“}\prod_{i_q=1}^{k_q}\text{”}\,\tilde{\mathbf{A}}^{(r^{(q)}_{i_q},s^{(q)}_{i_q})}\right)\right\}\right| \\
&\leqq \|[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\|^n\cdot\prod_{q=1}^{n}\left\|\text{“}\prod_{i_q=1}^{k_q}\text{”}\,\tilde{\mathbf{A}}^{(r^{(q)}_{i_q},s^{(q)}_{i_q})}\right\|_{\mathcal{T}_n} \\
&\leqq \|[I+i\varrho\zeta\tilde{\mathbf{A}}]^{-1}\|^n\cdot\prod_{q=1}^{n}\left\|\text{“}\prod_{i_q=1}^{k_q}\text{”}\,\tilde{\mathbf{A}}^{(r^{(q)}_{i_q},s^{(q)}_{i_q})}\right\|_{\mathcal{T}_2}.
\end{aligned} \tag{1.10.9}$$

Thus, from (1.10.4),

$$\begin{aligned}
&\left\|\mathrm{Tr}\left\{\prod_{q=1}^{n}\left([I+i\varrho\zeta\tilde{\mathbf{A}}(\underline{t},\underline{t})]^{-1}\cdot\text{“}\prod_{i_q=1}^{k_q}\text{”}\,\tilde{\mathbf{A}}^{(r^{(q)}_{i_q},s^{(q)}_{i_q})}(\underline{t},\underline{t})\right)\right\}\right\|_p \\
&\leqq|\cos\mathrm{Arg}\,\varrho\zeta|^{-n}\cdot\prod_{q=1}^{n}\left\|\,\left\|\text{“}\prod_{i_q=1}^{k_q}\text{”}\,\tilde{\mathbf{A}}^{(r^{(q)}_{i_q},s^{(q)}_{i_q})}(\underline{t},\underline{t})\right\|_{\mathcal{T}_2}\right\|_{\frac{|\underline{k}|}{k_q}p},
\end{aligned} \tag{1.10.10}$$

where, for $\underline{k}=(k_q)_{1\leqq q\leqq n}$, one set $|\underline{k}|=\sum_{q=1}^{n}k_q$. Then one notes that, if $(\underline{r}^{(q)},\underline{s}^{(q)})_{1\leqq q\leqq n}\in\mathfrak{S}_n^{(k)}(\underline{a},\underline{b})$, there exists a partition $\underline{a}=\bigcup_{q=1}^{n}\underline{a}^{(q)}$, $\underline{b}=\bigcup_{q=1}^{n}\underline{b}^{(q)}$, such that $(\underline{r}^{(q)},\underline{s}^{(q)})\in\mathfrak{S}_1^{(k_q)}(\underline{a}^{(q)},\underline{b}^{(q)})$, $(1\leqq q\leqq n)$, and one estimates the right hand side of (1.10.10) by (1.9.3), [with $(\underline{a}^{(q)},\underline{b}^{(q)})$ in place of $(\underline{a},\underline{b})$]. Next from (1.8.13) and (1.10.9), and using the inequalities

$$\max\{|\underline{a}|,|\underline{b}|,n\}\leqq|\underline{k}|-n\leqq|\underline{a}|+|\underline{b}|,\quad\text{if}\quad\mathfrak{S}_n^{(k)}(\underline{a},\underline{b})\neq\emptyset,$$

$$|\mathfrak{S}_n^{(k)}(\underline{a},\underline{b})|\leqq\frac{(|\underline{k}|-n)!}{(|\underline{k}|-n-|\underline{a}|)!}\,\frac{(|\underline{k}|-n)!}{(|\underline{k}|-n-|\underline{b}|)!}\leqq4^{(|\underline{a}|+|\underline{b}|)}\cdot(|\underline{a}|+|\underline{b}|)!\,,$$

$$\begin{aligned}
&\left|\left\{\underline{k}=(k_q)_{1\leqq q\leqq n};\,k_q\geqq2,\,|\underline{k}|\equiv\sum_{q=1}^{n}k_q=v\right\}\right| \\
&=\binom{v-n-1}{n-1}\leqq2^{(|\underline{a}|+|\underline{b}|)},\quad\text{if}\quad v\leqq|\underline{a}|+|\underline{b}|+n,
\end{aligned}$$

one has, (with $\eta < 1/4$ and $c' > 0$),

$$\|\mathfrak{U}_n(\underline{a},\underline{b})\|_p \leqq p^{(|\underline{a}|+|\underline{b}|)}[(|\underline{a}|+|\underline{b}|)!]^2 \prod_{i \in \underline{a} \cup \underline{b}} (c'\kappa_{i-1})^{-\eta}, \quad (n \geqq 2). \tag{1.10.11}$$

iv) In the same way, from (1.8.15) (with $n=0$), and (1.9.4),

$$\|\mathfrak{V}_0(\underline{a},\underline{b})\|_p \leqq \|\psi\|^2_{\mathscr{H}^1} \cdot p^{(|\underline{a}|+|\underline{b}|)/2}[(|\underline{a}|+|\underline{b}|)!]^{3/2} \prod_{i \in \underline{a} \cup \underline{b}} (c'\kappa_{i-1})^{-\eta}. \tag{1.10.12}$$

v) At last, if $n \geqq 1$, $(\underline{r}^{(q)}, \underline{s}^{(q)})_{0 \leqq q \leqq n} \in \mathfrak{T}_n^{(\underline{k})}(\underline{a},\underline{b})$,

$$\begin{aligned} &\left\|\left(\psi, \text{“}\prod_{i_0=1}^{k_0}\text{”}\, \tilde{\mathbf{A}}^{(r^{(0)}_{i_0}, s^{(0)}_{i_0})}(\underline{t},\underline{t}) \cdot \prod_{q=1}^{n} \left([I + i\varrho\zeta\tilde{\mathbf{A}}]^{-1} \cdot \text{“}\prod_{i_q=1}^{k_q}\text{”}\, \tilde{\mathbf{A}}^{(r^{(q)}_{i_q}, s^{(q)}_{i_q})}(\underline{t},\underline{t})\right)\psi\right)_{\mathscr{H}^1}\right\|_p \\ &\leqq \|\, \|\text{“}\prod_{i_0=1}^{k_0}\text{”}\, \tilde{\mathbf{A}}^{(s^{(0)}_{k_0-i_0+1}, r^{(0)}_{k_0-i_0+1})}(\underline{t},\underline{t}) \cdot \psi\|_{\mathscr{H}^1}\|_{(|k|/k_0)p} \\ &\cdot |\cos \operatorname{Arg} \varrho\zeta|^{-n} \cdot \prod_{q=1}^{n-1} \|\, \|\text{“}\prod_{i_q=1}^{k_q}\text{”}\, \tilde{\mathbf{A}}^{(r^{(q)}_{i_q}, s^{(q)}_{i_q})}(\underline{t},\underline{t})\|_{\mathscr{T}_2}\|_{(|\underline{k}|/k_q)p} \\ &\cdot \|\, \|\text{“}\prod_{i_n=1}^{k_n}\text{”}\, \tilde{\mathbf{A}}^{(r^{(n)}_{i_n}, s^{(n)}_{i_n})}(\underline{t},\underline{t}) \cdot \psi\|_{\mathscr{H}^1}\|_{(|\underline{k}|/k_n)p}\,. \end{aligned} \tag{1.10.13}$$

Therefore, from (1.8.15), (1.9.3) and (1.9.5),

$$\|\mathfrak{V}_n(\underline{a},\underline{b})\|_p \leqq \|\psi\|^2_{\mathscr{H}^1} \cdot p^{(|\underline{a}|+|\underline{b}|)}[(|\underline{a}|+|\underline{b}|)!]^2 \prod_{i \in \underline{a} \cup \underline{b}} (c'\kappa_{i-1})^{-\eta}, \quad (n \geqq 1). \tag{1.10.14}$$

