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Local Observables and Particle Statistics II*

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Abstract. Starting from the principles of local relativistic Quantum Theory without long range forces, we study the structure of the set of superselection sectors (charge quantum numbers) and its implications for the particle aspects of the theory. Without assuming the commutation properties (or even the existence) of unobservable fields connecting different sectors (charge-carrying fields), one has a particle-antiparticle symmetry, an intrinsic notion of statistics for identical particles, and a spin-statistics theorem. Particles in "pseudoreal sectors" cannot be their own antiparticles (a variant of Carruthers' theorem). We also show how scattering states and transition probabilities are obtained in this frame.

I. Introduction

In [1] we studied the structure of the set of charge quantum numbers (or superselection rules in elementary particle physics) as far as it follows from the general principles of local quantum physics. The setting and the main results may be sketched as follows. One considers the theory to be specified by the algebra $\mathfrak{A}$ which is generated by the local observables[1]. Assuming the existence of a state[2] ω_0, corresponding to the physical vacuum, we restrict our attention to the class of those states which become indistinguishable from ω_0 in asymptotic observations (observations outside a sufficiently large region of space). The pure states

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[1] The observables which can be measured in a space-time region $\mathcal{O}$ generate the subalgebra $\mathfrak{A}(\mathcal{O})$. The principle of locality is expressed in terms of these; it requires that two observables commute if they can be measured in spacelike separated regions. Therefore the anticommuting fields occurring in conventional quantum field theory are not affiliated with the algebra of observables.

[2] Mathematically a "state" means an expectation functional over the abstract algebra $\mathfrak{A}$.

within this class may then be divided into superselection sectors[3]; the parameters which distinguish different sectors may be interpreted physically as charge quantum numbers. One finds:

i) there is a composition law of sectors ("addition of charges")[4];

ii) there is a conjugate to each sector ("charge conjugation")[5];

iii) each sector has a "statistics parameter" λ whose value can be a positive or negative inverse integer or zero;

iv) if ξ denotes a sector and ξ^n its n^{th} power in the sense of i) then there is a unitary representation $\varepsilon^{(n)}$ of the permutation group of n elements which commutes with the representation of the observable algebra belonging to ξ^n. The equivalence class of $\varepsilon^{(n)}$ is determined for all n by the statistics parameter λ belonging to the sector ξ.

We did not treat in [1] any of the particle aspects of the theory. This will be the essential objective of the present paper. We shall therefore consider only Poincaré invariant sectors and furthermore exclude the case of infinite statistics ($\lambda = 0$). A brief discussion of the pathological situation arising if $\lambda = 0$ is given in the appendix.

Sections II, III and V will serve to show that the set of Poincaré covariant representations with finite statistics is closed under the operations i) and ii) above; that positivity of the energy in the vacuum sector implies positivity of the energy in each of these representations; and that the statistics parameters of conjugate sectors are equal.

In Section VI we consider sectors which contain single particle states. With the help of techniques developed by Epstein [2] one finds the expected generalizations of theorems well known in quantum field theory, namely:

1) Conjugate Sectors contain particles with the same mass, spin and multiplicity (particle-antiparticle symmetry).

2) The sign of the statistics parameter of a sector in which a particle of spin s occurs is $(-1)^{2s}$ (connection between spin and parastatistics).

3) Self-conjugate sectors may be divided into two classes called "real" and "pseudoreal". In a pseudoreal sector a particle cannot be its own antiparticle. (This may be regarded as a variant of Carruthers' Theorem [3].)

Finally in Section VII we discuss the construction of incoming and outgoing many particle states and describe collision theory. This will

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[3] One sector consists of all the vector states occurring in one equivalence class of irreducible representations of $\mathfrak{A}$. Thus the occurrence of superselection rules is tied to the existence of inequivalent representations of $\mathfrak{A}$.

[4] The product of two sectors may lead to a reducible representation. Thus the "addition of charges" is not always an arithmetic addition but may be analogous to the "addition of angular momenta" in quantum mechanics.

[5] Actually we need here finite statistics. See also Theorem 3.1.

establish the connection between our intrinsic definition of statistics (items iii) and iv) above) and the wave mechanical description of systems of identical particles: the operators $\varepsilon^{(n)}$ permute the arguments of the wave functions of asymptotic particle configurations; their physical significance stems from the fact that they determine the metric in the Hilbert space of asymptotic n-particle states.

The essential assumptions all concern the vacuum sector. They are:

1) Poincaré covariance.

2) Positivity of the energy.

3) Duality (see [1]), which combines the principle of locality with the requirement that the algebra of observables cannot be enlarged.

4) Weak additivity[6].

We use the same notation as in [1]. As discussed there the representations of interest may be related to the vacuum representation by localized morphisms. Thereby all representation spaces are identified with the space $\mathscr{H}_0$ of the vacuum representation. We shall now be concerned with those localized morphisms which are Poincaré covariant and lead to sectors with finite statistics. We denote the set of these by Δ_s.

Definition. $\varrho \in \Delta_s$ means that

a) ϱ is an irreducible localized morphism of $\mathfrak{A}$.

b) The statistics parameter $\lambda_\varrho \neq 0$.

c) There exists a strongly continuous representation $\mathscr{U}_\varrho$ of $\mathscr{P}$, the (covering group of the) Poincaré group by unitary operators acting in $\mathscr{H}_0$ such that

$$\mathscr{U}_\varrho(L)\,\varrho(A)\,\mathscr{U}_\varrho(L)^{-1} = \varrho\alpha_L(A); \qquad A \in \mathfrak{A}, \quad L \in \mathscr{P}. \tag{1.1}$$

Here α_L denotes the automorphism of $\mathfrak{A}$ which corresponds to the Poincaré transformation L (Poincaré invariance of the theory).

Note that the set of charge quantum numbers considered here is $\Delta_s/\mathscr{I}$ where $\mathscr{I}$ denotes the set of inner localized automorphisms of $\mathfrak{A}$. To help us describe the product composition of sectors we need to work with certain reducible morphisms as well. For this reason we introduce the set Δ_r of localized morphisms defined to be the smallest set containing Δ_s and closed under taking products and subrepresentations. It turns out, see Section II, that Δ_r consists of covariant morphisms with finite statistics.

II. Covariant Representations

We show that the product of covariant morphisms is covariant (Lemma 2.1). Further if ϱ is covariant with finite statistics then the

[6] This assumption is only used implicitly in as much as some of the results of [1] depend on Property B formulated in the introduction of [1]. This property has been derived by Borchers under the above assumptions.

Poincaré operators $\mathscr{U}_\varrho(L)$ are uniquely determined by ϱ and belong to the weak closure of the representation ϱ of the observable algebra (Lemma 2.2).

Let us denote by ϱ_L the morphism which results from "shifting" ϱ by the Poincaré transformation L:

$$\varrho_L = \alpha_L \varrho \alpha_L^{-1}. \tag{2.1}$$

If ϱ is covariant then it is clear that ϱ_L must be equivalent to ϱ. In fact we can easily write down a unitary element from $\mathfrak{A}$ implementing this equivalence. Let $\mathscr{U}_\varrho$ be as in Eq. (1.1) and, similarly, let $\mathscr{U}_0$ be the representation of $\mathscr{P}$ in the vacuum representation of $\mathfrak{A}$. We have

$$\mathscr{U}_0(L)\, A\, \mathscr{U}_0(L)^{-1} = \alpha_L(A) \tag{2.2}$$

because of our convention identifying $\mathfrak{A}$ with its vacuum representation. Now we get from (2.1), (1.1) and (2.2)

$$\varrho_L(A) = \alpha_L(\mathscr{U}_\varrho(L)^{-1}\, \varrho(A)\, \mathscr{U}_\varrho(L)) = \mathscr{U}_0(L)\, \mathscr{U}_\varrho(L)^{-1}\, \varrho(A)\, \mathscr{U}_\varrho(L)\, \mathscr{U}_0(L)^{-1}$$

or, introducing

$$X_L(\varrho) = \mathscr{U}_0(L)\, \mathscr{U}_\varrho(L)^{-1}, \tag{2.3}$$

we have

$$\varrho_L = \sigma_{X_L(\varrho)}\, \varrho. \tag{2.4}$$

If ϱ is localized in $\mathcal{O}$ then ϱ_L is localized in $L\mathcal{O}$ and $X_L(\varrho)$ may be regarded as a charge transfer operator from $\mathcal{O}$ to $L\mathcal{O}$. We see from (2.3) that it commutes with $\mathfrak{A}(\mathcal{O}')\cap\mathfrak{A}(L\mathcal{O}')$ and hence belongs to $\mathfrak{A}$; it is a special case of the objects considered in Section III of [1].

From (2.3) and (2.2) we get the "cocycle identity"

$$X_{L_2 L_1}(\varrho) = \alpha_{L_2}(X_{L_1}(\varrho))\, X_{L_2}(\varrho). \tag{2.5}$$

We may sum up the above discussion by saying: given a covariant ϱ we have a family of intertwiners [1; Section IV]

$$\boldsymbol{X}_L(\varrho) = (\varrho_L | X_L(\varrho) | \varrho) \tag{2.6}$$

such that the operators $X_L(\varrho)$ depend in a strongly continuous fashion on L and satisfy the identity (2.5). The converse statement is also true as one easily verifies: if ϱ is a localized morphism and if we can find a continuous family of intertwiners (2.6) such that the operators $X_L(\varrho)$ satisfy the identity (2.5) then ϱ is covariant and $\mathscr{U}_\varrho(L)$ can be obtained from (2.3)[7]. We use this remark to prove

2.1. **Lemma.** *If ϱ_1 and ϱ_2 are covariant localized morphisms so is $\varrho_2\varrho_1$.*

[7] Actually ϱ itself may be recovered from the cocycle $X_L(\varrho)$ [1; Lemma 3.1].

Proof. As shown in Section IV of [1] we get an intertwiner from $\varrho_2\varrho_1$ to $(\varrho_2\varrho_1)_L$ by taking the cross product:

$$X_L(\varrho_2) \times X_L(\varrho_1) = (\varrho_{2L}\varrho_{1L} | X_L(\varrho_2)\, \varrho_2(X_L(\varrho_1)) | \varrho_2\varrho_1)\,. \tag{2.7}$$

We have to show that the family

$$X_L(\varrho_2\varrho_1) = X_L(\varrho_2)\, \varrho_2(X_L(\varrho_1)) \tag{2.8}$$

is strongly continuous in L and that it satisfies the identity (2.5). Since the representation ϱ_2 is locally normal each of the two factors on the right hand side of (2.8) is strongly continuous and since each factor has unit norm their product is strongly continuous. The validity of (2.5) for the family $X_L(\varrho_2\varrho_1)$ defined by (2.8), given the validity of (2.5) for the $X_L(\varrho_i)$, is checked by direct calculation. It is an example of Eq. (4.7) in [1]. This proves the covariance of $\varrho_2\varrho_1$.

Remark. Using (2.3) and (2.8) the representation $\mathscr{U}_{\varrho_2\varrho_1}$ corresponding to the cocycle $X_L(\varrho_2\varrho_1)$ is given by

$$\mathscr{U}_{\varrho_2\varrho_1}(L) = \varrho_2(X_L(\varrho_1))^{-1}\, \mathscr{U}_{\varrho_2}(L)\,. \tag{2.9}$$

Next we show

2.2. **Lemma.** *Let ϱ be covariant with finite statistics, then ϱ is equivalent to a finite direct sum of morphisms from Δ_s. Further*

a) *There is only one strongly continuous unitary representation $\mathscr{U}_\varrho$ of $\mathscr{P}$ satisfying* (1.1).

b) $\mathscr{U}_\varrho(L) \in \varrho(\mathfrak{A})''$, $L \in \mathscr{P}$.

c) *If ϱ_1, ϱ_2 are covariant with finite statistics and $\boldsymbol{R} = (\varrho_2 | R | \varrho_1)$ then R also intertwines from $\mathscr{U}_{\varrho_1}(L)$ to $\mathscr{U}_{\varrho_2}(L)$.*

Proof. By assumption ϱ has finite statistics and hence is a finite direct sum of irreducible representations with finite statistics [1; Section VI]. The crucial point is now the absence of non-trivial finite dimensional, continuous, unitary representations of $\mathscr{P}$. Let us consider part c) of the lemma first. If R intertwiners from ϱ_1 to ϱ_2 then so does R_L given by

$$R_L = \mathscr{U}_{\varrho_2}(L)\, R\, \mathscr{U}_{\varrho_1}(L)^* \tag{2.10}$$

for any $L \in \mathscr{P}$. The set of intertwiners from ϱ_1 to ϱ_2 is a linear space $\mathscr{T}$. For any pair R', R from $\mathscr{T}$ we have $R'^* R \in \varrho_1(\mathfrak{A})'$. Choosing a state ω over $\varrho_1(\mathfrak{A})'$ we can define a scalar product in $\mathscr{T}$ by

$$\langle R', R\rangle = \omega(R'^* R)\,. \tag{2.11}$$

Moreover we can choose ω to be faithful and invariant under the Poincaré transformations (2.10) in $\mathscr{T}$. To see this we note that the central decomposition of $\varrho_1(\mathfrak{A})$ decomposes $\varrho_1(\mathfrak{A})'$ into a finite direct sum of finite dimensional full matrix rings. The Poincaré transformations define a *-automorphism of each of these full matrix rings since the centre of $\varrho_1(\mathfrak{A})'$, being discrete, must be pointwise invariant for reasons of continuity. If we pick ω to be proportional to the trace state on each of these full matrix rings we obtain a faithful, invariant state. But then (2.10) gives us a unitary, continuous representation of $\mathscr{P}$ in the finite dimensional Hilbert space $\mathscr{T}$. This representation must be trivial and thus, since ω is faithful, we have

$$R_L = R$$

or

$$R\mathscr{U}_{\varrho_1}(L) = \mathscr{U}_{\varrho_2}(L)R \tag{2.12}$$

which is the property claimed in part c) of the lemma. Part b) is an immediate consequence of (2.12). We need only put $\varrho_2 = \varrho_1 = \varrho$ and note that any element of $\varrho(\mathfrak{A})'$ is an interwiner from ϱ to ϱ. This also shows that any subrepresentation of ϱ is covariant so ϱ is a finite direct sum of morphisms from Δ_s. a) also follows from (2.12) because $\boldsymbol{I}_\varrho = (\varrho | I | \varrho)$ must intertwine two representations of $\mathscr{P}$ satisfying (1.1).

It follows from Lemma 2.2 that the set of covariant localized morphisms with finite statistics is closed under subrepresentations. However Lemma 2.1 and [1; Corollary 6.8] show that it is also closed under products.

In the introduction we defined Δ_r to be the smallest set of localized morphisms containing Δ_s and closed under taking products and subrepresentations. Thus Δ_r consists of covariant morphisms with finite statistics. If $\varrho \in \Delta_r$ then ϱ has pure para-Bose or pure para-Fermi statistics and not a mixture of the two [1; Eq. (6.16), (6.19)]. Thus a standard left inverse ϕ of ϱ satisfies $\phi(\varepsilon_\varrho) = \lambda_\varrho \cdot I$ [1; Proposition 6.5] and we may refer to λ_ϱ as the statistics parameter of $\varrho \in \Delta_r$. This feature makes it technically simpler to work with Δ_r rather than with all covariant morphisms with finite statistics.

We close this section with a remark on asymptotic behaviour under large translations.