Then inserting (1.10.1), (1.10.8), (1.10.11) in (1.8.11) and (1.10.12), (1.10.14) in (1.8.14), one obtains from (1.8.4),

$$\begin{aligned} &\left\|\left[\prod_{i=1}^{|\underline{a}|} (D'_{a_i} + i\varrho\zeta\tilde{\mathbf{c}}^{(a_i,0)}\partial_g) \cdot \prod_{k=1}^{|\underline{b}|} (D'_{b_k} + i\varrho\zeta\tilde{\mathbf{c}}^{(0,b_k)}\partial_g)\right] \log \tilde{Y}(J_j(t), J_j(t))\right\|_p \\ &\leqq \left(1 + \sum_{n=1}^{K} \|f_n\|^2_{\mathscr{H}^1}\right) \cdot p^{(|\underline{a}|+|\underline{b}|)}[(|\underline{a}|+|\underline{b}|)!]^2 \prod_{i \in \underline{a} \cup \underline{b}} (c''\kappa_{i-1})^{-\eta}. \end{aligned} \tag{1.10.15}$$

At last, as the derivations $(D_{a_i} + i\varrho\zeta(D_{a_i}\mathbf{c})\partial_g)_{1 \leqq i \leqq n}$ are mutually commutative, one has

$$\begin{aligned} &\left[\prod_{i=1}^{n} (D_{a_i} + i\varrho\zeta(D_{a_i}\mathbf{c})\partial_g)\right] Y \\ &= Y \cdot \sum_{p \in \mathfrak{P}(e_n)} \prod_{u \in P} \left(\left[\prod_{i_u \in u} (D_{a_{i_u}} + i\varrho\zeta(D_{a_{i_u}}\mathbf{c})\partial_g)\right] \log Y\right). \end{aligned} \tag{1.10.16}$$

Then, using

$$\begin{aligned} &\left\|\sum_{P \in \mathfrak{P}(e_n)} \prod_{u \in P} \left(\left[\prod_{i_u \in u} (D_{a_{i_u}} + i\varrho\zeta(D_{a_{i_u}}\mathbf{c})\partial_g)\right] \log Y\right)\right\|_p \\ &\leqq \sum_{P \in \mathfrak{P}(e_n)} \prod_{u \in P} \left\|\left[\prod_{i_u \in u} (D_{a_{i_u}} + i\varrho\zeta(D_{a_{i_u}}\mathbf{c})\partial_g)\right] \log Y\right\|_{(n/|u|)p}, \end{aligned} \tag{1.10.17}$$

one deduces (1.6.4) from (1.7.4), (1.8.7) and the estimates (1.4.3), (1.9.26) and (1.10.15), according to Hölder's inequality. □

2. Borel Summability

We now obtain an estimate (Theorem II, 2.1) of the derivatives of the functions $(\lambda, z) \mapsto Z^{\star}_{z,\lambda,g^2}(\mathbf{f})$ introduced by (1.5.1), from which in particular one deduces Borel summability of the Taylor series at zero of functions $z \mapsto Z^{\star}_{z,\lambda,g^2}(\mathbf{f})$, (where λ is fixed so that $|\lambda|$ is sufficiently small), and, as a consequence, a characterization of each of these among all possible holomorphic interpolations of the corresponding function $\frac{1}{N} \mapsto Z^{\star}_{\frac{1}{N},\lambda,g^2}(\mathbf{f})$, $(N \in \mathbb{N}_+)$, [defined by (1.2.10)].

An analogous result holds for the "normalized" functions $(\lambda, z) \mapsto S^{\star}_{z,\lambda,g^2}(\mathbf{f}) = Z^{\star}_{z,\lambda,g^2}(\mathbf{f})/Z^{0}_{z,\lambda,g^2}$.

2.1. Some Properties of the Analytic Continuation

For each subset $X \subset \mathbb{C}^2$ one sets $X^{(2)} = \left\{\left(\frac{\varrho^2}{8}, \zeta^2\right); (\varrho, \zeta) \in X\right\}$, one has

Theorem II. *For each $U \in \mathscr{U}$ (defined by (1.4.16)), the function $(\lambda, z) \mapsto Z^{\star}_{z,\lambda,g^2}(\mathbf{f})$, defined by (1.5.1), is indefinitely derivable on the closure $\overline{U^{(2)}}$ of $U^{(2)}$ and, for any bounded subset $B \subset \overline{U^{(2)}}$, there exists a constant $M > 0$ such that, for all $(\lambda, z) \in B$ and all $n_1, n_2 \in \mathbb{N}$,*

$$\left|\frac{\partial^{n_1}}{\partial \lambda^{n_1}} \frac{\partial^{n_2}}{\partial z^{n_2}} Z^{\star}_{z,\lambda,g^2}(\mathbf{f})\right| \leqq M^{n_1+n_2+1} \cdot (n_1! n_2!)^2 . \tag{2.1.1}$$

It follows[27] in particular that, if $|\lambda| < \frac{2}{3} m^2$, one has[28]

$$Z^{\star}_{z,\lambda,g^2}(\mathbf{f}) = (\lambda z)^{-1} \int_0^{\infty} e^{-x/\lambda z} \mathscr{B}_{\lambda}(\lambda^{-1} x) dx , \quad (\operatorname{Re} \lambda z > 0), \tag{2.1.2}$$

(the integral converges absolutely), where $\mathscr{B}_{\lambda}$ is the Borel transform of the function $z \mapsto Z^{\star}_{z,\lambda,g^2}(\mathbf{f})$, that is the holomorphic function which continues

$$\mathscr{B}_{\lambda}(u) = \sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} \left(\frac{\partial^n}{\partial z^n} Z^{\star}_{0,\lambda,g^2}(\mathbf{f})\right), \quad (|u| < M^{-1}). \tag{2.1.3}$$

---

27 See [5]

28 In this case, there exists $U \in \mathscr{U}$ such that $(z, \lambda) \in \overline{U^{(2)}}$ for all $z \in \mathbb{C}$ such that $\operatorname{Re} \lambda z \geqq 0$. In fact, the analyticity domain of the functions $z \mapsto Z^{\star}_{z,\lambda,g^2}(\mathbf{f})$ contains an angle strictly larger than the half-plane $\{z \in \mathbb{C};\ \operatorname{Re} \lambda z > 0\}$, (especially, if $|\operatorname{Arg} \lambda| \leqq \pi/2$ and $|\lambda| \leqq m^2/2$, it is the cut plane $\{z \neq 0; |\operatorname{Arg} \lambda z| \neq \pi\}$), and Borel summability extends to this angle. Besides this property allows us to make use of the obvious generalizations [4, p. 268] of the Borel transformation which use the entire functions $\mathscr{B}^{(\alpha)}_{\lambda}(u) = \sum_{n=0}^{\infty} \frac{u^n}{n! \Gamma(\alpha n + 1)} \left(\frac{\partial^n}{\partial z^n} Z^{\star}_{0,\lambda,g^2}(\mathbf{f})\right)$, with $\alpha \in \left]1, \frac{\theta}{\pi}\right[$ if the domain of analyticity contains an angle of measure $\theta > \pi$

This property allows a characterization of functions (1.5.1) among all possible analytic interpolations of the functions (1.3.8): for each $\lambda\in]0,\frac{2}{3}m^2[$, the unique function F, holomorphic in some disc $\Delta_r=\{z\in\mathbb{C};\operatorname{Re}z^{-1}>r^{-1}\}$, which has an asymptotic expansion $F(z)=\sum_{k=0}^{n-1}a_kz^k+R_n(z)$, $(n\in\mathbb{N})$, with, for some $M>0$, $|R_n(z)|\leqq M^{n+1}n!|z|^n$, $\forall z\in\Delta_r$, $\forall n\in\mathbb{N}$, and such that $F\left(\frac{1}{N}\right)=Z^{\star}_{\frac{1}{N},\lambda,g^2}(\mathbf{f})$, $\forall N\in\mathbb{N}$, $(N>r)$, satisfies $F(z)=Z^{\star}_{z,\lambda,g^2}(\mathbf{f})$, $\forall z\in\Delta_r$. [Indeed, first, the existence of the asymptotic expansion implies that F is indefinitely derivable at $z=0$ and that $a_n=\dfrac{F^{(n)}(0)}{n!}$. Next, one sees recursively from Taylor's formula that the successive derivatives $F^{(n)}(0)$ are completely determined by the values of F on the sequence $\left\{z=\frac{1}{N};N\in\mathbb{N},N>r\right\}$, therefore $F^{(n)}(0)=\dfrac{\partial^n}{\partial z^n}Z^{\star}_{0,\lambda,g^2}(\mathbf{f})$, $(n\in\mathbb{N})$. Last, Borel summability implies the equality of the two functions.] The holomorphic function $(\lambda,z)\mapsto Z^{\star}_{z,\lambda,g^2}(\mathbf{f})$ is thus determined on a subset sufficiently large to characterize it.