2.3. **Lemma.** *Let $\varrho \in \Delta_s$, $A \in \mathfrak{A}$ and x a translation. Then as x tends spacelike to infinity*

$$\mathscr{U}_\varrho(x)^{-1} A \mathscr{U}_\varrho(x) \equiv \beta_{-x}(A) \xrightarrow[\text{weakly}]{} \omega_0 \circ \phi(A)\, I\,.$$

Proof. For any $A \in \mathfrak{A}$ the commutator of $\beta_{-x}(A)$ with a fixed element $\varrho(B)$ will tend to zero in norm as x tends spacelike to infinity:

$$\| [\beta_{-x}(A), \varrho(B)] \| = \| [A, \beta_x \varrho(B)] \| = \| [A, \varrho\alpha_x(B)] \| \to 0\,.$$

Since $\varrho(\mathfrak{A})$ is weakly dense in $\mathscr{B}(\mathscr{H}_0)$ we have by a well known generalization of Schur's Lemma

$$\beta_{-x}(A) - \omega_0(\beta_{-x}(A))\, I \xrightarrow[\text{weakly}]{} 0 .$$

Since the vacuum is invariant under $\mathscr{U}_0$ we get by (2.3)

$$\omega_0(\beta_{-x}(A)) = \omega_0(X_x(\varrho)\, A\, X_x(\varrho)^{-1}) \to \omega_0 \circ \phi(A)$$

where we have used [1; Theorem 3.9] in the last step.

III. The Conjugate Charge

We complete here the discussion of the conjugation symmetry of sectors begun in [1; § III] by showing that the conjugate representation π_ϕ constructed there is equivalent to a morphism $\bar{\varrho} \in \Delta_s$ and that the statistics parameters of conjugate sectors are equal.

To understand how the morphism $\bar{\varrho}$ is constructed, it is worth recalling the analogy between Δ_s and the irreducible continuous unitary representations of a compact group [8]. In this analogy the conjugation of sectors corresponds to passing from a unitary representation of $\mathscr{G}$ to its complex conjugate representation. However for a unitary matrix group of dimension d, the conjugate representation may be constructed as follows: let ϱ' be the representation on the d-dimensional space of totally antisymmetric tensors of rank $(d-1)$, then $g \to \det(g)^{-1} \varrho'(g)$ is the conjugate representation. However $g \to \det(g)$ is itself nothing more than the representation γ on the 1-dimensional space of totally antisymmetric tensors of rank d. Since both ϱ' and γ are constructed using antisymmetrized tensor powers of the original representation, it is clear how the analogous morphisms ϱ' and γ may be constructed for $\varrho \in \Delta_s$.

Let E^n_s and E^n_a denote the totally symmetric and totally antisymmetric projections in the group algebra of $\mathbb{P}^{(n)}$. Given $\varrho \in \Delta_r$, one may compute the statistical dimensions of the corresponding subrepresentations of ϱ^n using [1; Lemma 5.3, Proposition 6.6 and Corollary 6.8] and one finds just the result predicted by the analogy except that symmetrization and antisymmetrization are reversed if ϱ has para-Fermi statistics. Let

$$E = \varepsilon_\varrho^{(d)}(E^d_a); \quad E' = \varepsilon_\varrho^{(d-1)}(E^{d-1}_a) \quad \text{if} \quad \lambda_\varrho = \frac{1}{d} > 0 ,$$

$$E = \varepsilon_\varrho^{(d)}(E^d_s); \quad E' = \varepsilon_\varrho^{(d-1)}(E^{d-1}_s) \quad \text{if} \quad \lambda_\varrho = -\frac{1}{d} < 0 .$$

[8] This analogy can be exhibited explicitly in the case of models where $\mathscr{G}$ plays the role of a gauge group [4].

There are morphisms γ and ϱ' equivalent to the subrepresentations of ϱ^d and ϱ^{d-1} projected out by E and E' respectively. Let $\boldsymbol{W}=(\varrho^d|W|\gamma)$ and $\boldsymbol{W}'=(\varrho^{d-1}|W'|\varrho')$ be isometric intertwiners; $WW^*=E$ and $W'W'^*=E'$. The crucial point is that since γ has statistical dimension 1, $d(\gamma)=1$, γ must be an automorphism[9]. Hence we set

$$\bar{\varrho}=\varrho'\gamma^{-1}; \tag{3.1}$$

clearly $d(\bar{\varrho})=d(\varrho)$ and keeping track of the sign of the statistics parameters using [1; Eq. (6.16), (6.19)], we see that $\lambda_\varrho=\lambda_{\bar{\varrho}}$. Now

$$\begin{aligned}\varrho(W'^*)\,WA&=\varrho(W'^*)\,W\gamma\gamma^{-1}(A)\\&=\varrho(W'^*)\,\varrho^d\gamma^{-1}(A)\,W=\varrho\bar{\varrho}(A)\,\varrho(W'^*)\,W.\end{aligned}$$

Now $\varrho(E')\geqq E$ since $\varrho(E')$ is the image under $\varepsilon_\varrho^{(d)}$ of the (anti)symmetrizer of the objects 2, 3 … d on which $\mathbb{P}^{(d)}$ operates [1, Theorem 4.2d], and the latter projection includes the total (anti)symmetrizer. Hence $\varrho(E')\,E=E$, $\varrho(W'^*)\,W$ is an isometry so $\varrho\bar{\varrho}$ contains the vacuum sector as a component which is a necessary condition for $\bar{\varrho}$ to play the role of a conjugate. Note that the covariance of ϱ plays no part in this construction, but if ϱ is covariant then by Lemmas 2.1 and 2.2 so are ϱ' and γ. Also by [5; Theorem 3.1], γ^{-1} is covariant so $\bar{\varrho}\in\Delta_r$.

The next step is to show that if ϱ is irreducible then $\pi_\phi\cong\bar{\varrho}$ where ϕ is the unique left inverse of ϱ.

3.1. **Theorem.** *Let $\xi\in\Delta_s/\mathcal{I}$. There exists a conjugate charge $\bar{\xi}\in\Delta_s/\mathcal{I}$ uniquely determined by the property that the product $\bar{\xi}\xi$ contains the vacuum sector as a component. Moreover*

a) *$\bar{\xi}\xi$ contains the vacuum sector precisely once.*

b) *The statistics parameters λ_ξ and $\lambda_{\bar{\xi}}$ are equal.*

Proof. Pick $\varrho\in\Delta_s$ from the class ξ and define $\bar{\varrho}$ by (3.1).

We have an intertwiner $\bar{\boldsymbol{R}}=(\varrho\bar{\varrho}\,|\bar{R}|\,\iota)$ defined by

$$\bar{R}=d(\varrho)^{1/2}\,\varrho(W'^*)\,W \tag{3.2}$$

and the defining property of a left inverse ϕ shows that if we set

$$R=d(\varrho)\,\phi(\bar{R}) \tag{3.3}$$

then we get an intertwiner $\boldsymbol{R}=(\bar{\varrho}\varrho\,|R|\,\iota)$. Also

$$\begin{aligned}R^*\bar{\varrho}(\bar{R})&=d(\varrho)\,\phi(\bar{R}^*)\,\bar{\varrho}(\bar{R})=d(\varrho)\,\phi(\bar{R}^*\varrho\bar{\varrho}(\bar{R}))\\&=d(\varrho)\,\phi(\bar{R}\bar{R}^*)=d(\varrho)^2\,W'^*\,\phi(E)\,W'.\end{aligned}$$

[9] By an oversight, this general result was not stated explicitly in [1]. The argument runs as follows: suppose $d(\varrho)=1$ then ϱ has pure Bose or pure Fermi statistics so $\lambda_\varrho=\pm 1$. Hence $\phi(I\mp\varepsilon_\varrho)=0$ where ϕ is a standard left inverse of ϱ. Since $I\mp\varepsilon_\varrho$ is positive and ϕ is faithful [1; Lemma 6.4], $\varepsilon_\varrho=\pm I$ and the result follows from [1; Proposition 2.7].

But by [1; Lemma 5.1], $d(\varrho)^2 \phi(E) = E'$. Hence $R^* \bar{\varrho}(\bar{R}) = W'^* E' W' = I$. We show that $\bar{\varrho} \cong \pi_\phi$ by showing that $R\Omega$ is a cyclic vector for $\bar{\varrho}$ inducing the state $\omega_0 \circ \phi$. Let $X \in \bar{\varrho}(\mathfrak{A})'$ then $X = X\bar{\varrho}(\bar{R})^* R = \bar{\varrho}(\bar{R})^* X R$. So $XR = 0$ implies $X = 0$ and hence $R\Omega$ is cyclic for $\bar{\varrho}$. Now $A \to R^* \bar{\varrho}(A) R$ satisfies [1; Eq. (3.5), (3.6')] of the definition of a left inverse and $R^* R$ is a multiple of the identity so $R^* \bar{\varrho}(A) R = R^* R\, \phi(A)$ where ϕ is the unique left inverse of ϱ [1; Theorem 3.9]. Hence $\bar{\varrho} \cong \pi_\phi$ showing in particular that $\bar{\varrho}$ is irreducible. Conversely if $\bar{\xi} \in \Delta_s/\mathscr{I}$ is such that $\bar{\xi}\xi$ contains the vacuum sector, then pick $\bar{\varrho} \in \Delta_s$ from the class $\bar{\xi}$ and $\boldsymbol{R} = (\bar{\varrho}\varrho | R | \iota) \neq 0$. Now $R\Omega \neq 0$ and is a cyclic vector for $\bar{\varrho}$ because $\bar{\varrho}$ is irreducible. Arguing as above $\bar{\varrho} \cong \pi_\phi$ so $\bar{\xi}$ is unique. It only remains to prove a): let $\boldsymbol{S} = (\bar{\varrho}\varrho | S | \iota)$ then $(\bar{\boldsymbol{R}}^* \times \boldsymbol{I}_\varrho) \circ (\boldsymbol{I}_\varrho \times \boldsymbol{S})$ intertwines ϱ and ϱ so $\bar{R}^* \varrho(S) = \mu \cdot I$ for some complex number μ. Hence $S = \bar{\varrho}(\bar{R}^*)\, \bar{\varrho}\varrho(S)\, R = \mu R$ completing the proof.

For technical reasons it is convenient to define conjugates for reducible morphisms and the remainder of this section, which may be omitted on first reading, is devoted to developing the necessary formalism. Rather than considering arbitrary localized morphisms with finite statistics we restrict ourselves to $\varrho \in \Delta_r$.

3.2. **Lemma.** *Let $\varrho \in \Delta_r$, ϕ a standard left inverse of ϱ and $\boldsymbol{S} = (\varrho\bar{\varrho} | S | \iota)$ then*

$$\phi(S) = \lambda_\varrho \varepsilon(\varrho, \bar{\varrho})\, S\,, \tag{3.4}$$

$$\phi(S)^* \bar{\varrho}(S) = \phi(SS^*) = \lambda_\varrho S^* \varrho(\varepsilon_{\bar{\varrho}})\, S\,. \tag{3.5}$$

Proof. Using [1; Theorems 4.2 and 4.3] we have

$$S = \varepsilon(\varrho, \varrho\bar{\varrho})\, \varrho(S) = \varrho(\varepsilon(\varrho, \bar{\varrho}))\, \varepsilon_\varrho\, \varrho(S)\,,$$
$$S = \varepsilon(\bar{\varrho}, \varrho\bar{\varrho})\, \bar{\varrho}(S) = \varrho(\varepsilon_{\bar{\varrho}})\, \varepsilon(\bar{\varrho}, \varrho)\, \bar{\varrho}(S)\,.$$

Applying ϕ to the first equation gives (3.4). The second equation and (3.4) give $\phi(S)^* \bar{\varrho}(S) = \lambda_\varrho S^* \varrho(\varepsilon_{\bar{\varrho}})\, S$. But $\phi(SS^*) = \phi(S^* \varrho\bar{\varrho}(S)) = \phi(S)^* \bar{\varrho}(S)$ completing the proof.

3.3. **Theorem.** *Let $\varrho \in \Delta_r$ then there exists $\bar{\varrho} \in \Delta_r$ and $\boldsymbol{R} = (\bar{\varrho}\varrho | R | \iota)$, $\bar{\boldsymbol{R}} = (\varrho\bar{\varrho} | \bar{R} | \iota)$ satisfying*

$$\bar{R}^* \varrho(R) = I\,; \qquad R^* \bar{\varrho}(\bar{R}) = I\,, \tag{3.6}$$

$$\bar{\boldsymbol{R}} = \operatorname{sign}(\lambda_\varrho)\, \varepsilon(\bar{\varrho}, \varrho) \circ \boldsymbol{R}\,, \tag{3.7}$$

$$R^* R = \bar{R}^* \bar{R} = d(\varrho) \cdot I\,. \tag{3.8}$$

Further ϱ and $\bar{\varrho}$ have the same statistics. If $\boldsymbol{R}' = (\bar{\varrho}'\varrho | R' | \iota)$ and $\bar{\boldsymbol{R}}' = (\varrho\bar{\varrho}' | \bar{R}' | \iota)$ satisfy (3.6) *and* (3.7) *with $\bar{\varrho}'$ replacing $\bar{\varrho}$ then there is a* *unique intertwiner* $\boldsymbol{U} = (\bar{\varrho}' \mid U \mid \bar{\varrho})$ *such that*

$$\boldsymbol{R}' = (\boldsymbol{U} \times \boldsymbol{I}_{\varrho}) \circ \boldsymbol{R}\,, \tag{3.9}$$

$$\bar{\boldsymbol{R}}' = (\boldsymbol{I}_{\varrho} \times \boldsymbol{U}) \circ \bar{\boldsymbol{R}}\,. \tag{3.10}$$

Proof. We define $\bar{\varrho}$ by (3.1), $\bar{R}$ by (3.2) and R by (3.3) where ϕ is to be a standard left inverse of ϱ. As in the proof of Theorem 3.1 we have the intertwining property of R and $\bar{R}$ and $R^*\bar{\varrho}(\bar{R}) = I$. Now (3.3) and (3.4) give (3.7) which in turn shows $R^*R = \bar{R}^*\bar{R}$. However we showed that $d(\varrho)^{-1/2}R = \varrho(W')^*W$ is an isometry so we have proved (3.8). Now setting

$$\bar{\phi}(A) = d(\varrho)^{-1}\bar{R}^*\varrho(A)\bar{R}, \qquad A \in \mathfrak{A} \tag{3.11}$$

defines a left inverse of $\bar{\varrho}$. But $I = R^*\bar{\varrho}(\bar{R}) = d(\varrho)\,\phi(\bar{R})^*\bar{\varrho}(\bar{R})$ so by (3.5), $\bar{\phi}(\varepsilon_{\bar{\varrho}}) = \lambda_{\varrho} \cdot I$ and $\bar{\phi}$ is standard. Using Lemma 3.2 with $\bar{\varrho}$ in place of ϱ we deduce $\bar{R} = d(\varrho)\,\bar{\phi}(R)$ from (3.7) and $\bar{R}^*\varrho(R) = d(\varrho)\,\bar{\phi}(RR^*)$ $= \bar{R}^*\varrho(RR^*)\bar{R}$. Hence $\bar{R}^*\varrho(R)$ is a projection. But $\phi(\bar{R}^*\varrho(R)) = \phi(\bar{R}^*)R = I$ by (3.3) and (3.8). But ϕ is faithful by [1; Lemma 6.4] so $\bar{R}^*\varrho(R) = I$ proving (3.6).

It only remains to prove the uniqueness result of the Theorem. Given an intertwiner $\boldsymbol{R}' = (\bar{\varrho}'\varrho \mid R' \mid \iota)$ set

$$\boldsymbol{U} = (\bar{\varrho}' \mid U \mid \bar{\varrho}) = (\boldsymbol{I}_{\bar{\varrho}'} \times \bar{\boldsymbol{R}}^*) \circ (\boldsymbol{R}' \times I_{\bar{\varrho}}) \tag{3.12}$$

then $UR = \bar{\varrho}'(\bar{R})^*R'R = \bar{\varrho}'(\bar{R}^*\varrho(R))R' = R'$ giving (3.9). Conversely (3.9) for an intertwiner $\boldsymbol{U} = (\bar{\varrho}' \mid U \mid \bar{\varrho})$ implies (3.12) since $\bar{\varrho}'(\bar{R})^*R' = \bar{\varrho}'(\bar{R})^*UR$ $= U\bar{\varrho}(\bar{R})^*R = U$. Similarly (3.10) is equivalent to

$$\boldsymbol{U} = (\boldsymbol{R}^* \times \boldsymbol{I}_{\bar{\varrho}'}) \circ (\boldsymbol{I}_{\bar{\varrho}} \times \bar{\boldsymbol{R}}')\,. \tag{3.13}$$

Now suppose $\boldsymbol{R}'$ and $\bar{\boldsymbol{R}}'$ satisfy (3.6) and (3.7) with $\bar{\varrho}'$ replacing $\bar{\varrho}$. Use (3.12) to define $\boldsymbol{U}$, then by (3.7) and [1; Theorem 4.3]

$$\begin{aligned}\bar{\boldsymbol{R}}' &= \operatorname{sign}(\lambda_{\varrho})\,\boldsymbol{\varepsilon}(\bar{\varrho}', \varrho) \circ (\boldsymbol{U} \times \boldsymbol{I}_{\varrho}) \circ \boldsymbol{R} \\ &= \operatorname{sign}(\lambda_{\varrho})\,(\boldsymbol{I}_{\varrho} \times \boldsymbol{U}) \circ \boldsymbol{\varepsilon}(\bar{\varrho}, \varrho) \circ \boldsymbol{R} = (\boldsymbol{I}_{\varrho} \times \boldsymbol{U}) \circ \bar{\boldsymbol{R}}\end{aligned}$$

proving (3.10). Now by (3.13)

$$UU^* = \bar{R}^*\bar{\varrho}(\bar{R}')\,U^* = \bar{R}^*U^*\bar{\varrho}'(\bar{R}')$$

and substituting for U from (3.12) we get using (3.7)

$$\begin{aligned}UU^* &= \bar{R}^*R'^*\bar{\varrho}'(\bar{R}\bar{R}') = R'^*\bar{\varrho}'(\varrho(\bar{R}^*)\,\bar{R}\bar{R}') \\ &= R'^*\bar{\varrho}'(\bar{R}') = I\,.\end{aligned}$$

The symmetry between $\bar{\varrho}$ and $\bar{\varrho}'$ shows that $U^*U = I$ completing the proof.

We call two localized morphisms ϱ and $\bar{\varrho}$ *conjugate* if we can find $\boldsymbol{R} = (\bar{\varrho}\varrho|R|\imath)$ satisfying (3.6) where $\bar{\boldsymbol{R}}$ is defined by (3.7)[10]. In closing we note that setting

$$\phi(A) = d(\varrho)^{-1} R^* \bar{\varrho}(A) R, \qquad A \in \mathfrak{A} \tag{3.14}$$

defines a standard left inverse of ϱ, compare (3.11), which by (3.9) with $\bar{\varrho}' = \bar{\varrho}$ is actually independent of the choice of $\boldsymbol{R}$[11].

IV. The Field Bundle

In the subsequent sections we are often faced with the task of extending certain standard results of local quantum field theory (particle-antiparticle symmetry, spin and statistics, construction and metric of collision states) to cover the case where we have superselection rules and are not given a priori unobservable fields with specified commutation relations for spacelike separations. If we introduce a concept called the "field bundle" which is a simple and intrinsic construct in our setting and has many structural features in common with a field algebra, we can work in close analogy with the standard procedure. Indeed once this analogy between a field bundle and a field algebra is understood many of the proofs become routine.

By using morphisms we have described many inequivalent representations in the same Hilbert space $\mathscr{H}_0$. Thus, picking a vector in $\mathscr{H}_0$ determines a state on $\mathfrak{A}$ only if we also specify the representation in which it is to be understood. Let us therefore consider as a "generalized state vector" a pair $\{\varrho, \Psi\}$ of which the first member is a morphism from Δ_r and the second member a vector in $\mathscr{H}_0$. Correspondingly we shall consider a set $\mathscr{B}$ of operators acting on this vector bundle. An element of $\mathscr{B}$ is a pair $\boldsymbol{B} = \{\varrho; B\}$ where $\varrho \in \Delta_r$ and $B \in \mathfrak{A}$. It acts on $\boldsymbol{\Psi} = \{\varrho'; \Psi\}$ by

$$\boldsymbol{B}\boldsymbol{\Psi} = \{\varrho'\varrho; \varrho'(B)\Psi\}. \tag{4.1}$$

This leads to the (associative) multiplication law within $\mathscr{B}$

$$\boldsymbol{B}_2\boldsymbol{B}_1 = \{\varrho_1\varrho_2; \varrho_1(B_2)B_1\}; \quad \boldsymbol{B}_i = \{\varrho_i; B_i\}, \quad i = 1, 2. \tag{4.2}$$

---

[10] This defines a conjugation $\xi \to \bar{\xi}$ in $\Delta_r/\mathscr{I}$ which is just the extension commuting with direct sums of the conjugation defined in $\Delta_s/\mathscr{I}$ in Theorem 3.1.

For those familiar with category theory, we remark that if $\mathfrak{A}$ is treated as a category with a single object whose morphisms are the elements of $\mathfrak{A}$ then ϱ and $\bar{\varrho}$ become endofunctors and (3.6) tells us that $\bar{\varrho}$ is a right and left adjoint for ϱ. This implies that the symmetric monoidal category of localized morphisms with finite statistics [1; Footnote 15] is actually a closed category [6].

[11] It can even be shown that a localized morphism with finite statistics has a unique standard left inverse.

We shall naturally define the action of the Poincaré group on the state vector bundle by writing

$$\mathcal{U}(L)\{\varrho;\Psi\}=\{\varrho;\mathcal{U}_\varrho(L)\,\Psi\}\,. \tag{4.3}$$

The corresponding action on $\mathcal{B}$ will be denoted by the same symbol α_L as the action $\mathfrak{A}$ (since the former may be considered to be an extension of the latter). We have

$$\alpha_L\{\varrho;B\}=\{\varrho;X_L(\varrho)^{-1}\alpha_L(B)\}=\{\varrho;\mathcal{U}_\varrho(L)\,B\,\mathcal{U}_0(L)^{-1}\}\,. \tag{4.4}$$

$\mathcal{B}$ has a local structure $\mathcal{O}\to\mathcal{B}(\mathcal{O})$ and we can define a conjugation $B\to B^\dagger$ in $\mathcal{B}$ corresponding to the Hermitean conjugation in a field algebra. We have to realize, however, that from a physical point of view there is some redundancy in the state vector bundle and in $\mathcal{B}$. Thus the generalized state vectors $\{\varrho;\Psi\}$ and $\{\sigma_U\varrho;U\Psi\}$ have the same physical meaning if U is a unitary from $\bigcup_{\mathcal{O}}\mathfrak{A}(\mathcal{O})$ and, similarly, $\{\varrho;B\}$ and $\{\sigma_U\varrho;UB\}$ are to be identified in physics[12].

The redundancy can easily be handled by considering the action of intertwiners on the state vector and field bundles. Given $\boldsymbol{T}=(\varrho'\,|\,T\,|\,\varrho)$ it is natural to define its action on elements in the fibre over ϱ in the two bundles by

$$\boldsymbol{T}\{\varrho;\Psi\}=\{\varrho';T\Psi\}\,, \tag{4.5}$$

$$\boldsymbol{T}\circ\{\varrho;B\}=\{\varrho';TB\}\,. \tag{4.6}$$

If T is unitary, applying $\boldsymbol{T}$ to a $\boldsymbol{\Psi}$ or a $\boldsymbol{B}$ does not affect the physical meaning.

We can now exhibit the local structure of $\mathcal{B}$.

4.1. *Definition.* $\boldsymbol{B}=\{\varrho;B\}\in\mathcal{B}(\mathcal{O})$ if there exists a unitary $\boldsymbol{U}=(\varrho'\,|\,U\,|\,\varrho)$ such that ϱ' has support in $\mathcal{O}$ and $UB\in\mathfrak{A}(\mathcal{O})$.

This takes care of the redundancy and one checks that

$$\boldsymbol{B}\in\mathcal{B}(\mathcal{O})\quad\text{implies}\quad\alpha_L(\boldsymbol{B})\in\mathcal{B}(L\mathcal{O})\,. \tag{4.7}$$

The following formulae are useful in the subsequent computations. They hold whenever the left hand side is defined.

$$(\boldsymbol{T}_1\circ\boldsymbol{T}_2)\circ\boldsymbol{B}=\boldsymbol{T}_1\circ(\boldsymbol{T}_2\circ\boldsymbol{B})\,, \tag{4.8}$$

$$(\boldsymbol{T}_2\circ\boldsymbol{B}_2)(\boldsymbol{T}_1\circ\boldsymbol{B}_1)=(\boldsymbol{T}_1\times\boldsymbol{T}_2)\circ(\boldsymbol{B}_2\boldsymbol{B}_1)\,, \tag{4.9}$$

$$\boldsymbol{B}\boldsymbol{T}\boldsymbol{\Psi}=(\boldsymbol{T}\times\boldsymbol{I}_\varrho)\,\boldsymbol{B}\boldsymbol{\Psi}\,,\quad\text{where } \boldsymbol{B} \text{ lies in the fibre over } \varrho\,; \tag{4.10}$$

$$\boldsymbol{T}\circ\boldsymbol{B}\boldsymbol{\Psi}=(\boldsymbol{I}_\varrho\times\boldsymbol{T})\,\boldsymbol{B}\boldsymbol{\Psi}\,,\quad\text{where } \boldsymbol{\Psi} \text{ lies in the fibre over } \varrho\,. \tag{4.11}$$

[12] There is a cohomological obstacle to removing this redundancy and this is related to the problem of passing from a field bundle to a field algebra – although in a field algebra this redundancy has been removed only to be replaced by another one involving the action of the gauge group.

In (4.10), (4.11) the intertwiners on the right hand side act on the vector $\boldsymbol{B}\boldsymbol{\Psi}$; this has to be distinguished from the composition of an intertwiner (by the symbol $\circ$) with $\boldsymbol{B}$ and the subsequent action on the vector.

The local commutation relations in $\mathscr{B}$ are given by

4.2. **Proposition.** *Let $\boldsymbol{B}_i = \{\varrho_i; B_i\} \in \mathscr{B}(\mathcal{O}_i)$, $i=1,2$ and $\mathcal{O}_1$ be spacelike to $\mathcal{O}_2$ then*

$$\boldsymbol{B}_1 \boldsymbol{B}_2 = \varepsilon(\varrho_1, \varrho_2) \circ (\boldsymbol{B}_2 \boldsymbol{B}_1). \tag{4.12}$$

Proof. By Definition 4.1, there exist $\boldsymbol{B}'_i = \{\varrho'_i; B'_i\} = \boldsymbol{U}_i \circ \boldsymbol{B}_i$ such that ϱ'_i and B'_i have supports in $\mathcal{O}_i$ and $\boldsymbol{U}_i = (\varrho'_i | U_i | \varrho_i)$ are unitary intertwiners. By (4.9)

$$\boldsymbol{B}_i \boldsymbol{B}_j = (\boldsymbol{U}_i^* \circ \boldsymbol{B}'_i)(\boldsymbol{U}_j^* \circ \boldsymbol{B}'_j) = (\boldsymbol{U}_j^* \times \boldsymbol{U}_i^*) \circ \boldsymbol{B}'_i \boldsymbol{B}'_j$$

Now under the given support assumptions, $\boldsymbol{B}'_1 \boldsymbol{B}'_2 = \boldsymbol{B}'_2 \boldsymbol{B}'_1$ by (4.2) and $(\boldsymbol{U}_2^* \times \boldsymbol{U}_1^*) \circ (\boldsymbol{U}_1 \times \boldsymbol{U}_2) = \varepsilon(\varrho_1, \varrho_2)$ by [1; Theorem 4.2]. Applying (4.8) then gives (4.12) as required.

The conjugation in $\mathscr{B}$ is an antilinear mapping from the fibre over ϱ to the fibre over $\bar{\varrho}$. For each $\varrho \in \Delta_r$ we suppose $\bar{\varrho}$ and $\boldsymbol{R}$ chosen as in Theorem 3.3 and define for $\boldsymbol{B} = \{\varrho, B\}$

$$\boldsymbol{B}^\dagger = \{\bar{\varrho}; \bar{\varrho}(B)^* R\}. \tag{4.13}$$

Thus the conjugation depends on the choice of $\bar{\varrho}$ and $\boldsymbol{R}$ but not in an essential way because we can pass from one definition to another by the unitary intertwiner U of Theorem 3.3. We adopt the convention that $\bar{\bar{\varrho}} = \varrho$ and use $\bar{R}$ as the intertwiner associated with $\bar{\varrho}$. With this convention[13]

$$\boldsymbol{B}^{\dagger\dagger} = \boldsymbol{B}. \tag{4.14}$$

The conjugation preserves locality in the following sense.

4.3. **Lemma.** *If $\boldsymbol{B} \in \mathscr{B}(\mathcal{O})$ then $\boldsymbol{B}^\dagger \in \mathscr{B}(\mathcal{O}_1)$ where $\mathcal{O}_1$ is any double cone containing $\mathcal{O}$ in its interior.*

Proof. First we see that if $\boldsymbol{U} = (\varrho' | U | \varrho)$ is a unitary intertwiner and $\boldsymbol{B} = \{\varrho; B\}$ then

$$(\boldsymbol{U} \circ \boldsymbol{B})^\dagger = \boldsymbol{U}^\dagger \circ \boldsymbol{B}^\dagger \tag{4.15}$$

where $\boldsymbol{U}^\dagger = (\bar{\varrho}' | U^\dagger | \bar{\varrho})$ is a unitary intertwiner between the conjugate morphisms, because $\boldsymbol{B} \to (\boldsymbol{U} \circ \boldsymbol{B})^\dagger$ would also be a possible way of defining conjugation on the fibre over ϱ. This shows that the localization of $\boldsymbol{B}^\dagger$ depends only on the physical significance of $\boldsymbol{B}$. We can therefore assume without loss of generality that ϱ and B have their supports in $\mathcal{O}$. If we compare the construction of $\bar{\varrho}$ given in Section III with the proof of Lemma 2.5 of [1] we see that we may take $\bar{\varrho}$ to have support in any $\mathcal{O}_1$ containing $\mathcal{O}$ in its interior. This implies that $\boldsymbol{B}^\dagger \in \mathscr{B}(\mathcal{O}_1)$ since both $\bar{\varrho}(B^*)$ and R then have their supports in $\mathcal{O}_1$.

[13] If ϱ and $\bar{\varrho}$ are equivalent then this convention only allows us to choose $\bar{\varrho} = \varrho$ in a real sector. This is related to Theorem 6.5.