Analogous properties hold for functions (1.4.19): if one adds to the assumptions of Theorem II the condition $|Z^0_{z,\lambda,g^2}|\geqq\varepsilon$, one obtains, (from the Leibnitz formula),

$$\left|\frac{\partial^{n_1}}{\partial\lambda^{n_1}}\frac{\partial^{n_2}}{\partial z^{n_2}}S^{\star}_{z,\lambda,g^2}(\mathbf{f})\right|\leqq M'^{n_1+n_2+1}\cdot(n_1!n_2!)^2,\qquad(n_1,n_2\in\mathbb{N}).\tag{2.1.4}$$

Especially, if one supposes as above $|\lambda|<\frac{2}{3}m^2$, the function $z\mapsto Z^0_{z,\lambda,g^2}$ is continuous on $\{z\in\mathbb{C};\operatorname{Re}\lambda z\geqq 0\}$ and nonvanishing at zero, since from (1.4.18),

$$Z^0_{0,\lambda,g^2}=[\det{}_2(I+4\lambda B_g)]^{-1/2},\tag{2.1.5}$$

where $B_g\in\mathscr{T}_2(L^2)$ is the real, self adjoint operator, defined by[29]

$$B_g\psi=g\cdot(G_m^2*(g\cdot\psi)),\qquad(\psi\in L^2;G_m\text{ is the kernel of }\Sigma_m^{-1}).\tag{2.1.6}$$

Therefore, there exists $r>0$ such that the function $z\mapsto 1/Z^0_{z,\lambda,g^2}$ is bounded on $\Delta_r(\lambda)=\{z\in\mathbb{C};\operatorname{Re}(\lambda z)^{-1}>r^{-1}\}$, and from (2.1.4), one deduces that

$$S^{\star}_{z,\lambda,g^2}(\mathbf{f})=(\lambda z)^{-1}\int_0^\infty e^{-x/\lambda z}\mathscr{A}_\lambda(\lambda^{-1}x)\,dx,\qquad(z\in\Delta_r(\lambda)),\tag{2.1.7}$$

where $\mathscr{A}_\lambda$ is the Borel transform of $z\mapsto S^{\star}_{z,\lambda,g^2}(\mathbf{f})$. Here also, this property characterizes the functions $(\lambda,z)\mapsto S^{\star}_{z,\lambda,g^2}(\mathbf{f})$ among all possible analytic interpolations of (1.1.6).

---

29 B_g is the limit in $\mathscr{T}_2(L^2)$ of the sequence of operators defined by (1.4.10) where one substitutes a unit step to χ. One has $\|B_g\|\leqq 1/4\pi m^2$, (if $\|g\|_\infty\leqq 1$), therefore $(I+4\operatorname{Re}\lambda\cdot B_g)>0$ and the right hand side of (2.1.5) is well defined according to (1.3.7)

2.2. Reduction of the Proof of Theorem II

With the notations of Sect. 1.6, for $(\varrho, \zeta) \in \mathbf{D}_{(4\pi m^2)}$, one sets

$$\mathscr{Z}^{(n_1,n_2)}(\underline{t}) = \frac{\partial^{n_1}}{\partial \varrho^{n_1}} \frac{\partial^{n_2}}{\partial \zeta^{n_2}} Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2, k(\underline{t})}(\mathbf{f}), \qquad (n_1, n_2 \in \mathbb{N}). \tag{2.2.1}$$

Then Theorem II follows from

Proposition. *For any $U \in \mathscr{U}$, each function (of $(\varrho, \zeta) \in U$) $\mathscr{Z}^{(n_1,n_2)}(\underline{t})$ has a continuous continuation to $\bar{U}$, and, for any bounded subset $B \subset \bar{U}$ and any $\varepsilon > 0$, $\eta < 1/4$, there exist constants $M_1 > 0$, $b > 0$, $c > 0$, such that*

$$\left|\left[\prod_{i=j}^{n} D_i^{q_i}\right] \mathscr{Z}^{(n_1,n_2)}(J_j(t))\right|$$
$$\leqq M_i^{n_1+n_2}(n_1!n_2!)^{3/2} e^{b\kappa_j^{\varepsilon}} \left[\left(\sum_{i=j}^{n} q_i\right)!\right]^2 \prod_{i=j}^{n} (c\kappa_{i-1})^{-\eta q_i}, \tag{2.2.2}$$

for all $(\varrho, \zeta) \in B$, $(n_1, n_2) \in \mathbb{N}^2$, $n \geqq j \geqq 0$, $\underline{q} = (q_i)_{j \leqq i \leqq n} \in \mathbb{N}^{n-j+1}$, $(q_0 = 0)$, $t \in [0, 1]$.

Indeed, from the existence of the continuation to $\bar{U}$ of each $\mathscr{Z}^{(n_1,n_2)}(\underline{t})$, one first deduces that the function $\mathscr{Z}^{(0,0)}(\underline{t})$ is indefinitely derivable on $\bar{U}$ and that the equality (2.2.1) extends to $\bar{U}$. Then, (as in 1.6), one deduces from (2.2.2) that, for each $(n_1, n_2) \in \mathbb{N}^2$, the sequence $(\mathscr{Z}^{(n_1,n_2)}(\mathbf{1}_n))_{n \in \mathbb{N}}$ converges uniformly for $(\varrho, \zeta) \in B$. It follows that $Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2}(\mathbf{f}) = \lim_{n \to \infty} \mathscr{Z}^{(0,0)}(\mathbf{1}_n)$, (from Theorem I), is indefinitely derivable with respect to $(\varrho, \zeta) \in \bar{U}$ and satisfies

$$\left|\frac{\partial^{n_1}}{\partial \varrho^{n_1}} \frac{\partial^{n_2}}{\partial \zeta^{n_2}} Z^{\star}_{\zeta^2, \frac{\varrho^2}{8}, g^2}(\mathbf{f})\right| \leqq a M_1^{n_1+n_2}(n_1!n_2!)^{3/2}, \qquad \forall (\varrho, \zeta) \in B, \tag{2.2.3}$$

where $a > 0$ is a constant independent of $(n_1, n_2) \in \mathbb{N}^2$. Then (2.1.1) follows, because if ψ is an even, $(\psi(\zeta) = \psi(-\zeta))$, indefinitely derivable function, and if $\Psi(z) = \psi(\sqrt{z})$, one has, (recursively on k),

$$\Psi^{(k)}(z) = 2^{-(2k-1)} \int_0^1 \frac{(1-u^2)^{k-1}}{(k-1)!} \psi^{(2k)}(u\sqrt{z}) du, \qquad (k \geqq 1), \tag{2.2.4}$$

from which one deduces

$$|\Psi^{(k)}(z)| \leqq \frac{k!}{(2k)!} \sup_{0 \leqq u \leqq 1} |\psi^{(2k)}(u\sqrt{z})|, \tag{2.2.5}$$

and especially,

$$|\Psi^{(k)}| \leqq a(4^{\alpha} M_1^2)^k (k!)^{1+2\alpha}, \quad \text{if} \quad |\psi^{(k)}| \leqq a M_1^k (k!)^{1+\alpha}, \qquad (\alpha \geqq 0). \tag{2.2.6}$$

2.3. Computation of $\mathscr{Z}^{(n_1,n_2)}$ and its Derivatives

The computation of the left hand side of (2.2.2) relies on the "integration by parts" formula (1.7.2): one shows, with the help of (1.4.8), that derivations under the integral sign are allowed, and one obtains, [with the notation (1.7.3)],

$$\left[\prod_{k=j}^{n} D_k^{q_k}\right]\mathscr{Y}^{(n_1,n_2)} = \int_{\mathscr{S}'} \exp(\tfrac{1}{2}\varrho^2\zeta^2\mathbf{c}^2\|g\|_2^2 + i\varrho\zeta\mathbf{c}\langle\sigma,g\rangle)$$
$$\cdot\left[\prod_{k=j}^{n}(D_k + i\varrho\zeta(D_k\mathbf{c})\partial_g)^{q_k}\right]\left(\frac{\partial}{\partial\varrho} + i\zeta\mathbf{c}\partial_g\right)^{n_1}\left(\frac{\partial}{\partial\zeta} + i\varrho\mathbf{c}\partial_g\right)^{n_2} Y(\sigma)\cdot\nu(d\sigma). \tag{2.3.1}$$

As the derivations $\left(\frac{\partial}{\partial\varrho} + i\zeta\mathbf{c}\partial_g\right)$, $\left(\frac{\partial}{\partial\zeta} + i\varrho\mathbf{c}\partial_g\right)$ and $(D_k + i\varrho\zeta(D_k\mathbf{c})\partial_g)$ are mutually commutative, to compute the derivatives of Y one applies derivations $\left(\frac{\partial}{\partial\varrho} + i\zeta\mathbf{c}\partial_g\right)$ and $\left(\frac{\partial}{\partial\zeta} + i\varrho\mathbf{c}\partial_g\right)$ to the right hand side of (1.8.11) and (1.8.14), (with $\underline{u} = \underline{t}$), as also to the function $\log Y$ itself.

First one has, (with $\mathbf{A}_{[g]} \equiv \partial_g\mathbf{A}$),

$$\left(\frac{\partial}{\partial\varrho} + i\zeta\mathbf{c}\partial_g\right)\log Y_0 = -\frac{\varrho}{2}\operatorname{Tr}\{I + i\varrho\zeta\mathbf{A}]^{-1}(\mathbf{A}^2 - \mathbf{c}\mathbf{A}_{[g]})\}\,, \tag{2.3.2}$$

and, for $n \geqq 1$,

$$\left(\frac{\partial}{\partial\zeta} + i\varrho\mathbf{c}\partial_g\right)^n \log Y_0 = \frac{(-1)^{n+1}}{2\zeta^{n+2}}(n+1)!\log\det{}_{n+2}(I + i\varrho\zeta\mathbf{A})$$
$$+\tfrac{1}{2}\sum_{k=1}^{n}(-1)^k(k-1)!(n+1-k)!(i\varrho)^{n+2}\operatorname{Tr}\{\mathbf{A}^k[I + i\varrho\zeta\mathbf{A}]^{-k}\}$$
$$+\tfrac{1}{2}\sum_{k=1}^{n}(-1)^{k-1}(k-1)!(i\varrho)^{k+2}\left(\frac{\partial}{\partial\zeta} + i\varrho\mathbf{c}\partial_g\right)^{n-k}[\operatorname{Tr}\{\mathbf{A}^k[I + i\varrho\zeta\mathbf{A}]^{-k}\mathbf{c}\mathbf{A}_{[g]}\}]\,, \tag{2.3.3}$$

(proof by induction on n). One can then achieve the computation using the Leibnitz formula, equalities (1.8.30), (1.8.31) and

$$\left(\frac{\partial}{\partial\zeta} + i\varrho\mathbf{c}\partial_g\right)^n\{[I + i\varrho\zeta\mathbf{A}]^{-1}\}$$
$$= \sum_{\substack{k\in\mathbb{N}\\ k\leqq n\leqq 2k}}\ \sum_{\substack{u\in\mathscr{P}(e_k)\\ |u|=n-k}} \alpha_n^k(u)\left\{[I + i\varrho\zeta\mathbf{A}]^{-1}\prod_{j=1}^{k} T_j^u(\varrho,\zeta)[I + i\varrho\zeta\mathbf{A}]^{-1})\right\}, \tag{2.3.4}$$

(as also the analogous formula obtained by exchange of ϱ and ζ), where $\mathscr{P}(e_k)$ is the set of subsets of $e_k = \{1, \ldots, k\}$,

$$T_j^u(\varrho,\zeta) = \begin{cases} 2\varrho^2\mathbf{c}\mathbf{A}_{[g]}, & \text{if } j\in u \\ -i\varrho\mathbf{A} + \varrho^2\zeta\mathbf{c}\mathbf{A}_{[g]}, & \text{if } j\notin u \end{cases}, \quad (u\in\mathscr{P}(e_k))\,, \tag{2.3.5}$$

and where the integers $\alpha_n^k(u)$ satisfy

$$\sum_{\substack{u\in\mathscr{P}(e_k)\\ |u|=n-k}} \alpha_n^k(u) = \frac{n!}{2^{n-k}}\binom{k}{n-k}, \quad (k\leqq n\leqq 2k)\,, \tag{2.3.6}$$

[indeed, one shows easily that the family $\beta_n^k \equiv \sum_{\substack{u\in\mathscr{P}(e_k)\\|u|=n-k}} \alpha_n^k(\mathrm{u})$, $(n, k\in\mathbb{N},\ k\leqq n\leqq 2k)$, satisfies the recurrency relation $\beta_n^k = k\beta_{n-1}^{k-1} + (2k-n-1)\beta_{n-1}^k$, with $\beta_0^0=1$, $(\beta_0^k=0$, if $k>0)$, which implies (2.3.6)].

2.4. Bounds on the Integrant

Next one has[30]

Lemma. *For all $n_1, n_2\in\mathbb{N}$, one has*

$$\left\|\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)^{n_1}\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{n_2}[I+i\varrho\zeta\mathbf{A}]^{-1}\right\| \leqq n_1!n_2!|\cos\operatorname{Arg}\varrho\zeta|^{-(n_1+n_2+1)}\left[(2+|\varrho|)^2(2+|\zeta|)^2\left(1+\|\mathbf{A}\|_{\mathscr{T}_2}+\mathbf{c}\frac{\|g\|_4^2}{m}\right)\right]^{n_1+n_2}, \tag{2.4.1}$$

and

$$\left|\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)^{n_1}\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{n_2}\log Y_0\right| \leqq n_1!n_2!|\cos\operatorname{Arg}\varrho\zeta|^{-(n_1+n_2)} \cdot\left[(2+|\varrho|)^2(2+|\zeta|)^2\left(1+\|\mathbf{A}\|_{\mathscr{T}_2}+\mathbf{c}\left(\|g\|_2+\frac{\|g\|_4^2}{m}\right)\right)\right]^{n_1+n_2+2}. \tag{2.4.2}$$