The most important use of the conjugate will be in connection with scalar products. A scalar product is obviously defined within each fibre of the state vector bundle: if $\boldsymbol{\Psi}_1 = \{\varrho; \Psi_1\}$; $\boldsymbol{\Psi}_2 = \{\varrho; \Psi_2\}$ we put

$$(\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2) = (\Psi_1, \Psi_2). \tag{4.16}$$

We then have

4.4. **Proposition.** *Let* $\boldsymbol{B} = \{\varrho; B\}$, $\boldsymbol{\Phi} = \{\varrho'; \Phi\}$ *and* $\boldsymbol{\Psi} = \{\varrho'\varrho; \Psi\}$ *then*

$$(\boldsymbol{B}\boldsymbol{\Phi}, \boldsymbol{\Psi}) = (\boldsymbol{I}_{\varrho'} \times \overline{\boldsymbol{R}}\boldsymbol{\Phi}, \boldsymbol{B}^\dagger \boldsymbol{\Psi}). \tag{4.17}$$

Proof. Using the definitions we get

$$(\boldsymbol{I}_{\varrho'} \times \overline{\boldsymbol{R}}\boldsymbol{\Phi}, \boldsymbol{B}^\dagger \boldsymbol{\Psi}) = (\varrho'(\overline{R})\,\Phi, \varrho'\varrho\bar{\varrho}(B^*)\,\varrho'\varrho(R)\,\Psi).$$

However by the intertwining property of $\overline{R}$ and (3.6) we get

$$(\boldsymbol{I}_{\varrho'} \times \overline{\boldsymbol{R}}\boldsymbol{\Phi}, \boldsymbol{B}^\dagger \boldsymbol{\Psi}) = (\Phi, \varrho'(B^*)\,\Psi) = (\boldsymbol{B}\boldsymbol{\Phi}, \boldsymbol{\Psi})$$

as required.

The action of the Poincaré group $\mathscr{P}$ on $\mathscr{B}$ preserves all the structure involved.

4.5. **Proposition.** *We have a representation* $L \to \alpha_L$ *of* $\mathscr{P}$ *by automorphisms of* $\mathscr{B}$: α_L *acts linearly on the fibres and*

$$\alpha_L(\alpha_{L'}(\boldsymbol{B})) = \alpha_{LL'}(\boldsymbol{B}), \tag{4.18}$$

$$\alpha_L(\boldsymbol{B}_2\boldsymbol{B}_1) = \alpha_L(\boldsymbol{B}_2)\,\alpha_L(\boldsymbol{B}_1), \tag{4.19}$$

$$\alpha_L(\boldsymbol{B}^\dagger) = \alpha_L(\boldsymbol{B})^\dagger, \tag{4.20}$$

for $\boldsymbol{B}, \boldsymbol{B}_1, \boldsymbol{B}_2 \in \mathscr{B}$ *and* $L, L' \in \mathscr{P}$.

Proof. Only (4.20) requires a little proof. If $\boldsymbol{B} = \{\varrho, B\}$ we have by (4.4) and (4.13)

$$\begin{aligned}\alpha_L(\boldsymbol{B})^\dagger &= \{\bar{\varrho}; \bar{\varrho}(\alpha_L(B^*)\,X_L(\varrho))\,R\} \\ &= \{\bar{\varrho}; \mathscr{U}_{\bar{\varrho}}(L)\,\bar{\varrho}(B^*)\,\mathscr{U}_{\bar{\varrho}}(L)^{-1}\,\bar{\varrho}(X_L(\varrho))\,R\}.\end{aligned}$$

But by Lemma 2.2c and (2.9)

$$R\mathscr{U}_0(L) = \mathscr{U}_{\bar{\varrho}\varrho}(L)\,R = \bar{\varrho}(X_L(\varrho))^{-1}\,\mathscr{U}_{\bar{\varrho}}(L)\,R, \tag{4.21}$$

so

$$\begin{aligned}\alpha_L(\boldsymbol{B})^\dagger &= \{\bar{\varrho}; \mathscr{U}_{\bar{\varrho}}(L)\,\bar{\varrho}(B^*)\,R\mathscr{U}_0(L)^{-1}\} \\ &= \alpha_L(\boldsymbol{B})^\dagger \quad \text{by (4.4).}\end{aligned}$$

We often apply classical functional analysis to the action of the Poincaré group on $\mathscr{B}$. The following remarks should suffice to make it clear that this is no more difficult than in the case of a field algebra.

All questions relating to the linear structure or topology on $\mathscr{B}$ or on the state vector bundle are to be understood as referring to the individual fibres. Thus we define

$$\|\{\varrho; B\}\| = \|B\|, \tag{4.22}$$

and note

$$\|B_1 B_2\| \leqq \|B_1\| \cdot \|B_2\|, \tag{4.23}$$

$$\|\{\varrho; B\}^\dagger\| \leqq d(\varrho)^{1/2} \|B\|. \tag{4.24}$$

The strong and weak operator topology on $\mathscr{B}$ refer to its action on the state vector bundle and are thus generated by the seminorms of the form $B \to \|B\boldsymbol{\Psi}\|$ and $B \to |(\boldsymbol{\Phi}, B\boldsymbol{\Psi})|$ respectively. The product in $\mathscr{B}$ is jointly continuous on the unit ball in the strong operator topology and conjugation is continuous in the weak operator topology. $\mathscr{B}(\mathcal{O})$ is fibrewise closed in the weak operator topology. The action of the Poincaré group is isometric and continuous in the strong operator topology.

V. The Energy-Momentum Spectrum

The energy-momentum spectrum in a representation given by $\varrho \in \Delta_r$ will be denoted by $S(\varrho)$[14]. We want to compare the spectrum in the subrepresentations of $\varrho_1 \varrho_2$ with $S(\varrho_1)$ and $S(\varrho_2)$.

The basic idea is to imitate the standard field theory technique by using elements of the field algebra to transfer momentum. As we shall see in the following lemma, if $B = \{\varrho; B\}$ then $\tilde{B}(q) = \int \alpha_x(B)\, e^{-iqx}\, d^4x$ adds precisely the momentum q to a state vector and changes the representation by a factor ϱ.

5.1. **Lemma.** *Let $B' = \{\varrho; B'\}$ and let f be an L^1-function on space-time whose Fourier transform*

$$\tilde{f}(q) = \int f(x)\, e^{iqx}\, d^4x$$

has support in an open set $\mathcal{N}$ of momentum space. Let

$$B = \int \alpha_x(B') f(x)\, d^4x. \tag{5.1}$$

If $\boldsymbol{\Psi} = \{\varrho_1; \Psi\}$ has momentum support in the open set $\mathcal{N}_1$ then $B\boldsymbol{\Psi}$ has momentum support in $\mathcal{N} + \mathcal{N}_1$. Further if $\mathcal{N} \cap S(\varrho)$ is not empty there exists a $B' = \{\varrho; B'\}$ such that $B\boldsymbol{\Omega} \neq 0$.

Proof. We first note that (5.1) defines an element of $\mathscr{B}$: this would be clear if f had compact support and $B' \in \mathscr{B}(\mathcal{O}_1)$ for some $\mathcal{O}_1$ because

[14] The following conventions will be used: if $x \in \mathscr{P}$ denotes a translation by x then $\mathscr{U}_\varrho(x) = e^{iPx}$. $Px = P_0 x_0 - \boldsymbol{P} \cdot \boldsymbol{x}$; the operators P_μ are called the momentum operators of the representation ϱ. $S(\varrho)$ is the closed set of $\mathbb{R}^4$ (momentum space) carrying the joint spectrum of the four operators P_μ.

the fibres of each $\mathscr{B}(\mathcal{O})$ are weak operator closed. As functions with compact support are dense in L^1 and local elements are norm dense in $\mathscr{B}$, $\boldsymbol{B}$ is the norm limit of a Cauchy sequence of local integrals and hence $\boldsymbol{B} \in \mathscr{B}$. Now $\mathscr{U}(x)\,\boldsymbol{B}\boldsymbol{\Psi} = \alpha_x(\boldsymbol{B})\,\mathscr{U}(x)\,\boldsymbol{\Psi}$ so Fourier analysis gives the stated result on the momentum support of $\boldsymbol{B}\boldsymbol{\Psi}$. Also by (5.1)

$$\boldsymbol{B\Omega} = \int d^4x\, f(x)\,\mathscr{U}(x)\,\boldsymbol{B}'\boldsymbol{\Omega} = \{\varrho; \int d^4x\, f(x)\,\mathscr{U}_\varrho(x)\,B'\Omega\}\,.$$

If $\mathscr{N} \cap S(\varrho)$ is not empty we may choose f so that $T = \int d^4x\, f(x)\,\mathscr{U}_\varrho(x) \neq 0$ and since Ω is cyclic we may choose B' such that $TB'\Omega \neq 0$.

We shall now show the following additivity property of the spectrum.

5.2. **Theorem.** a) *Let* $\varrho_1, \varrho_2 \in \Delta_r$ *then* $S(\varrho_1) + S(\varrho_2) \subset S(\varrho_1\varrho_2)$.

b) *Let* $\varrho_1, \varrho_2 \in \Delta_s$ *and* ϱ *a subrepresentation of* $\varrho_1\varrho_2$ *then*

$$S(\varrho_1) + S(\varrho_2) \subset S(\varrho)\,.$$

Proof. The proof of a) is standard. Pick arbitrary open sets $\mathscr{N}_1$ and $\mathscr{N}_2$ in momentum space intersecting $S(\varrho_1)$ and $S(\varrho_2)$ respectively. Pick $\boldsymbol{\Psi}_1 = \{\varrho_1, \Psi_1\}$ with momentum support in $\mathscr{N}_1$. By Lemma 5.1 there exists a $\boldsymbol{B} = \{\varrho_2; B\} \in \mathscr{B}$ such that $\boldsymbol{\Psi}_2 = \boldsymbol{B\Omega}$ has support in $\mathscr{N}_2$ and $\boldsymbol{\Psi} = \boldsymbol{B}\boldsymbol{\Psi}_1$ has momentum support in $\mathscr{N}_1 + \mathscr{N}_2$. Moreover for each fixed y, $\boldsymbol{\Psi}_y \equiv \boldsymbol{B}\mathscr{U}(y)\,\boldsymbol{\Psi}_1$ will also have momentum support in $\mathscr{N}_1 + \mathscr{N}_2$. We have to show that for some y, $\boldsymbol{\Psi}_y \neq 0$. Now

$$\begin{aligned}\|\boldsymbol{\Psi}_y\|^2 &= (\Psi_1, \mathscr{U}_{\varrho_1}(y)^{-1}\,\varrho_1(B^*B)\,\mathscr{U}_{\varrho_1}(y)\,\Psi_1)\\ &= (\Psi_1, \varrho_1(\alpha_{-y}(B^*B))\,\Psi_1) \to \|\boldsymbol{\Psi}_1\|^2\,\omega_0(B^*B) = \|\boldsymbol{\Psi}_1\|^2 \cdot \|\boldsymbol{\Psi}_2\|^2 \neq 0\end{aligned}$$

as y tends spacelike to infinity. To prove b) we have to consider the component of $\boldsymbol{\Psi}_y$ in the subspace corresponding to the subrepresentation ϱ. Let $S = (\varrho_1\varrho_2|S|\varrho)$ be a non-zero intertwiner. By Lemma 2.2, we know that S also intertwines the respective Poincaré representations: $S^*\mathscr{U}_{\varrho_1\varrho_2}(L) = \mathscr{U}_\varrho(L)\,S^*$. Hence $\boldsymbol{S}^*\boldsymbol{\Psi}_y$ also has momentum support in $\mathscr{N}_1 + \mathscr{N}_2$ and

$$\|\boldsymbol{S}^*\boldsymbol{\Psi}_y\|^2 = (\Psi_1, \mathscr{U}_{\varrho_1}(y)^{-1}\,\varrho_1(B^*)\,SS^*\varrho_1(B)\,\mathscr{U}_{\varrho_1}(y)\,\Psi_1)\,. \tag{5.2}$$

We again want to show that for some y, $\boldsymbol{S}^*\boldsymbol{\Psi}_y \neq 0$ and may apply Lemma 2.3 to obtain

$$\|\boldsymbol{S}^*\boldsymbol{\Psi}_y\|^2 \to \|\boldsymbol{\Psi}_1\|^2\,\omega_0 \circ \phi_1(\varrho_1(B^*)\,SS^*\varrho_1(B)) = \|\boldsymbol{\Psi}_1\|^2\,\omega_0(B^*\phi_1(SS^*)\,B)$$

as y tends spacelike to infinity. We note that $\phi_1(SS^*)$ intertwines from ϱ_2 to ϱ_2, it is therefore a scalar as ϱ_2 was assumed irreducible. We may choose S isometric and then use [1; Proposition 6.6 and, Corollary 6.8] to compute

$$\phi_1(SS^*) = \phi_2\phi_1(SS^*) = d(\varrho)\,d(\varrho_1\varrho_2)^{-1}\,I$$

giving

$$\|S^* \Psi_y\|^2 \to \|\Psi_1\|^2 \, \|\Psi_2\|^2 \, d(\varrho)\, d(\varrho_1 \varrho_2)^{-1} \neq 0 \tag{5.3}$$

as y tends spacelike to infinity. This completes the proof.

Now the vacuum representation ι is a subrepresentation of $\bar{\varrho}\varrho$. Since, by assumption, $S(\iota)$ is contained in the closed forward light cone $\overline{V^+}$ and both $S(\varrho)$ and $S(\bar{\varrho})$ are Lorentz invariant sets we deduce[15]

5.3. **Corollary.** *If $\varrho \in \Delta_s$ then $S(\varrho) \subset \overline{V^+}$.*

Of course since any $\varrho \in \Delta_r$ is equivalent to a (finite) direct sum of irreducibles from Δ_s we deduce from Lemma 2.2c that $S(\varrho) \subset \overline{V^+}$ for $\varrho \in \Delta_r$ as well.