Proof. 1) First, from (2.3.4), (2.3.5), (2.3.6) and (1.10.4), one has

$$\begin{aligned}\left\|\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{n}[I+i\varrho\zeta\mathbf{A}]^{-1}\right\| &\leqq n!\sum_{\substack{k\in\mathbb{N}\\k\leqq n\leqq 2k}}\binom{k}{n-k}|\cos\operatorname{Arg}\varrho\zeta|^{-(k+1)}\\ &\quad\cdot(|\varrho|\,\|\mathbf{A}\|+|\zeta|\,|\varrho|^2\mathbf{c}\|\mathbf{A}_{[g]}\|)^{2k-n}(|\varrho|^2\mathbf{c}\|\mathbf{A}_{[g]}\|)^{n-k}\\ &\leqq n!\,|\cos\operatorname{Arg}\varrho\zeta|^{-(n+1)}(1+|\varrho|\,\|\mathbf{A}\|_{\mathscr{T}_2}+(1+|\zeta|)|\varrho|^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathscr{T}_2})^n.\end{aligned} \tag{2.4.3}$$

Next one applies $\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)$ to the right hand side of (2.3.4). Then, using (2.4.3) where one exchanges ϱ and ζ, and that the only nonvanishing derivatives of the T_j^u satisfy

$$\left\|\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)^{r}\{\varrho^2\mathbf{c}\mathbf{A}_{[g]}\}\right\|\leqq(2+|\varrho|)^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathscr{T}_2},\quad(0\leqq r\leqq 2), \tag{2.4.4}$$

$$\left\|\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)^{r}\{-i\varrho\mathbf{A}+\zeta\varrho^2\mathbf{A}_{[g]}\}\right\|\leqq(1+|\varrho|)\|\mathbf{A}\|_{\mathscr{T}_2}+(2+|\varrho|)^2|\zeta|\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathscr{T}_2},\ (0\leqq r\leqq 2), \tag{2.4.5}$$

---

30 In what follows, we select (for simplicity) quite rough estimates

one has, from the Leibnitz formula

$$\left\|\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)^{n_1}\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{n_2}[I+i\varrho\zeta\mathbf{A}]^{-1}\right\|$$
$$\leqq \sum_{\substack{k_2\in\mathbb{N}\\ k_2\leqq n_2\leqq 2k_2}} n_2!\binom{k_2}{n_2-k_2}\sum_{\substack{k_1\in\mathbb{N}\\ n_1-2k_2\leqq k_1\leqq n_1}}\binom{n_1}{k_1}(n_1-k_1)!\binom{2k_2}{n_1-k_1}$$
$$\sum_{\substack{q_j\in\mathbb{N},\,(1\leqq j\leqq k_2+1)\\ \sum_{j=1}^{k_2+1} q_j=k_1}}\cdot\frac{k_1!}{\prod_{j=1}^{k_2+1}(q_j!)}$$
$$\cdot\prod_{j=1}^{k_2+1} q_j!\,|\cos\mathrm{Arg}\,\varrho\zeta|^{-(q_j+1)}(1+|\zeta|\,\|\mathbf{A}\|_{\mathcal{T}_2}+(1+|\varrho|)|\zeta|^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_2})^{q_j}$$
$$\cdot((2+|\varrho|)^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_2})^{n_2-k_2}\cdot((1+|\varrho|)\|\mathbf{A}\|_{\mathcal{T}_2}+(2+|\varrho|)^2|\zeta|\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_2})^{2k_2-n_2}, \tag{2.4.6}$$

from which one deduces

$$\left\|\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)^{n_1}\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{n_2}[I+i\varrho\zeta\mathbf{A}]^{-1}\right\|$$
$$\leqq n_1!n_2!2^{n_1+n_2}|\cos\mathrm{Arg}\,\varrho\zeta|^{-(n_1+n_2+1)}$$
$$\cdot(2+(1+|\varrho|+|\zeta|)\,\|\mathbf{A}\|_{\mathcal{T}_2}+(2+|\varrho|)^2(2+|\zeta|)^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_2})^{n_1+n_2}. \tag{2.4.7}$$

2.i) In the case $n_1\geqq 1$ one computes the derivatives of $\log Y_0$ by differentiating the right hand side of (2.3.2). First, if $n_1=1$, $n_2=0$, one has

$$\left|\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)\log Y_0\right|\leqq\tfrac{1}{2}|\cos\mathrm{Arg}\,\varrho\zeta|^{-1}|\varrho|(\|\mathbf{A}\|^2_{\mathcal{T}_2}+\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_1}), \tag{2.4.8}$$

and, if $n_1=2$, $n_2=0$,

$$\left|\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)^2\log Y_0\right|\leqq\tfrac{1}{2}|\cos\mathrm{Arg}\,\varrho\zeta|^{-1}(\|\mathbf{A}\|^2_{\mathcal{T}_2}+\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_1})$$
$$+\tfrac{1}{2}|\cos\mathrm{Arg}\,\varrho\zeta|^{-2}(1+(1+|\varrho|+|\zeta|)\,\|\mathbf{A}\|_{\mathcal{T}_2}+|\varrho|(1+|\zeta|)^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_2})^3. \tag{2.4.9}$$

Next, in the other cases, ($n_1\geqq 3$ or $n_1\geqq 1$, $n_2\geqq 1$), one estimates the nonvanishing derivatives of $\frac{\varrho}{2}(\mathbf{A}^2-\mathbf{c}\mathbf{A}_{[g]})$ by

$$\left\|\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)^{r_1}\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{r_2}\left\{\frac{\varrho}{2}(\mathbf{A}^2-\mathbf{c}\mathbf{A}_{[g]})\right\}\right\|_{\mathcal{T}_2}$$
$$\leqq(2+(1+|\varrho|+|\zeta|)\|\mathbf{A}\|_{\mathcal{T}_2}+(2+|\varrho|)^2(2+|\zeta|)^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_2})^2,$$
$$(0\leqq r_1\leqq 3,\quad 0\leqq r_2\leqq 2), \tag{2.4.10}$$

and, according to (2.4.7), one deduces, from the Leibnitz formula,

$$\left|\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)^{n_1}\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{n_2}\log Y_0\right|\leqq n_1!n_2!2^{n_1+n_2+1}|\cos\operatorname{Arg}\varrho\zeta|^{-(n_1+n_2)}$$
$$\cdot(2+(1+|\varrho|+|\zeta|)\|\mathbf{A}\|_{\mathcal{T}_2}+(2+|\varrho|)^2(2+|\zeta|)^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_2})^{n_1+n_2+2},$$
$$(n_1\geqq 3\quad\text{or}\quad n_1\geqq 1, n_2\geqq 1). \tag{2.4.11}$$

ii) If $n_1=0$, one estimates the right hand side of (2.3.3). From the inequality [easily deduced from (1.4.5), (1.4.6) and (1.4.7)]

$$\left|\log(1+u)+\sum_{k=1}^{n+1}\frac{(-u)^k}{k}\right|\leqq\frac{2^{n+2}}{n+1}|\sin\operatorname{Arg}u|^{-1}|u|^{n+2},\qquad(n\geqq 0,|\operatorname{Im}\log(1+u)|<\pi), \tag{2.4.12}$$

and from $\|\mathbf{A}\|_{\mathcal{T}_{n+2}}\leqq\|\mathbf{A}\|_{\mathcal{T}_2}$, $(n\geqq 0)$, one deduces

$$\left|\frac{(n+1)!}{\zeta^{n+2}}\log\det{}_{n+2}(I+i\varrho\zeta\mathbf{A})\right|\leqq n!2^{n+2}|\cos\operatorname{Arg}\varrho\zeta|^{-1}|\varrho|^{n+2}\|\mathbf{A}\|_{\mathcal{T}_2}^{n+2}. \tag{2.4.13}$$

Next, from (1.10.4),

$$\sum_{k=1}^{n}(k-1)!(n+1-k)!|\varrho|^{n+2}|\operatorname{Tr}\{\mathbf{A}^{n+2}[I+i\varrho\zeta\mathbf{A}]^{-k}\}|$$
$$\leqq 2n!|\cos\operatorname{Arg}\varrho\zeta|^{-n}|\varrho|^{n+2}\|\mathbf{A}\|_{\mathcal{T}_2}^{n+2}, \tag{2.4.14}$$

and last from (2.4.3) one obtains

$$\left\|\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{n}\{\varrho\mathbf{A}[I+i\varrho\zeta\mathbf{A}]^{-1}\}\right\|_{\mathcal{T}_2}$$
$$\leqq n!|\cos\operatorname{Arg}\varrho\zeta|^{-(n+1)}(1+|\varrho|\,\|\mathbf{A}\|_{\mathcal{T}_2}+(1+|\zeta|)|\varrho|^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_2})^{n+1}. \tag{2.4.15}$$

Therefore, from the Leibnitz formula,

$$\left\|\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{n}\{\varrho\mathbf{A}[I+i\varrho\zeta\mathbf{A}]^{-1}\}^k\right\|_{\mathcal{T}_2}$$
$$\leqq\sum_{\substack{q_j\in\mathbb{N},(1\leqq j\leqq k)\\ \sum_{j=1}^{k}q_j=n}}\frac{n!}{\prod_{j=1}^{k}(q_j)!}\prod_{j=1}^{k}\left\|\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{q_j}\{\varrho\mathbf{A}[I+i\varrho\zeta\mathbf{A}]^{-1}\}\right\|_{\mathcal{T}_2}$$
$$\leqq n!\binom{n+k-1}{k-1}|\cos\operatorname{Arg}\varrho\zeta|^{-(n+k)}(1+|\varrho|\,\|\mathbf{A}\|_{\mathcal{T}_2}+(1+|\zeta|)|\varrho|^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_2})^{n+k}, \tag{2.4.16}$$

$\left[\text{because }\left|\left\{(q_j)_{1\leqq j\leqq k}\in\mathbb{N}^k;\ \sum_{j=1}^{k}q_j=n\right\}\right|=\binom{n+k-1}{k-1}\right]$. Thus

$$\sum_{k=1}^{n}(k-1)!|\varrho|^{k+2}\left|\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{n-k}\operatorname{Tr}\{\mathbf{A}^k[I+i\varrho\zeta\mathbf{A}]^{-k}\mathbf{c}\mathbf{A}_{[g]}\}\right|$$
$$\leqq n!|\cos\operatorname{Arg}\varrho\zeta|^{-n}(1+|\varrho|\,\|\mathbf{A}\|_{\mathcal{T}_2}+(1+|\zeta|)|\varrho|^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathcal{T}_2})^{n+1}, \tag{2.4.17}$$

then, joining (2.4.13), (2.4.14), (2.4.17), one has from (2.3.3),

$$\left|\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^n \log Y_0\right|$$
$$\leqq n!\,|\cos \operatorname{Arg}\varrho\zeta|^{-n}(1+|\varrho|\,\|\mathbf{A}\|_{\mathscr{T}_2}+(1+|\zeta|)|\varrho|^2\mathbf{c}\|\mathbf{A}_{[g]}\|_{\mathscr{T}_2})^{n+2}. \qquad (2.4.18)$$

3) From (1.8.2) (with $h=\delta$, $\underline{u}=\underline{t}$), one has

$$\|\mathbf{A}_{[g]}(\underline{t})\|^2_{\mathscr{T}_2}=\int_{E^2} g(x)^2 g(y)^2([\Sigma_m^{-1}\check{k}_{(\underline{t})}{*}k_{(\underline{t})}](x-y))^2dxdy$$
$$\leqq \|g^2\|_2^2\|\widehat{[\Sigma_m^{-1}\check{k}_{(\underline{t})}{*}k_{(\underline{t})}]^2}\|_\infty$$
$$\leqq (2\pi)^{-2}\|g\|_4^4\|\widehat{\Sigma_m^{-1}\check{k}_{(\underline{t})}{*}k_{(\underline{t})}}\|_2^2$$
$$\leqq (2\pi)^{-2}\|g\|_4^4\int_E\frac{dp}{(|p|^2+m^2)^2}=\frac{1}{2^4\pi^3m^2}\|g\|_4^4, \qquad (2.4.19)$$

(because $\|k_{(\underline{t})}\|_\infty\leqq 1$, by assumption). On the other hand $\mathbf{A}_{[g]}(\underline{t})\geqq 0$, thus, from (1.8.17),

$$\|\mathbf{A}_{[g]}(\underline{t})\|_{\mathscr{T}_1}=\operatorname{Tr}\{\mathbf{A}_{[g]}(\underline{t})\}=\mathbf{c}(\underline{t})\|g\|_2^2. \qquad (2.4.20)$$

Then one verifies (2.4.1) and (2.4.2), inserting respectively (2.4.19) in (2.4.7), and (2.4.19), (2.4.20) in (2.4.8), (2.4.9), (2.4.11) and (2.4.18). □

2.5. L^P-Estimates

Now, supposing $U\in\mathscr{U}$, [see (1.4.16)], and $B\subset\bar{U}$ a bounded subset, are given, one has

Lemma. *For any $\eta<1/4$, there exist constants $M_0>0$, $c_1>0$ such that for all $n_1, n_2, n\in\mathbb{N}$, $a_i\in\mathbb{N}_+$, $(1\leqq i\leqq n)$, $j\in\mathbb{N}$, $j\leqq\min(a_i)_{1\leqq i\leqq n}$, $(j=0$ if $n=0)$, $t\in[0,1]$ and $(\varrho,\zeta)\in B$, one has*

$$\left\|\prod_{i=1}^n(D_{a_i}+i\varrho\zeta(D_{a_i}\mathbf{c})\partial_g)\cdot\left(\frac{\partial}{\partial\varrho}+i\zeta\mathbf{c}\partial_g\right)^{n_1}\cdot\left(\frac{\partial}{\partial\zeta}+i\varrho\mathbf{c}\partial_g\right)^{n_2}\log Y(J_j(t))\right\|_p$$
$$\leqq n_1!\,n_2!\,(M_0[1+\|\,\|\mathbf{A}(J_j(t))\|_{\mathscr{T}_2}\|_{(n_1+n_2)p_1}+\mathbf{c}(J_j(t))])^{n_1+n_2+2}$$
$$\cdot p_2^n(n!)^2\prod_{i=1}^n(c_1\kappa_{a_i-1})^{-\eta}, \qquad (2.5.1)$$

where $p, p_1, p_2\in[1,+\infty[$ and $\dfrac{1}{p}=\dfrac{1}{p_1}+\dfrac{1}{p_2}$.

Indeed, for $n=0$, (2.5.1) simply follows from (2.4.1) and (2.4.2).

Next one computes $\partial_g^n\left(\text{“}\prod_{i=1}^k\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right)$ by iterating (1.8.30); one deduces $\partial_g^n\left(\operatorname{Tr}\left\{\text{“}\prod_{i=1}^k{}^{\circ}\text{”}\,\tilde{\mathbf{A}}^{(r_i,s_i)}\right\}\right)$ thanks to (1.8.31); then, with the assumptions of Lem-

ma 1.9, and adapting elementarily the proof of this lemma[31], one estimates the norms of the derivatives of functions in the left hand side of inequalities (1.9.1)–(1.9.5): so one verifies that each one of these can be generalized by
- applying ∂_g^n to the function of the left hand side,
- multiplying the right hand side by $n!$, and substituting there a new constant (independent of n), to c,

so, for example, (1.9.3) is generalized by

$$\| \|\partial_g^n \cdot \text{``}\prod_{i=1}^{k}\text{''} \tilde{\mathbf{A}}^{(r_i,s_i)}\|_{\mathscr{T}_2}\|_p \leqq n! p^{k/2}(k!)^{1/2} \prod_{i\in \underline{a}\cup \underline{b}} (c_1' \kappa_{i-1})^{-\eta}. \tag{2.5.2}$$