VI. Particles and Antiparticles

In this section we shall consider single particle states and study the particle-antiparticle symmetry. The starting point of this discussion is the relationship between $\mathscr{U}_\varrho$ and $\mathscr{U}_{\bar{\varrho}}$. If $\varrho \in \Delta_r(\mathcal{O})$ we have by (2.3)

$$(\Omega, \mathscr{U}_\varrho(L)\, \Omega) = (\Omega, X_L(\varrho)^{-1}\, \Omega)\,.$$

On the other hand by (3.14) and (4.21)

$$(R\Omega, \mathscr{U}_{\bar{\varrho}}(L)\, R\Omega) = (R\Omega, \bar{\varrho}(X_L(\varrho))\, R\Omega) = d(\varrho) \cdot (\Omega, \phi(X_L(\varrho))\, \Omega)\,.$$

Now by the definition of ε_ϱ [1; Theorem 4.2], we have

$$X_L(\varrho) = \varepsilon_\varrho \varrho(X_L(\varrho)), \qquad L\mathcal{O} \subset \mathcal{O}'\,, \tag{6.1}$$

$$\phi(X_L(\varrho)) = \lambda_\varrho X_L(\varrho)\,, \qquad L\mathcal{O} \subset \mathcal{O}'\,. \tag{6.2}$$

Hence

$$(R\Omega, \mathscr{U}_{\bar{\varrho}}(L)^{-1}\, R\Omega) = \operatorname{sign}(\lambda_\varrho)\, (\Omega, \mathscr{U}_\varrho(L)\, \Omega)\,; \qquad L\mathcal{O} \subset \mathcal{O}'\,. \tag{6.3}$$

The significance of this simple relation can be better understood as a special case of

6.1. **Lemma.** a) *Let $B_i = \{\varrho; B_i\} \in \mathscr{B}(\mathcal{O})$ then*

$$(B_1 \boldsymbol{\Omega}, \mathscr{U}(L)\, B_2 \boldsymbol{\Omega}) = \operatorname{sign}(\lambda_\varrho)\, (B_2^\dagger \boldsymbol{\Omega}, \mathscr{U}(L^{-1})\, B_1^\dagger \boldsymbol{\Omega})\,, \qquad L\mathcal{O} \subset \mathcal{O}'\,. \tag{6.4}$$

b) *Let $\boldsymbol{\Phi}_i = \{\varrho', \Phi_i\}$ and $B_i = \{\varrho; B_i\} \in \mathscr{B}(\mathcal{O}_i)$, $i = 1, 2$, then*

$$(B_1 \boldsymbol{\Phi}_1, B_2 \boldsymbol{\Phi}_2) = \operatorname{sign}(\lambda_\varrho)\, (B_2^\dagger \boldsymbol{\Phi}_1, B_1^\dagger \boldsymbol{\Phi}_2)\,; \qquad \mathcal{O}_1 \subset \mathcal{O}_2'\,. \tag{6.5}$$

Proof. If we replace B_2 by $\alpha_L(B_2)$ in b) and use (4.4) and (4.20) we get a) on specializing to $\boldsymbol{\Phi}_1 = \boldsymbol{\Phi}_2 = \boldsymbol{\Omega}$. Now to prove b) we have

$$(B_1 \boldsymbol{\Phi}_1, B_2 \boldsymbol{\Phi}_2) = (\boldsymbol{I}_{\varrho'} \times \bar{\boldsymbol{R}} \boldsymbol{\Phi}_1, B_1^\dagger B_2 \boldsymbol{\Phi}_2)\,.$$

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[15] The idea of using additivity of the spectrum to prove positivity is due to Borchers [7].

Since $B_1^\dagger$ and B_2 are spacelike by Lemma 4.3, we have by (4.12)

$$\begin{aligned}(B_1\Phi_1, B_2\Phi_2) &= (I_{\varrho'}\times \bar{R}\Phi_1, I_{\varrho'}\times \varepsilon(\bar{\varrho},\varrho)\, B_2 B_1^\dagger \Phi_2)\\ &= \operatorname{sign}(\lambda_\varrho)\,(I_{\varrho'}\times R\Phi_1, B_2 B_1^\dagger\Phi_2) \qquad \text{by (3.7)}\\ &= \operatorname{sign}(\lambda_\varrho)\,(B_2^\dagger\Phi_1, B_1^\dagger\Phi_2)\end{aligned}$$

completing the proof of the Lemma.

Note that (6.5), a result of combining the commutation relation with the adjoint in the field bundle, takes just the same form as in standard field theory with Bose or Fermi statistics.

Now setting $B_1 = B_2$ in (6.4) and taking L to be a spacelike translation $x \in \mathbb{R}^4$ let us denote the left and right hand sides of (6.4) by $h_1(x)$ and $\operatorname{sign}(\lambda_\varrho)\, h_2(x)$ respectively. The Fourier transforms $\tilde{h}_1(p)$, $\tilde{h}_2(p)$ are bounded positive Borel measures with support in the cones $\overline{V^+}$ and $\overline{V^-}$ respectively. According to Lemma 6.1a, the function $h(x) = h_1(x) - \operatorname{sign}(\lambda_\varrho)\, h_2(x)$ vanishes on the complement $\mathcal{O}_1'$ of some double cone $\mathcal{O}_1$:

$$h_1(x) = \operatorname{sign}(\lambda_\varrho)\, h_2(x)\,, \qquad x\in\mathcal{O}_1'\,.$$

We are therefore in a situation where the techniques of the Jost-Lehmann-Dyson representation may be applied. We may define for any Borel set $S \subset \mathbb{R}^+$ the functions

$$h_i^S(x) = \int_{\sqrt{p^2}\in S} e^{ipx}\, d\tilde{h}_i(p)\,. \tag{6.6}$$

The Borel measure $S \to h_i^S(0)$ is just the projection of $\tilde{h}_i(p)$ onto the space of mass values under the Borel map $p \to \sqrt{p^2}$.

6.2. **Lemma.** a) *Let S be an arbitrary Borel set of mass values, then*

$$h_1^S(x) = \operatorname{sign}(\lambda_\varrho)\, h_2^S(x)\,, \qquad x\in\mathcal{O}_1'\,.$$

b) *The Borel measures defined by projection of $\tilde{h}_1(p)$ and $\tilde{h}_2(p)$ on the mass axis are equivalent (i.e. they have the same null sets).*

Proof. a) Let $\tilde{H}_i(p,m)$ be the Borel measure on $\mathbb{R}^5$ induced by $\tilde{h}_i(p)$ under the Borel map $p \to (p, \sqrt{p^2})$. Then

$$\tilde{K}(p,m) = \tilde{H}_1(p,m) + \tilde{H}_1(p,-m) - \operatorname{sign}(\lambda_\varrho)\, \tilde{H}_2(p,m) - \operatorname{sign}(\lambda_\varrho)\, \tilde{H}_2(p,-m)$$

has Fourier transform

$$K(x,s) = \int e^{ipx-ims}\, d\tilde{K}(p,m) = 2\int e^{ipx}\cos(\sqrt{p^2}\, s)\, d\tilde{h}(p)\,.$$

Using properties of the wave equation in 5-dimensions one shows [8]

$$K(x,s) = 0 \quad \text{for} \quad x\in\mathcal{O}_1'\,, \quad s\in\mathbb{R}^1\,.$$

Elementary integration theory then shows that for any Borel set $S \subset \mathbb{R}^1$

$$\int e^{ipx} \chi_s(m)\, d\tilde{K}(p,m) = 0 \quad \text{for} \quad x \in \mathcal{O}_1'$$

where χ_s is the characteristic function of S. Hence for $S \subset \mathbb{R}^+$ we have

$$h_1^S(x) = \operatorname{sign}(\lambda_\varrho)\, h_2^S(x) \quad \text{for} \quad x \in \mathcal{O}_1' .$$

b) We must show that $h_1^S(0) = 0$ if and only if $h_2^S(0) = 0$. If $h_1^S(0) = 0$, then $|h_1^S(x)| \leqq h_1^S(0) = 0$, so from a) $h_2^S(x) = 0$ for $x \in \mathcal{O}_1'$. But $h_2^S(x)$ is the boundary value of a function analytic in the backward tube hence $h_2^S(x) = 0$ for all x, in particular $h_2^S(0) = 0$. The converse follows by symmetry.

An immediate consequence is

6.3. **Theorem.** *The mass spectra of conjugate representations are quasiequivalent, i.e.*

$$E_\varrho(S) = 0 \quad \textit{if and only if} \quad E_{\bar{\varrho}}(S) = 0$$

where $E_\varrho(S)$, $E_{\bar{\varrho}}(S)$ denote the spectral projections of the mass operator in the representations ϱ and $\bar{\varrho}$.

The first result in this direction was obtained by Borchers [7; Theorem VII-7].

The facts described above allow us to define an unbounded antilinear mapping J_ϱ from state vectors $\{\varrho; \Psi\}$ of the sector ϱ to state vectors in the conjugate sector. If $\boldsymbol{B} = \{\varrho, B\}$ is strictly local, i.e. $\boldsymbol{B} \in \bigcup_{\mathcal{O}} \mathcal{B}(\mathcal{O})$, we define

$$J_\varrho \boldsymbol{B} \boldsymbol{\Omega} = \boldsymbol{B}^\dagger \boldsymbol{\Omega} . \tag{6.7}$$

Since Ω is cyclic and separating for $\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$, the domain of J_ϱ is dense and the definition is consistent. Also, due to the cyclicity of Ω, the range of J_ϱ is dense. One checks immediately that J_ϱ commutes on its domain with the Poincaré operators:

$$J_\varrho \mathcal{U}_\varrho(L) = \mathcal{U}_{\bar{\varrho}}(L)\, J_\varrho . \tag{6.8}$$

This, together with Lemma 6.2, allows us to extend the domain of J_ϱ setting

$$J_\varrho E_\varrho(S)\, \boldsymbol{B} \boldsymbol{\Omega} = E_{\bar{\varrho}}(S) \boldsymbol{B}^\dagger \boldsymbol{\Omega} \tag{6.9}$$

and extending J_ϱ to an antilinear operator. Here S is an arbitrary Borel set in the mass spectrum. In particular we consider now the case where ϱ contains single particle states of mass m. The projectors $E_\varrho(\{m\})$, $E_{\bar{\varrho}}(\{m\})$ on the single point m are then non-zero. The corresponding subspaces in the two sectors will be denoted by $\mathcal{H}_\varrho^m$, $\mathcal{H}_{\bar{\varrho}}^m$. J_ϱ is defined on a dense domain in $\mathcal{H}_\varrho^m$ and maps it into a dense set in $\mathcal{H}_{\bar{\varrho}}^m$.

Let $\mathscr{H}^{(m,s)}$ be the space of an irreducible representation $\mathscr{U}^{(m,s)}$ of the Poincaré group with mass m, spin s and positive energy. A vector of this space will be described in the standard manner by a wave function $f_\alpha(p)$ where the index $\alpha=(r_1, \ldots, r_{2s})$ corresponds to a symmetric spinor of rank $2s$ (it has $2s+1$ independent components) and the scalar product is given by

$$(g, f) = \int (g(p), M(p) f(p))\, d\Omega_m^+(p) \tag{6.10}$$

where

$$M(p) = \bigotimes_1^{2s} \left(\frac{p_0 \cdot I - \boldsymbol{p} \cdot \boldsymbol{\sigma}}{m} \right); \tag{6.11}$$

$\boldsymbol{\sigma}=(\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and $d\Omega_m^+(p)=\delta(p^2-m^2)\,\varepsilon(p_0)\,d^4p$ is the invariant measure on the positive mass hyperboloid. We note

$$M(-p) = (-1)^{2s} M(p). \tag{6.12}$$

There is a natural involution $F \to F'$ defined on spinor-valued functions which are analytic on the whole complex hyperboloid by putting

$$F'_\alpha(p) = F_\alpha(-p).$$

If both F and F' are in $\mathscr{H}^{(m,s)}$ when restricted to the positive mass hyperboloid V_m^+ we set

$$\begin{aligned} f_\alpha(p) &= F_\alpha(p), & p \in V_m^+ \\ (J'^{(m,s)} f)_\alpha(p) &= \sum_\beta T_{\alpha\beta}(p)\, \bar{F}'_\beta(p); & p \in V_m^+ \end{aligned} \tag{6.13}$$

where

$$T(p) = \bigotimes_1^{2s} \left(\frac{p_0 \cdot I + \boldsymbol{p} \cdot \boldsymbol{\sigma}}{m} \cdot \sigma_2 \right). \tag{6.14}$$

The operator $J'^{(m,s)}$ defined in this manner is a densely defined (unbounded) antilinear involution on $\mathscr{H}^{(m,s)}$ commuting with $\mathscr{U}^{(m,s)}$.

The representation of $\mathscr{P}$ on $\mathscr{H}_\varrho^m$ is equivalent to a direct sum of representations $\mathscr{U}^{(m,s_i)}$ where we allow for the possibility that ϱ contains more than one type of particle with mass m (the s_i are the spins of these particles: each s_i may also occur several times in the direct sum). However we discuss only the case of finite mass degeneracy where there are just a finite number of types of particles with mass m. We may identify $\mathscr{H}_\varrho^m$ with $\mathscr{K}_\varrho = \bigoplus_i \mathscr{K}^{(m,s_i)}$ and hence regard $J'_\varrho = \bigoplus_i J'^{(m,s_i)}$ as a densely defined involution on $\mathscr{H}_\varrho^m$ commuting with $\mathscr{U}_\varrho$:

$$J'_\varrho \mathscr{U}_\varrho(L) = \mathscr{U}_\varrho(L)\, J'_\varrho, \qquad L \in \mathscr{P}. \tag{6.15}$$

Similarly we may identify $\mathscr{H}_{\bar\varrho}^m$ with $\mathscr{K}_{\bar\varrho} = \bigoplus_j \mathscr{K}^{(m,\bar s_j)}$ (where a priori the $\bar s_j$ might be different from the s_i).

6.4. **Theorem.** *If the mass m occurs with finite degeneracy in the representation* ϱ, *then*

a) *there is a unitary operator C mapping* $\mathscr{H}_\varrho^m$ *onto* $\mathscr{H}_{\bar{\varrho}}^m$ *(kinematical charge conjugation) with the following properties: C establishes the unitary equivalence of the Poincaré representations in the conjugate single particle spaces (equality of spins and multiplicities):*

$$C\mathscr{U}_\varrho^m(L) = \mathscr{U}_{\bar{\varrho}}^m(L)\, C \tag{6.16}$$

and C is the unique closed extension of the mapping $J'_\varrho \boldsymbol{\Psi} \to J_\varrho \boldsymbol{\Psi}$.

b) *For all values* s_i *of the spin occurring in* $\mathscr{H}_\varrho^m$ *we have (connection between spin and statistics)*

$$(-1)^{2s_i} = \operatorname{sign}(\lambda_\varrho)\,. \tag{6.17}$$

Proof. Taking S in Lemma 6.2 to be the single point $\{m\}$ we obtain two functions $h_1^m(x)$ and $h_2^m(x)$ which coincide up to a sign on the causal complement of some double cone and have Fourier transforms $\tilde{h}_1^m$ and $\tilde{h}_2^m$ whose supports are the positive and negative mass hyperboloids respectively. It follows, see e.g. Epstein [2; p. 757–758], that there is an analytic function k on the complex mass hyperboloid such that

$$\begin{aligned} \tilde{h}_1^m(p) &= k(p)\,\delta(p^2-m^2), \qquad p \text{ real}, \qquad p_0>0 \\ \operatorname{sign}(\lambda_\varrho)\, \tilde{h}_2^m(p) &= k(p)\,\delta(p^2-m^2), \qquad p \text{ real}, \qquad p_0<0\,. \end{aligned} \tag{6.18}$$

Let $B \in \mathscr{B}(\mathcal{O})$ and consider the state vectors

$$\boldsymbol{\Psi} = E_\varrho(\{m\})\, B\boldsymbol{\Omega}\,; \qquad \boldsymbol{\Psi}' = J_\varrho \boldsymbol{\Psi}\,.$$

If $f \in \mathscr{K}_\varrho$ and $f' \in \mathscr{K}_{\bar{\varrho}}$ are the wave functions of $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}'$ respectively, then using the definition of $\tilde{h}_i^m$ and (6.18) we have for real p on the positive mass hyperboloid

$$(f(p), M(p) f(p)) = k(p)\,, \tag{6.19}$$

$$(f'(p), M(p) f'(p)) = \operatorname{sign}(\lambda_\varrho)\, k(-p)\,. \tag{6.20}$$

In these equations we have used matrix notation so that the left hand sides involve sums over spinor indices and over the spins of the particles involved. The difficult but vital step in the proof is to show that the wave function f can be extended to an analytic function F on the whole complex mass hyperboloid, see Epstein [2; Appendix II][16]. Therefore

[16] Actually Epstein considers only the case of a single particle with mass m and spin s. However the proof in [2] depends only on the following properties of the representations $\mathscr{D}^{(s)}$ of $GL(2,\mathbb{C})$ and $\hat{\mathscr{D}}^{(s)}$ of $\overline{L_+(\mathbb{C})}$: $\mathscr{D}^{(s)}(L^*) = \mathscr{D}^{(s)}(L)^*$; $\mathscr{D}^{(s)}(L^T) = \mathscr{D}^{(s)}(L)^T$ and $\Lambda \to \hat{\mathscr{D}}^{(s)}(\Lambda)$ is holomorphic. These properties hold equally well for finite direct sums of the $\mathscr{D}^{(s)}$ and $\hat{\mathscr{D}}^{(s)}$. The proof would also require B to have been regularized over the Poincaré group and this suffices for our purpose. However using this as an intermediate step, one can show that f can always be extended to an analytic function on the whole complex mass hyperboloid.

we have from (6.19) and (6.20)

$$(F(p), M(p)\,F(p)) = k(p)\,,$$

$$\begin{aligned}\|\boldsymbol{\Psi}'\|^2 &= \operatorname{sign}(\lambda_\varrho) \int k(-p)\,d\Omega_m^+(p) \\ &= \operatorname{sign}(\lambda_\varrho) \int (F(-p), M(-p)\,F(-p))\,d\Omega_m^+(p)\,.\end{aligned} \tag{6.21}$$

To simplify the argument leading to b) we note that the spins s_i occurring in $\mathscr{H}_\varrho$ must either be all half integral or all integral since the superselection rule between integral and half-integral spin is a consequence of the unique action of α_L on the observable algebra (α_L represents the Poincaré group not its covering group). Now $M(p)$ is a positive-definite matrix for p on the positive mass hyperboloid and $\|\boldsymbol{\Psi}'\|^2$ is positive so using (6.12) we get b). To prove a) we note that

$$\begin{aligned}&\int (F'(p), M(p)\,F'(p))\,d\Omega_m^+(p) \\ &= \int (F(-p), M(p)\,F(-p))\,d\Omega_m^+(p) = \|\boldsymbol{\Psi}'\|^2 < \infty\,.\end{aligned}$$

Hence $\boldsymbol{\Psi}$ is in the domain of J'_ϱ and

$$\|J'_\varrho \boldsymbol{\Psi}\| = \|J_\varrho \boldsymbol{\Psi}\|\,.$$

Thus $J'_\varrho \boldsymbol{\Psi} \to J_\varrho \boldsymbol{\Psi}$ is a densely defined isometry with dense range and its extension C is a unitary operator intertwining $\mathscr{U}_\varrho^m$ and $\mathscr{U}_{\bar\varrho}^m$ by (6.8) and (6.15). This completes the proof.

It is known from models that the connexion between spin and statistics does not necessarily hold in the presence of infinite mass and spin degeneracies [9, 10]. Presumably a) above and Theorem 6.5 below are valid without the restriction to finite degeneracies.

We turn now to self-conjugate sectors where we may choose $\varrho = \bar\varrho$[17]. Since ϱ is irreducible, $\bar R$ is a scalar multiple of R by Theorem 3.1a and from (3.7) we have

$$\bar R = \pm R\,. \tag{6.22}$$

The sign in (6.22) determines the sign of J_ϱ^2. If $B = \{\varrho; B\} \in \mathscr{B}(\mathscr{O})$,

$$\begin{aligned}J_\varrho B \boldsymbol{\Psi} &= \{\varrho; \varrho(B^*)\,R\Omega\}\,, \\ J_\varrho^2 B \boldsymbol{\Omega} &= \{\varrho; \varrho(R^*)\,\varrho^2(B)\,R\Omega\} = \{\varrho; \varrho(R^*)\,RB\Omega\}\,,\end{aligned}$$

and by (3.6) and (6.22) we get

$$J_\varrho^2 = \pm I \tag{6.23}$$

on its domain of definition. Naturally (6.23) also holds for J_ϱ considered as a densely defined operator on $\mathscr{H}_\varrho^m$. We call the self-conjugate sector ξ corresponding to ϱ real or pseudoreal according as J_ϱ^2 is $+I$ or $-I$.

[17] Here we temporarily abandon a previous convention. Compare footnote 13.

Now if $\Psi = E_\varrho(\{m\})\, B\Omega$ where B is strictly localized and regularized over $\mathscr{P}$, then by (6.23) and the definition of C

$$J'_\varrho C J'_\varrho J_\varrho \Psi = \pm J'_\varrho \Psi .$$

Hence if $\bar{C}$ denotes the extension of $J'_\varrho C J'_\varrho$ to $\mathscr{H}^m_\varrho$ we have

$$\bar{C} = \pm C^{-1} . \tag{6.24}$$

6.5. **Theorem.** *Let ξ be a pseudoreal sector, m a mass occurring in it with finite multiplicity. Then a particle with mass m and charge ξ cannot be its own antiparticle. Consequently there must be an even number of particles with mass m and spin s in the sector ξ.*

Proof. A particle coincides with its own antiparticle if the corresponding irreducible subspace of $\mathscr{H}^m_\varrho$ is an eigenspace of C. Let the corresponding eigenvalue be λ. But this irreducible subspace must then also be an eigenspace of $\bar{C} \supset J'_\varrho C J'_\varrho$ with eigenvalue $\bar{\lambda}$, and of $\bar{C}C$ with eigenvalue $\bar{\lambda}\lambda = 1$. Thus $\bar{C}C = I$ and comparing with (6.24) the sector must be real contrary to hypothesis.

In models where each sector is associated with a representation of the gauge group [11], [4] a self-conjugate sector is associated with a representation unitarily equivalent to its complex conjugate. The condition $\bar{R} = \pm R$ translates directly into group theoretical language using the results of [4] and it can be shown that, in the terminology of Wigner [12], $\bar{R} = R$ if the representation of the gauge group is (potentially) real and $\bar{R} = -R$ if it is pseudoreal. If the gauge group is the isospin group SU(2), the pseudoreal representations correspond to half integral isospin and we may recognize that Theorem 6.5 is just a variant of Carruthers' Theorem [3].

VII. Collision Theory

The construction of scattering states is done in close analogy to standard procedure in quantum field theory[18]. We sketch the line of argument but suppress full proofs of those statements which can be easily obtained combining the techniques of Section IV with known results of quantum field theory. We shall emphasize instead those aspects where unfamiliar features appear namely the metric of the scattering states and the definition of transition probabilities and amplitudes.

The first step is the construction of almost local creation operators for each type of particle. Suppose the sector ξ contains single particle states with mass m and assume as usual that m is isolated from the other parts of the mass spectrum in the sector ξ. Pick a $\varrho \in \Delta_s$ in the class ξ,

[18] See e.g. [13, Chapter VI].

an element $B' = \{\varrho; B'\} \in \mathscr{B}$ and a function g on Minkowski space, absolutely integrable and such that the support of its Fourier transform intersects $S(\varrho)$ only on the mass hyperboloid $p^2 = m^2$; put

$$B = \int \alpha_x(B')\, g(x)\, d^4x\,. \tag{7.1}$$

Then $B\Omega$ describes (if it does not vanish) a single particle state with charge ξ and mass m. If B' is chosen strictly local and g is a test function in class $\mathscr{S}$ then B will be almost local (quasilocal of infinite order) and $\alpha_x(B)$ differentiable in the norm topology with respect to x. Let f be a smooth, positive frequency solution of the Klein-Gordon equation with mass m, specifically

$$\begin{gathered} f(x) = \int \tilde{f}(\boldsymbol{p})\, e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - \omega_p x_0)}\, d^3\boldsymbol{p}\,, \\ \tilde{f}(\boldsymbol{p}) \in \mathscr{D}^{(3)}; \quad \omega_{\boldsymbol{p}} = (\boldsymbol{p}^2 + m^2)^{1/2}\,. \end{gathered} \tag{7.2}$$

Then, with B as above, we define

$$B_f(t) = \int_{x_0 = t} \alpha_x(B)\, f(x)\, d^3\boldsymbol{x} \tag{7.3}$$

and show, as usual, that the state vector

$$\boldsymbol{\Psi} = B_f(t)\,\boldsymbol{\Omega} = \{\varrho; \Psi\} \tag{7.4}$$

is independent of t. Furthermore, if we vary B', g and f within the stated restrictions we obtain by (7.4) a set $\mathscr{L}_\varrho^m$ of state vectors $\boldsymbol{\Psi}$ which is dense in $\mathscr{H}_\varrho^m$. In fact $\boldsymbol{\Psi} \in \mathscr{L}_\varrho^m$ if $\boldsymbol{\Psi} = F(P)\, B'\boldsymbol{\Omega}$ where B' is strictly local and $F \in \mathscr{D}^{(4)}$ with $\operatorname{supp} F \cap S(\varrho) \subset \{p/p^2 = m^2\}$.

As a next step we look at the time dependence of objects of the form

$$B_n(t) \ldots B_1(t)\,\boldsymbol{\Omega} \quad \text{with} \quad B_k(t)\,\boldsymbol{\Omega} = \boldsymbol{\Psi}_k\,. \tag{7.5}$$

Here we suppose that we have picked n "creation operators"

$$B_i = \{\varrho_i, B_i\} \in \mathscr{B}$$

constructed as described above but not necessarily referring to particles of the same mass or charge. We have written $B_i(t)$ as a shorthand for $B_{f_i}^{(i)}(t)$ where the f_i are smooth solutions of the respective Klein-Gordon equations. Furthermore we shall choose the f_i to have disjoint supports in velocity space[19]. This device, suggested in [14] simplifies the subsequent discussion and suffices for our purpose.

The study of the asymptotic behaviour of smooth solutions of the Klein-Gordon equation given by Ruelle [15] leads us to conclude as in [14, 16] that, for large $|t|$, the $B_k(t)$ are localized spacelike to each other

[19] The velocity of the i^{th} particle is related to its momentum by $\boldsymbol{v} = (\boldsymbol{p}^2 + m^2)^{-1/2}\,\boldsymbol{p}$. Thus we have to choose the supports of the $\tilde{f}_i$ appropriately.

up to terms which decrease in norm faster than any inverse power of $|t|$. Therefore applying the spacelike commutation relations of the field bundle (Proposition 4.2) and iterating it using [1; Theorem 4.2] we have for any permutation p of the n factors and any $N>0$:

$$|t|^N \cdot \| B_{p^{-1}(n)}(t) \dots B_{p^{-1}(1)}(t) - \varepsilon_p(\varrho_1, \dots, \varrho_n) \circ (B_n(t) \dots B_1(t)) \| \xrightarrow[|t|\to\infty]{} 0. \quad (7.6)$$

This leads to

7.1. **Proposition.** *Given single particle states $\Psi_k \in \mathscr{L}^{m_k}_{\varrho_k}$, $k=1,2,\dots,n$, whose velocity supports are mutually disjoint, pick $B_k(t) \equiv B^{(k)}_{\tilde{f}_k}(t)$ as above with $B_k(t)\Omega = \Psi_k$ so that the velocity supports of the $\tilde{f}_k$ are mutually disjoint then the strong limits of (7.5) exist for $t \to \pm\infty$ and depend only on the one particle states Ψ_k. We shall write*

$$\begin{aligned} \lim_{t\to-\infty} B_n(t) \dots B_1(t)\,\Omega &= \Psi_n \overset{\text{in}}{\times} \dots \overset{\text{in}}{\times} \Psi_1\,, \\ \lim_{t\to+\infty} B_n(t) \dots B_1(t)\,\Omega &= \Psi_n \overset{\text{out}}{\times} \dots \overset{\text{out}}{\times} \Psi_1\,. \end{aligned} \quad (7.7)$$

The rate of convergence in (7.7) is faster than any inverse power of $|t|$. Further the scattering states have the expected behaviour under permutations of the arguments and Poincaré transformations:

$$\varepsilon_p(\varrho_1, \dots, \varrho_n)\, \Psi_n \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \Psi_1 = \Psi_{p^{-1}(n)} \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \Psi_{p^{-1}(1)}, \quad p \in \mathbb{P}^{(n)}; \quad (7.8)$$

$$\mathscr{U}(L)\,(\Psi_n \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \Psi_1) = (\mathscr{U}(L)\,\Psi_n) \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} (\mathscr{U}(L)\,\Psi_1), \quad L \in \mathscr{P}. \quad (7.9)$$

The corresponding equations hold if "in" is replaced everywhere by "out".

Sketch of Proof. The time derivative of any $B_k(t)$ is the sum of two terms each of the same form as a $B_k(t)$ and it annihilates Ω since $B_k(t)\,\Omega$ is time independent. Differentiating (7.5) with respect to t and using the asymptotic commutation relations (7.6) to shift the time derivatives to the right, we find that the norm of the time derivative of (7.5) decreases for large $|t|$ faster than any inverse power of $|t|$. This proves that (7.5) converges to a limit as $t \to \pm\infty$ faster than any inverse power of $|t|$. If $C_f(t)$ is another creation operator for Ψ_k chosen as above so that $(C_f(t) - B_k(t))\,\Psi = 0$ and if the velocity support of $\tilde{f}$ is disjoint from that of $\tilde{f}_j$ for $j \neq k$ then arguing as before we see that the time limits are unchanged if $B_k(t)$ is replaced by $C_f(t)$ in (7.5). Since our creation operators $C_f(t)$ for Ψ_k may be chosen such that the velocity support of $\tilde{f}$ is contained in an arbitrary neighbourhood of that of Ψ_k, a finite chain of such replacements will lead us from any one choice of the $B_k(t)$ to any other. Thus the limits in (7.7) depend only on the one particle states. Eq. (7.8) is an immediate consequence of (7.6) and (7.7). To prove (7.9) we first note that

$$\mathscr{U}(L)\,(\boldsymbol{B}_n(t) \ldots \boldsymbol{B}_1(t)\,\boldsymbol{\Omega}) = \alpha_L(\boldsymbol{B}_n(t)) \ldots \alpha_L(\boldsymbol{B}_1(t))\,\boldsymbol{\Omega}\,. \qquad (7.10)$$

Also $\alpha_L(\boldsymbol{B}_k(t))\,\boldsymbol{\Omega} = \mathscr{U}(L)\,\boldsymbol{\Psi}_k \in \mathscr{L}_{\varrho_k}^{m_k}$. The velocity supports of the $\mathscr{U}(L)\,\boldsymbol{\Psi}_k$ are still mutually disjoint (the transformation law of velocity is independent of the mass). Choosing creation operators $\boldsymbol{C}_{g_k}^{(k)}(t) \equiv \boldsymbol{C}_k(t)$ for the $\mathscr{U}(L)\,\boldsymbol{\Psi}_k$ with the velocity support of $\tilde{g}_k$ in a sufficiently small neighbourhood of that of $\mathscr{U}(L)\,\boldsymbol{\Psi}_k$ we may arrange that, for L in some neighbourhood of the identity in $\mathscr{P}$, $\boldsymbol{C}_k(t)$ and $\alpha_L(\boldsymbol{B}_j(t))$ are localized spacelike to each other for $j \neq k$ up to terms which decrease in norm faster than any inverse power of $|t|$. Hence, by the usual arguments, we may replace the $\alpha_L(\boldsymbol{B}_k(t))$ by $\boldsymbol{C}_k(t)$ in (7.10) successively for $k = 1, 2, \ldots n$ without altering the limits of (7.10) as $t \to \pm\infty$. This proves (7.9) for L in some neighbourhood of the identity in $\mathscr{P}$. The same result holds if $\boldsymbol{\Psi}_k$ is replaced by $\mathscr{U}(L)\,\boldsymbol{\Psi}_k$, $k = 1, 2, \ldots, n$ in (7.9). But $\mathscr{P}$ is path-connected so (7.9) must hold for all $L \in \mathscr{P}$ completing the proof of the Proposition.