Then one applies derivations $\left(\frac{\partial}{\partial \varrho} + i\zeta \mathbf{c}\partial_g\right)$ and $\left(\frac{\partial}{\partial \zeta} + i\varrho \mathbf{c}\partial_g\right)$ to functions (1.8.3) and (1.8.15), and, according to the estimates of 1.10, one verifies (2.5.1) using, on the one hand (2.4.1), and on the other hand (2.5.2) and the four analogous inequalities described above, to estimate the various factors. □

2.6. *Proof of Proposition 2.2*

Then Proposition 2.2 comes from inequality (2.5.1) as follows: First, if w_j denotes a set with n_j elements, $(j=1,2)$, and $\mathfrak{P} = \mathfrak{P}(e_n \cup w_1 \cup w_2)$ the set of partitions of the disjoint union of $e_n = \{1, \dots, n\}$ with w_1 and w_2, one has

$$\left[\prod_{i=1}^{n} (D_{a_i} + i\varrho\zeta(D_{a_i}\mathbf{c})\partial_g) \cdot \left(\frac{\partial}{\partial \varrho} + i\zeta\mathbf{c}\partial_g\right)^{n_1} \left(\frac{\partial}{\partial \zeta} + i\varrho\mathbf{c}\partial_g\right)^{n_2}\right] Y = \left(\sum_{P\in\mathfrak{P}} \prod_{u\in P} X(u)\right) \cdot Y, \tag{2.6.1}$$

with

$$X(u) = \left[\prod_{i_u \in u\cap e_n} (D_{a_{i_u}} + i\varrho\zeta(D_{a_{i_u}}\mathbf{c})\partial_g) \cdot \left(\frac{\partial}{\partial \varrho} + i\zeta\mathbf{c}\partial_g\right)^{|u\cap w_1|} \cdot \left(\frac{\partial}{\partial \zeta} + i\varrho\mathbf{c}\partial_g\right)^{|u\cap w_2|}\right] \log Y. \tag{2.6.2}$$

Next, $p\in[1, +\infty[$ being fixed, for each $P\in\mathfrak{P}$ and $u\in P$, one sets

$$p_1(P,u) = \frac{2(n_1 + n_2 + 2|P|)}{|u\cap w_1| + |u\cap w_2| + 2} p, \qquad p_2(P,u) = \frac{2n}{|u\cap e_n|} p, \tag{2.6.3}$$

and $p(P,u) = \left(\frac{1}{p_1(P,u)} + \frac{1}{p_2(P,u)}\right)^{-1}$, so that $\sum_{u\in P} \frac{1}{p(P,u)} = \frac{1}{p}$, and thus, from Hölder's inequality

$$\left\|\prod_{u\in P} X(u)\right\|_p \leqq \prod_{u\in P} \|X(u)\|_{p(P,u)}. \tag{2.6.4}$$

---

31 Modified in that way, the proof introduces, except the "kernels" (1.9.19) and (1.9.21), only those which are associated with factors of the form $\tilde{\mathbf{A}}_{[g]}^{(i_1,i_2)}$ in the expansion of $\partial_g^n\left(\text{``}\prod_{i=1}^{k}\text{''} \tilde{\mathbf{A}}^{(r_i,s_i)}\right)$, namely $\int_E g(x)^2 \prod_{n=1}^{2} F^{(i_n)}(x-y_n)dx$, which are estimated by (1.9.23), (1.9.24) and (1.9.25). One notes that it is not possible to deduce simply the result from (1.9.3) and (1.9.5) and Hölder's inequality, because factors of the form "$\prod_{i=j_1}^{j_2}$" $\tilde{\mathbf{A}}^{(r_i,s_i)}$ do not necessarily satisfy the assumptions of the lemma

Then one estimates $\|X(u)\|_{p(P,u)}$ by (2.5.1) where one gives to p_1 and p_2 the values defined by (2.6.3): one obtains (with suitable constants M'_0, c'_1),

$$\left\|\prod_{P\in\mathfrak{P}}\prod_{u\in P}X(u)\right\|_p \leqq M_0'^{(n_1+n_2)}n!\prod_{i=1}^{n}(c'_1\kappa_{a_i-1})^{-\eta}$$
$$\cdot\sum_{P\in\mathfrak{P}}(1+\|\,\|\mathbf{A}\|_{\mathscr{F}_2}\|_{2(n_1+n_2+2|P|)p}+\mathbf{c})^{n_1+n_2+2|P|}\prod_{u\in P}|u|!\,. \tag{2.6.5}$$

But one has

$$\|\mathbf{A}\|^2_{\mathscr{F}_2}=\Pi_2\cdot\|\mathbf{A}\|^2_{\mathscr{F}_2}+E_\nu[\|\mathbf{A}\|^2_{\mathscr{F}_2}]\,, \tag{2.6.6}$$

where Π_2 denotes the orthogonal projection on the "two particles space" $\mathscr{F}_2\in L^2(\mathscr{S}',\nu)$. Thus, from (1.9.6),

$$\begin{aligned}\|\,\|\mathbf{A}\|_{\mathscr{F}_2}\|_{2q}&=\|\,\|\mathbf{A}\|^2_{\mathscr{F}_2}\|_q^{1/2}\leqq(\|\Pi_2\|\mathbf{A}\|^2_{\mathscr{F}_2}\|_q+E_\nu[\|\mathbf{A}\|^2_{\mathscr{F}_2}])^{1/2}\\&\leqq(q\|\Pi_2\|\mathbf{A}\|^2_{\mathscr{F}_2}\|_2+E_\nu[\|\mathbf{A}\|^2_{\mathscr{F}_2}])^{1/2}\\&\leqq q^{1/2}\|\Pi_2\|\mathbf{A}\|^2_{\mathscr{F}_2}\|_2^{1/2}+E_\nu[\|\mathbf{A}\|^2_{\mathscr{F}_2}]^{1/2}.\end{aligned} \tag{2.6.7}$$

Moreover, from (1.4.9) and (1.4.14),

$$E_\nu[\|\mathbf{A}_{g,k_{(\underline{t})}}\|^2_{\mathscr{F}_2}]=\mathbf{c}(\underline{t})^2\|g\|_2^2\,, \tag{2.6.8}$$

and, on the other hand,

$$\begin{aligned}\|\Pi_2\|\mathbf{A}_{g,k_{(\underline{t})}}\|^2_{\mathscr{F}_2}\|_2^2&=2\int_{E^2}|[\Sigma_m^{-1}\check{k}_{(\underline{t})}*k_{(\underline{t})}(x_1-x_2)]^2g(x_1)g(x_2)|dx_1dx_2\\&\leqq\frac{2}{(2\pi)^6}\|g\|_2^2\|\widehat{\Sigma_m^{-1}\check{k}_{(\underline{t})}*k_{(\underline{t})}}\|^4_{4/3}\\&\leqq\frac{2}{(2\pi)^6}\|g\|_4^4\left(\int_E\frac{dp}{(|p|^2+m^2)^{4/3}}\right)^3=\frac{3^3}{2^5\pi^3}\frac{\|g\|_4^4}{m^2}.\end{aligned} \tag{2.6.9}$$

Then, inserting (2.6.7), (2.6.8) and (2.6.9) in (2.6.5),

$$\left\|\sum_{P\in\mathfrak{P}}\prod_{u\in P}X(u)\right\|_p \leqq M_0''^{(n_1+n_2)}n!\prod_{i=1}^{n}(c''_1\kappa_{a_i-1})^{-\eta}$$
$$\cdot\sum_{P\in\mathfrak{P}}(1+(n_1+n_2+2|P|)^{1/2}+\mathbf{c})^{n_1+n_2+2|P|}\prod_{u\in P}|u|!\,. \tag{2.6.10}$$