Although the definition of the products $\overset{\text{in}}{\times}$ and $\overset{\text{out}}{\times}$ in (7.7) is not manifestly covariant, we have shown in the course of proving (7.9) that it is independent of the choice of Lorentz frame.

We study next the metric of scattering states. As in the case of field theory this will follow from the approximate factorization (clustering properties) of vacuum expectation values to which the next two lemmas refer.

7.2. **Lemma.** *Let*

$$\boldsymbol{B}_i = \{\varrho_i, B_i\} \in \mathscr{B}(\mathscr{O}_i), \qquad i = 1, 2,$$

with $\mathscr{O}_1$ spacelike to $\mathscr{O}_2$ and

$$r = \sup\{|t| / \mathscr{O}_1 + (t, \mathbf{0}) \subset \mathscr{O}_2'\}\,.$$

If $\boldsymbol{T}$ is an arbitrary intertwiner from ϱ_1 to ϱ_2 and E_0 denotes the projector on the subspace of Poincaré invariant vectors in the representation ϱ_1 then for any $n > 0$ there is a number a_n, independent of the B_i and $\boldsymbol{T}$, such that

$$|(\boldsymbol{B}_2\boldsymbol{\Omega}, \boldsymbol{T}(I - E_0)\,\boldsymbol{B}_1\boldsymbol{\Omega})| < a_n r^{-n} d(\varrho_1) \|B_1\| \cdot \|B_2\| \cdot \|\boldsymbol{T}\|\,. \qquad (7.11)$$

Proof. Recall that we assumed at the beginning of this section that the single particle hyperboloids are isolated. This demands in particular that there is a lowest mass $m_0 \neq 0$ in the theory because otherwise (by Section V) the spectrum in the vacuum sector would contain all mass values and then there could be no gap above the lowest hyperboloid in any sector. We now follow closely an adaptation of the technique developed by Ruelle [15] and Araki [17] as used in [18]. Therefore the explanation can be kept brief. We choose an infinitely differentiable function $\tilde{f}$ of one variable such that $\tilde{f}(\omega)=1$ for $\omega \geqq m_0$ and $\tilde{f}(\omega)=0$ for $\omega \leqq 0$. Define for $\gamma>0$

$$f_\gamma(t)=(2\pi)^{-1}\int \tilde{f}(\omega)\, e^{-i\omega t-\gamma\omega}\, d\omega\,; \qquad \boldsymbol{B}_{1\gamma}=\int f_\gamma(t)\,\alpha_t(\boldsymbol{B})\, dt\,.$$

Then we have (see Lemma 5.1)

$$(I-E_0)\,\boldsymbol{B}_1\boldsymbol{\Omega}=\lim_{\gamma\to 0}\boldsymbol{B}_{1\gamma}\boldsymbol{\Omega}\,; \qquad \boldsymbol{B}_{1\gamma}^\dagger\boldsymbol{\Omega}=0\,. \tag{7.12}$$

Thus by Proposition 4.4 and Eq. (4.10)

$$\begin{aligned}(\boldsymbol{B}_2\boldsymbol{\Omega}, \boldsymbol{T}(I-E_0)\,\boldsymbol{B}_1\boldsymbol{\Omega}) &= \lim_{\gamma\to 0}(\boldsymbol{T}^*\,\boldsymbol{B}_2\boldsymbol{\Omega}, \boldsymbol{B}_{1\gamma}\boldsymbol{\Omega}) \\ &= \lim_{\gamma\to 0}((\boldsymbol{T}^*\times \boldsymbol{I}_{\varrho_1})\,\boldsymbol{B}_{1\gamma}^\dagger\boldsymbol{B}_2\boldsymbol{\Omega}, \bar{\boldsymbol{R}}_{\varrho_1}\boldsymbol{\Omega})\end{aligned} \tag{7.13}$$

according to (7.12)

$$\boldsymbol{B}_{1\gamma}^\dagger\boldsymbol{B}_2\,\Omega=\int \bar{f}_\gamma(t)\,(\alpha_t(\boldsymbol{B}_1)^\dagger\boldsymbol{B}_2+\varepsilon(\bar{\varrho}_1,\varrho_2)\,\boldsymbol{B}_2\,\alpha_t(\boldsymbol{B}_1)^\dagger)\,dt\boldsymbol{\Omega}\,.$$

The integrand vanishes for $|t|<r$ according to (4.12). Thus

$$\|\boldsymbol{B}_{1\gamma}^\dagger\boldsymbol{B}_2\boldsymbol{\Omega}\|<2d(\varrho_1)^{1/2}\,\|\boldsymbol{B}_1\|\,\|\boldsymbol{B}_2\|\int_{|t|>r}|f_\gamma(t)|\,dt\,. \tag{7.14}$$

The choice of $\tilde{f}$ depends on none of the symbols on the left hand side of (7.11) and, from the construction of f_γ, one sees that $\lim_{\gamma\to 0}\int_{|t|>r}|f_\gamma(t)|\,dt$ decreases faster than any inverse power of r. Combining this information with (7.14), (7.13) and remembering that $\|\bar{\boldsymbol{R}}_{\varrho_1}\|=d(\varrho_1)^{1/2}$ we get (7.11).

7.3. **Lemma.** *Let*

$$\boldsymbol{B}_i=\{\varrho_i, B_i\}\in\mathscr{B}(\mathcal{O}_i), \qquad i=1,\dots,4$$

where $\mathcal{O}_1\cup\mathcal{O}_3$ *is spacelike to* $\mathcal{O}_2\cup\mathcal{O}_4$, $\boldsymbol{T}=(\varrho_3\varrho_4|T|\varrho_1\varrho_2)$. *Set*

$$r=\sup\{|t|/\mathcal{O}_1\cup\mathcal{O}_3+(t,\mathbf{0})\subset(\mathcal{O}_2\cup\mathcal{O}_4)'\}\,.$$

a) *If* $\varrho_1=\varrho_3\in\Delta_s$ *then*

$$\begin{aligned}&|(\boldsymbol{B}_4\boldsymbol{B}_3\boldsymbol{\Omega}, \boldsymbol{T}\boldsymbol{B}_2\boldsymbol{B}_1\boldsymbol{\Omega})-(\boldsymbol{B}_3\boldsymbol{\Omega}, \boldsymbol{B}_1\boldsymbol{\Omega})\,(\boldsymbol{B}_4\boldsymbol{\Omega}, \phi_1(\boldsymbol{T})\,\boldsymbol{B}_2\boldsymbol{\Omega})| \\ &\qquad < a_n r^{-n}\,\|\boldsymbol{T}\|\prod_{i=1}^{4}\|\boldsymbol{B}_i\|\,.\end{aligned} \tag{7.15}$$

for any $n>0$ *with* a_n *depending only on the charges of the* $\boldsymbol{B}_i$. *Here* ϕ_1 *is the left inverse of* ϱ_1 *and* $\phi_1(\boldsymbol{T})=(\varrho_4|\phi_1(T)|\varrho_2)$.

b) *If* ϱ_1 *and* ϱ_3 *from* Δ_s *are inequivalent then similarly*

$$|(\boldsymbol{B}_4\boldsymbol{B}_3\boldsymbol{\Omega}, \boldsymbol{T}\boldsymbol{B}_2\boldsymbol{B}_1\boldsymbol{\Omega})|<a_n r^{-n}\,\|\boldsymbol{T}\|\prod_{i=1}^{4}\|\boldsymbol{B}_i\|\,. \tag{7.15*}$$

Proof. Shifting B_3 we get, using (4.12), (4.17) and (4.10), (4.11)

$$(B_4 B_3 \boldsymbol{\Omega}, \boldsymbol{T} B_2 B_1 \boldsymbol{\Omega})$$
$$= \left(B_4 \boldsymbol{\Omega}, (\boldsymbol{I}_{\varrho_4} \times \bar{\boldsymbol{R}}_{\varrho_3}^*) (\varepsilon(\varrho_4, \varrho_3)^* \times \boldsymbol{I}_{\bar{\varrho}_3}) (\boldsymbol{T} \times \boldsymbol{I}_{\bar{\varrho}_3}) (\boldsymbol{I}_{\varrho_1} \times \varepsilon(\bar{\varrho}_3, \varrho_2)) B_2 B_3^\dagger B_1 \boldsymbol{\Omega}\right).$$

We split the vector $B_3^\dagger B_1 \boldsymbol{\Omega}$ into its component in the subspace of Poincaré invariant vectors (projector E_0) and the orthogonal component. The contribution of the latter may be rearranged in a form to which Lemma 7.2 applies with $B_2^\dagger B_4$ replacing B_2, $B_3^\dagger B_1$ replacing B_1 in (7.11). The complicated intertwiner in the middle is composed of ε-factors, $\bar{\boldsymbol{R}}_{\varrho_3}^*$ and $\boldsymbol{T}$; thus its norm is bounded by $d(\varrho_3)^{1/2} \|\boldsymbol{T}\|$. The whole contribution containing $I - E_0$ is therefore bounded by the estimates on the right hand side of (7.14). We still have to compute the contribution containing E_0. If $\varrho_1, \varrho_3 \in \Delta_s$ are inequivalent then $E_0 = 0$ (Theorem 3.1). This proves part b) of the lemma. If $\varrho_3 = \varrho_1 \in \Delta_s$ then E_0 is 1-dimensional and the normalized invariant vector is given by $d(\varrho_1)^{-1/2} \bar{\boldsymbol{R}}_{\varrho_1} \boldsymbol{\Omega}$ [see Lemma 2.2c and (3.8)]. Therefore this contribution factorizes as

$$(B_4 \boldsymbol{\Omega}, \boldsymbol{T}' B_2 \boldsymbol{\Omega}) (\bar{\boldsymbol{R}}_{\varrho_1} \boldsymbol{\Omega}, B_3^\dagger B_1 \boldsymbol{\Omega})$$
$$\begin{aligned} \boldsymbol{T}' = d(\varrho_1)^{-1} (\boldsymbol{I}_{\varrho_4} \times \bar{\boldsymbol{R}}_{\varrho_1}^*) \circ (\varepsilon(\varrho_1, \varrho_4) \times \boldsymbol{I}_{\bar{\varrho}_1}) \\ \circ (\boldsymbol{T} \times \boldsymbol{I}_{\bar{\varrho}_1}) \circ (\boldsymbol{I}_{\varrho_1} \times \varepsilon(\bar{\varrho}_1, \varrho_2)) \circ (\bar{\boldsymbol{R}}_{\varrho_1} \times \boldsymbol{I}_{\varrho_2}). \end{aligned} \tag{7.16}$$

The second factor in (7.16) equals $(B_3 \boldsymbol{\Omega}, B_1 \boldsymbol{\Omega})$. It remains to show that $\boldsymbol{T}' = \phi_1(\boldsymbol{T})$. For this purpose it is convenient to change the order in some of the cross products using [1; Theorem 4.3]. Thus

$$\boldsymbol{T} \times \boldsymbol{I}_{\bar{\varrho}_1} = \varepsilon(\bar{\varrho}_1, \varrho_1 \varrho_4) \circ (\boldsymbol{I}_{\bar{\varrho}_1} \times \boldsymbol{T}) \circ \varepsilon(\varrho_1 \varrho_2, \bar{\varrho}_1),$$
$$\boldsymbol{I}_{\varrho_4} \times \bar{\boldsymbol{R}}_{\varrho_1}^* = (\bar{\boldsymbol{R}}_{\varrho_1}^* \times \boldsymbol{I}_{\varrho_4}) \circ \varepsilon(\varrho_1 \bar{\varrho}_1, \varrho_4).$$

The product of several ε-factors always leads to one resultant ε which corresponds to the overall permutation made [1; Theorem 4.2]. Therefore we can write

$$\begin{aligned} \boldsymbol{T}' = d(\varrho_1)^{-1} (\bar{\boldsymbol{R}}_{\varrho_1}^* \times \boldsymbol{I}_{\varrho_4}) \circ (\varepsilon(\bar{\varrho}_1, \varrho_1) \times \boldsymbol{I}_{\varrho_4}) \circ (\boldsymbol{I}_{\bar{\varrho}_1} \times \boldsymbol{T}) \\ \circ (\varepsilon(\varrho_1, \bar{\varrho}_1) \times \boldsymbol{I}_{\varrho_2}) \circ (\bar{\boldsymbol{R}}_{\varrho_1} \times \boldsymbol{I}_{\varrho_2}). \end{aligned}$$

Using (3.7) and (3.14) we get

$$\begin{aligned} \boldsymbol{T}' &= d(\varrho_1)^{-1} (\boldsymbol{R}_{\varrho_1}^* \times \boldsymbol{I}_{\varrho_4}) \circ (\boldsymbol{I}_{\bar{\varrho}_1} \times \boldsymbol{T}) \circ (\boldsymbol{R}_{\varrho_1} \times \boldsymbol{I}_{\varrho_2}) \\ &= d(\varrho_1)^{-1} (\varrho_4 | R_{\varrho_1}^* \bar{\varrho}_1(T) R_{\varrho_1} | \varrho_2) = (\varrho_4 | \phi_1(T) | \varrho_2) = \phi_1(\boldsymbol{T}). \end{aligned}$$

This completes the proof.

This last lemma and the consistent use of single particle states with disjoint supports in velocity space allow us to derive the metric of scattering states without having to prove cluster properties of general truncated vacuum expectation values of elements of the field bundle.

7.4. **Proposition.** *Let*

$$\boldsymbol{\Phi}_k \in \mathscr{L}^m_\varrho, \qquad k=1,2,\ldots,n$$

and

$$\boldsymbol{\Psi}_k \in \mathscr{L}^m_\varrho, \qquad k=1,2,\ldots,n$$

be two sets of single particle states, each set having mutually disjoint supports in momentum space, then

$$(\boldsymbol{\Phi}_n \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \boldsymbol{\Phi}_1, \boldsymbol{\Psi}_n \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \boldsymbol{\Psi}_1) = \sum_{p \in \mathbb{P}^{(n)}} \omega^n_\lambda(p)\,(\boldsymbol{\Phi}_n, \boldsymbol{\Psi}_{p(n)}) \ldots (\boldsymbol{\Phi}_1, \boldsymbol{\Psi}_{p(1)})\,. \tag{7.17}$$

The summation extends over all permutations and ω^n_λ is the trace state on the permutation group multiplicative on disjoint cycles and taking the value λ^{k-1} on a k-cycle (see [1; Proposition 5.2]). *The same relation holds if "in" is replaced by "out" in* (7.17).