Last, for any $\varepsilon>0$, one has

$$\begin{aligned}&(1+(n_1+n_2+2|P|)^{1/2}+\mathbf{c})^{n_1+n_2+2|P|}\\&\quad\leqq\sum_{k=0}^{n_1+n_2+2|P|}\binom{n_1+n_2+2|P|}{k}\cdot\exp\left(\frac{n_1+n_2}{2}+|P|\right)(k!)^{1/2}\\&\quad\cdot\left(\frac{1}{2\varepsilon}\right)^{\frac{n_1+n_2+2|P|-k}{2}}[(n_1+n_2+2|P|-k)!]^{1/2}e^{\varepsilon(1+\mathbf{c})^2}\\&\quad\leqq(6e/\varepsilon)^{\frac{n_1+n_2}{2}+|P|}[(n_1+n_2)!]^{1/2}|P|!e^{\varepsilon(1+\mathbf{c})^2},\end{aligned} \tag{2.6.11}$$

[one used inequalities $\xi^k \leqq e^{\xi} k!$, and $\xi^k \leqq (2\varepsilon)^{-\frac{k}{2}}(k!)^{1/2} e^{\varepsilon\xi^2}$, ($\xi>0$, $k\in\mathbb{N}$, $\varepsilon>0$)], and on the other hand,

$$\sum_{P\in\mathfrak{P}} |P|! \prod_{u\in P} |u|! \leqq 3^{(n_1+n_2+n)}(n_1+n_2+n)!\,, \tag{2.6.12}$$

(proof by induction), thus finally

$$\begin{aligned}&\Big\|\sum_{P\in\mathfrak{P}} \prod_{u\in P} X(u)\Big\|_p \\ &\quad\leqq M_1'^{(n_1+n_2)}(n_1!n_2!)^{3/2}(n!)^2 \prod_{i=1}^{n} (c'\kappa_{a_i-1})^{-\eta} e^{\varepsilon(1+c)^2}\,,\end{aligned} \tag{2.6.13}$$

from which one deduces (2.2.2), using (2.3.1), (2.6.1), the estimates (1.4.3), (1.9.26), and Hölder's inequality.

Appendix

We describe the domain defined by $(z^{1/2}, (8\lambda)^{1/2}) \in \mathbf{D}_{(4\pi m^2)}$, [see (1.4)].

A.1. Boundary of the Domain in the Plane of the Complex Variable λ/m^2:

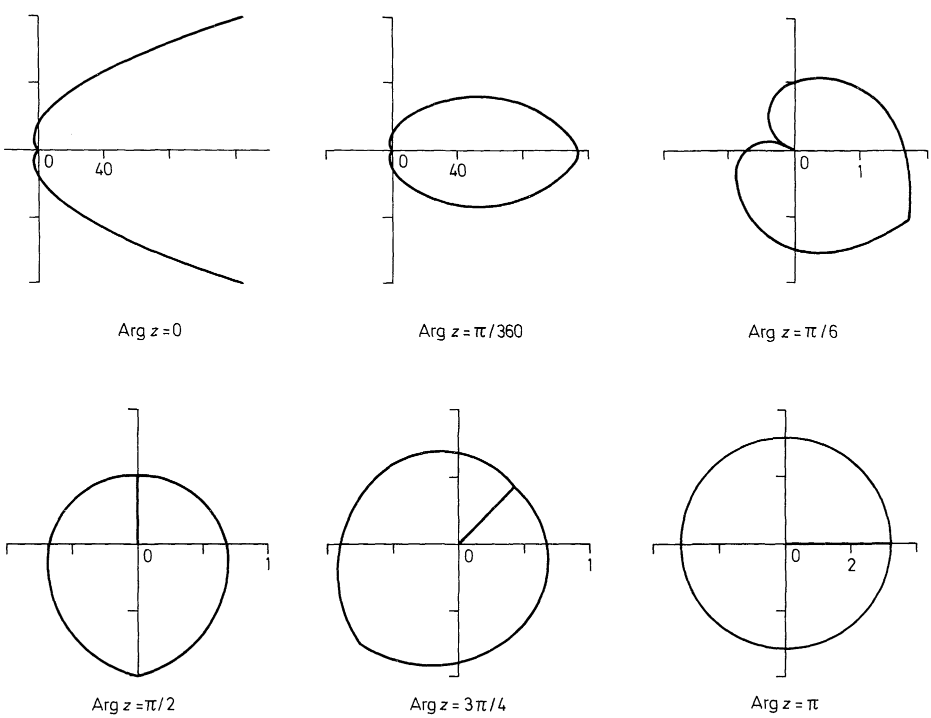

Fig. 1

A.2. Admissible values of $|\lambda|/m^2$ lie below the graph of the following functions of $\operatorname{Arg} z\lambda$:

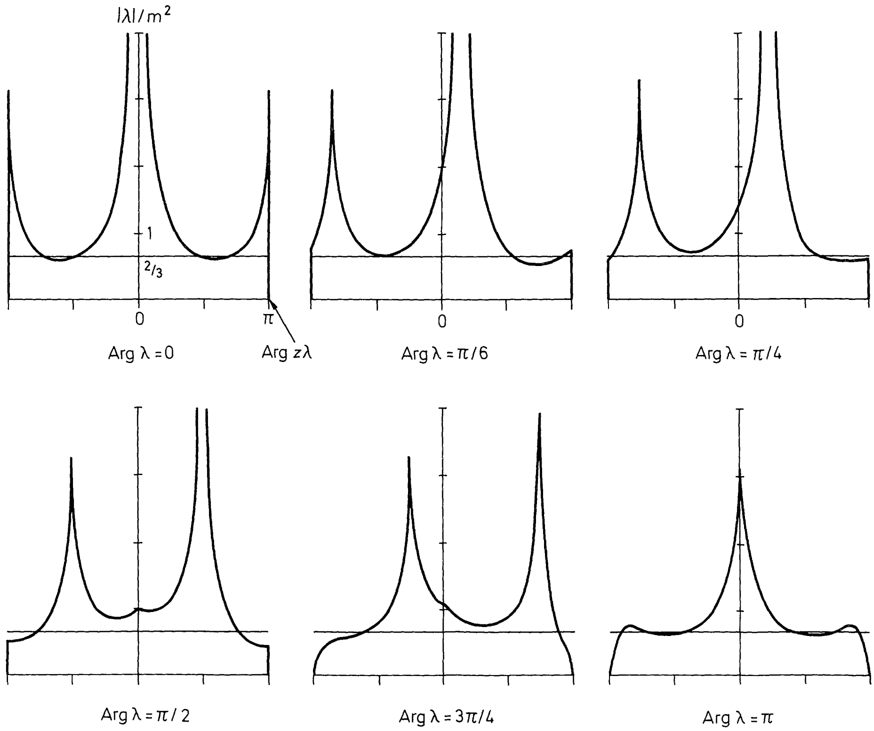

Fig. 2

One sees that, for $|\operatorname{Arg} z\lambda| \leqq \pi/2$, all λ's such that $|\lambda|/m^2 < 2/3$ are allowed (see Sect. 2.1, footnote 28).

Acknowledgements. We would like to thank J. Magnen and R. Seneor for pointing out to our attention this possible extension of the work of A. Kupiainen, and for valuable indications.

References

1. Glimm, J., Jaffe, A.: Quantum field theory models. In: Statistical mechanics and quantum field theory, De Witt, C., Stora, R. (eds.) p. 1, New York: Gordon and Breach 1971
2. Kupianen, A.: $1/N$ expansion for a quantum field model. Commun. Math. Phys. **74**, 199–221 (1980)
3. Nelson, E.: A quartic interaction in two dimensions. In: Mathematical theory of elementary particles, Goodmann, R., Segal, I., (eds.), p. 69, Cambridge MA: M.I.T. Press 1966
4. Renouard, P.: Analyticité et sommabilité de Borel des fonctions de Schwinger du modèle de Yukawa en dimension $d = 2$.I. Ann. I.H.P. **27**, 237 (1977)
5. Sokal, A.: An improvement of Watson's theorem on Borel summability. J.M.P. **21**, 261 (1980)

Communicated by K. Osterwalder

Received September 24, 1981