Proof. Let S_k and S'_k denote the momentum supports of $\boldsymbol{\Psi}_k$ and $\boldsymbol{\Phi}_k$ respectively. Since each set of state vectors has mutually disjoint supports in momentum (velocity) space, we can subdivide the supports in the following way: let $\{\mathscr{U}_k\}_{k=1,2,\ldots,n}$ be an open covering of $\mathbb{R}^3$ such that $\mathscr{U}_k \cap S_j$ is empty if $k \neq j$ and $\{\theta_k\}_{k=1,2,\ldots,n}$ a partition of the unity subordinate to the covering $\{\mathscr{U}_k\}$, then we set $\boldsymbol{\Phi}_{k,m} = \theta_m(\boldsymbol{P})\,\boldsymbol{\Phi}_k \in \mathscr{L}^m_\varrho$. Defining $\{\mathscr{U}'_k\}$ and $\{\theta'_k\}$ similarly we set $\boldsymbol{\Psi}_{j,l} = \theta'_l(\boldsymbol{P})\,\boldsymbol{\Psi}_j$. The supports of $\boldsymbol{\Phi}_{k,m}$ and $\boldsymbol{\Psi}_{j,l}$ are disjoint unless $k=l$ and $j=m$. Now $\boldsymbol{\Phi}_k = \sum_m \boldsymbol{\Phi}_{k,m}$, $\boldsymbol{\Psi}_j = \sum_l \boldsymbol{\Psi}_{j,l}$ and since (7.7) is linear in each state vector we see that it suffices to prove (7.17) for the case where $\boldsymbol{\Phi}_k$ may be paired off with $\boldsymbol{\Psi}_{p(k)}$ so that the sets $S_k \cup S_{p(k)}$ are mutually disjoint. Now picking $B_k(t) \equiv B^{(k)}_{f_k}(t)$ and $C_k(t) \equiv C^{(k)}_{\tilde{g}_k}(t)$ with $\boldsymbol{\Psi}_k = B_k(t)\,\boldsymbol{\Omega}$ and $\boldsymbol{\Phi}_k = C_k(t)\,\boldsymbol{\Omega}$ and taking the supports of $\tilde{f}_k$ and $\tilde{g}_k$ in a sufficiently small neighborhood of S_k and S'_k respectively, $C_k(t)$ and $B_{p(k)}(t)$ will be localized spacelike to $B_{p(j)}(t)$ and $C_j(t)$ for $j \neq k$ up to terms which decrease in norm faster than any inverse power of $|t|$. The left hand side of (7.17) is the limit of

$$(C_n(t) \ldots C_1(t)\,\boldsymbol{\Omega}, B_n(t) \ldots B_1(t)\,\boldsymbol{\Omega}) \quad \text{as} \quad t \to -\infty\,.$$

Using (7.6) and applying Lemma 7.2 repeatedly we have

$$\begin{aligned}
&\lim_{|t|\to\infty} (C_n(t) \ldots C_1(t)\,\boldsymbol{\Omega}, B_n(t) \ldots B_1(t)\,\boldsymbol{\Omega}) \\
&\quad = \lim_{|t|\to\infty} (C_n(t) \ldots C_1(t)\,\boldsymbol{\Omega}, \varepsilon^{(n)}_\varrho(p)\, B_{p(n)}(t) \ldots B_{p(1)}(t)\,\boldsymbol{\Omega}) \\
&= (C_n(t)\boldsymbol{\Omega}, \phi^{n-1}(\varepsilon^{(n)}_\varrho(p))\, B_{p(n)}(t)\boldsymbol{\Omega}) \ldots (C_1(t)\boldsymbol{\Omega}, B_{p(1)}(t)\boldsymbol{\Omega}) \\
&= \omega^n_\lambda(p)\,(\boldsymbol{\Phi}_n, \boldsymbol{\Psi}_{p(n)}) \ldots (\boldsymbol{\Phi}_1, \boldsymbol{\Psi}_{p(1)})
\end{aligned}$$

where we have used [1; Proposition 5.2] to deduce the last equality. If q is a permutation distinct from p we can find at least one k such that the momentum supports of $\boldsymbol{\Phi}_k$ and $\boldsymbol{\Psi}_{q(k)}$ are disjoint giving $(\boldsymbol{\Phi}_k, \boldsymbol{\Psi}_{q(k)})=0$. Therefore the sum on the right hand side of (7.17) reduces in the case considered to the one term computed above and the proposition is proved.

We may use this result to extend the definition of the asymptotic products to arbitrary particle configurations. To this end we introduce a variant of the usual tensor power of a Hilbert space which takes the statistics parameter λ into account and reduces if $\lambda=+1$ or -1 to the usual totally symmetric or totally antisymmetric tensor power respectively. Let $\mathscr{K}^{\otimes n}$ be the n-th tensor power of a Hilbert space $\mathscr{K}$; it carries a unitary representation $\varepsilon^{(n)}$ of $\mathbb{P}^{(n)}$ defined by

$$\varepsilon^{(n)}(p)\,(\Psi_n \otimes \cdots \otimes \Psi_1) = \Psi_{p^{-1}(n)} \otimes \cdots \otimes \Psi_{p^{-1}(1)}\,.$$

The operator

$$M_\lambda = \sum_{p \in \mathbb{P}^{(n)}} \omega_\lambda^n(p)\,\varepsilon^{(n)}(p) \tag{7.18}$$

is positive semidefinite. If E_λ denotes the projection onto the range of M_λ then we define $\mathscr{K}^{\otimes_\lambda n}$ to be the Hilbert space spanned by the vectors

$$\Psi_n \otimes_\lambda \cdots \otimes_\lambda \Psi_1 = E_\lambda \Psi_n \otimes \cdots \otimes \Psi_1 \tag{7.19}$$

with the scalar product

$$(\Phi, \Psi)_{\mathscr{K}^{\otimes_\lambda n}} = (\Phi, M_\lambda \Psi)_{\mathscr{K}^{\otimes n}}\,. \tag{7.20}$$

If U is a unitary operator on $\mathscr{K}$ then we define $U^{\otimes_\lambda n}$ by

$$U^{\otimes_\lambda n}(\Psi_n \otimes_\lambda \cdots \otimes_\lambda \Psi_1) = (U\Psi_n) \otimes_\lambda \cdots \otimes_\lambda (U\Psi_1) \tag{7.21}$$

and note that $U \to U^{\otimes_\lambda n}$ is a unitary representation of the unitary group of $\mathscr{K}$. We also have a unitary representation $\varepsilon_\lambda^{(n)}$ of $\mathbb{P}^{(n)}$ defined on $\mathscr{K}^{\otimes_\lambda n}$ by

$$\varepsilon_\lambda^{(n)}(p)\,(\Psi_n \otimes_\lambda \cdots \otimes_\lambda \Psi_1) = \Psi_{p^{-1}(n)} \otimes_\lambda \cdots \otimes_\lambda \Psi_{p^{-1}(1)}\,. \tag{7.22}$$

From Proposition 7.4 we see that the mapping

$$\boldsymbol{\Psi}_n \otimes_\lambda \cdots \otimes_\lambda \boldsymbol{\Psi}_1 \to \boldsymbol{\Psi}_n \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \boldsymbol{\Psi}_1 \tag{7.23}$$

is defined on a total set in $(\mathscr{H}_\varrho^m)^{\otimes_\lambda n}$ and is scalar-product preserving. It thus extends by linearity and continuity to a linear isometry V^{in} on $(\mathscr{H}_\varrho^m)^{\otimes_\lambda n}$ whose image one would naturally call the incoming n-particle states with mass m and charge ξ, the class of ϱ. $(\mathscr{H}_\varrho^m)^{\otimes_\lambda n}$ carries a unitary representation $\mathscr{U}^{\otimes_\lambda n}$ of $\mathscr{P}$ induced via (7.21) by the action on the 1-particle space and a representation $\varepsilon_\lambda^{(n)}$ of the permutation group. We have

$$V^{\text{in}}\mathscr{U}^{\otimes_\lambda n}(L)=\mathscr{U}(L)\,V^{\text{in}},\qquad L\in\mathscr{P}, \tag{7.24}$$

$$V^{\text{in}}\varepsilon_\lambda^{(n)}(p)=\varepsilon_\varrho^{(n)}(p)\,V^{\text{in}},\qquad p\in\mathbb{P}^{(n)}. \tag{7.25}$$

The tensor power $\otimes_\lambda n$ thus not only describes the metric of the scattering states but also the transformation properties of these states under the Poincaré group and the permutation group.

In the course of this contruction we have also extended the definition of the asymptotic product so that (7.17) is valid for arbitrary particle configurations. We note in passing that the technique used in Proposition 7.4 to prove (7.17) also shows that if $\boldsymbol{\Phi}_k$, $\boldsymbol{\Psi}_k$ $k=1,2,\ldots,n$ are as in Proposition 7.4 and if $\boldsymbol{S}=(\varrho^n|S|\varrho^n)$ then

$$(\boldsymbol{\Phi}_n\overset{\text{in}}{\times}\cdots\overset{\text{in}}{\times}\boldsymbol{\Phi}_1,\boldsymbol{S}\boldsymbol{\Psi}_n\overset{\text{in}}{\times}\cdots\overset{\text{in}}{\times}\boldsymbol{\Psi}_1)=\sum_{p\in\mathbb{P}^{(n)}}\mathrm{Tr}_{\varrho^n}(S\varepsilon_\varrho^{(n)}(p))\prod_{j=1}^{n}(\boldsymbol{\Phi}_j,\boldsymbol{\Psi}_{p(j)}) \tag{7.26}$$

where Tr_{ϱ^n} is the trace state on $\varrho^n(\mathfrak{A})'$ defined in [1; Proposition 6.6]. Naturally (7.26) is then valid by extension for arbitrary particle configurations. It is also easy to see what happens if the 1-particle states in (7.17) are not all taken from the same sector. Here we need Lemma 7.3b to show that a pairing of $\boldsymbol{\Phi}_k$ with $\boldsymbol{\Psi}_{p(k)}$ involving particles from different sectors does not contribute.

In discussing the structure of scattering states involving several types of particles we shall reduce the redundancy in the description as much as possible by conventions. We choose once and for all one morphism for each sector, we keep the factors $\boldsymbol{\Psi}_i$ referring to the same type of particle together and we keep the different types of particle in a fixed order. At this point it may be worthwhile to clarify what is meant by "type of particle" in this context when one has a degeneracy for some charge and mass. We may then adopt either one of two attitudes. The first is to consider the linear space spanned by all the state vectors with this charge, and mass as the single particle space of *one* type of particle. Alternatively we could split this space in some way into its irreducible components under the Poincaré group and consider each component as a different type of particle. If the second course is adopted we have to avoid considering linear combinations of state vectors corresponding to different such components in asymptotic products and we have to adopt an ordering convention as discussed above. Bearing these conventions in mind we may summarize the preceding discussion as

7.5. **Theorem.** *The Hilbert space of state vectors describing outgoing (or incoming) configurations of n_1 particles of type 1, n_2 particles of type 2 etc. is isomorphic to*

$$\mathscr{H}_1^{\otimes_{\lambda_1}n_1}\otimes\mathscr{H}_2^{\otimes_{\lambda_2}n_2}\otimes\cdots.$$

The next (and last) topic to be discussed is the definition of transition probabilities and their relation to transition amplitudes. Let us use the term "configuration of particles" to mean a set of single particle states and let us abbreviate such a configuration by one letter α. Picking for each single particle state entering into α a normalized state vector $\boldsymbol{\Psi}_k$ we denote the normalized asymptotic product state vector by $\boldsymbol{\Psi}_\alpha^{\mathrm{ex}}$ and the corresponding state by $\omega_\alpha^{\mathrm{ex}}$ ("ex" standing for either "in" or "out"). The fibre of $\boldsymbol{\Psi}_\alpha^{\mathrm{ex}}$ will be denoted by ϱ_α.

Our task is now to relate collision cross-sections to the mathematical objects $\boldsymbol{\Psi}_\alpha^{\mathrm{ex}}, \omega_\alpha^{\mathrm{ex}}$. In a collision experiment one prepares a state which (ideally) corresponds to the information that the particle configuration α existed at $t \to -\infty$. This information results from an (optimal) study of the motion of each individual particle entering into α at a time before the collision when the particles are still far separated. It is natural to identify the physical state so prepared with the state $\omega_\alpha^{\mathrm{in}}$ in our formalism. In field theoretical treatments of collision theory involving only particles with simple charges (no parastatistics) this identification is not questioned. There, however, $\omega_\alpha^{\mathrm{in}}$ is a pure state whereas in the general case (non-simple charges) $\omega_\alpha^{\mathrm{in}}$ is a mixture and every pure component of it describes the same asymptotic motion of the individual particles at $t \to -\infty$. Therefore in making this identification one asserts that preparing the configuration by means of uncorrelated sources for the individual particles produces the specific mixture $\omega_\alpha^{\mathrm{in}}$ (uniquely determined by α) in which the various pure components, corresponding to the same configuration, occur with precisely specified weights ("the natural weights"). This assertion should be checked by discussing a realistic experimental arrangement for preparing the state but such a study falls outside the scope of this paper.

Given the state $\omega_\alpha^{\mathrm{in}}$ one is interested in the probability of finding the configuration β at $t \to +\infty$. This may be called the "transition probability $w_{\beta\alpha}$ from the configuration α to the configuration β" but it would be somewhat misleading to call it the transition probability from the "initial state $\omega_\alpha^{\mathrm{in}}$" to the "final state $\omega_\beta^{\mathrm{out}}$" since specifying an asymptotic configuration does not (in general) determine a unique state and the experimental arrangement does not treat the configurations α and β on an equal footing. The particle sources determine a *state* ($\omega_\alpha^{\mathrm{in}}$); the final detector arrangement determines a *projection* E_β^{out} selecting the configuration β. According to the general rules of quantum physics

$$w_{\beta\alpha} = \omega_\alpha^{\mathrm{in}}(E_\beta^{\mathrm{out}})\,. \tag{7.27}$$

How is E_β^{out} expressed in our mathematical frame? If we idealize the detector arrangement as an asymptotic measurement at $t \to +\infty$ it is clear that E_β^{out} cannot belong to the quasilocal algebra $\mathfrak{A}$ but may be regarded as the limit of a sequence in $\mathfrak{A}$ which converges in the weak topology induced by all the states of interest. It is convenient then to consider the following "universal" representation of $\mathfrak{A}$: choose in each class $\xi \in \Delta_s/\mathcal{I}$ one morphism ϱ_ξ and define π as the direct sum of all the representations

$$\pi = \bigoplus_{\xi} \varrho_\xi . \tag{7.28}$$

All states of interest may be described as density matrices in this representation (positive trace class operators of trace one in the representation space $\mathcal{H}_\pi$). Thus E_β^{out} can be represented as the projection on the subspace of $\mathcal{H}_\pi$ spanned by all the state vectors of pure states corresponding to the configuration β at $t \to +\infty$. It is thus the "support projection" [19] of $\omega_\beta^{\text{out}}$, i.e. the smallest projection in $\pi(\mathfrak{A})''$ satisfying

$$\omega_\beta^{\text{out}}(E_\beta^{\text{out}}) = 1 . \tag{7.29}$$

Thus E_β^{out} is uniquely determined by $\omega_\beta^{\text{out}}$ and $w_{\beta\alpha}$ by the two states $\omega_\alpha^{\text{in}}, \omega_\beta^{\text{out}}$.

To make this explicit let us discuss the decomposition of $\omega_\alpha^{\text{ex}}$. The representation ϱ_α (in which $\omega_\alpha^{\text{ex}}$ is a vector state) will in general contain a representation ϱ_ξ with a multiplicity l_ξ. We have for such a ξ a basis of l_ξ orthogonal isometric intertwiners $\boldsymbol{T}_i^\xi = (\varrho_\alpha | T_i^\xi | \varrho_\xi)$:

$$\boldsymbol{T}_i^{\xi *} \circ \boldsymbol{T}_j^\xi = \delta_{ij} \boldsymbol{I}_{\varrho_\xi} ; \qquad i, j = 1, 2, \ldots, l_\xi , \tag{7.30}$$

$$\sum_\xi \sum_{i=1}^{l_\xi} \boldsymbol{T}_i^\xi \circ \boldsymbol{T}_i^{\xi *} = \boldsymbol{I}_{\varrho_\alpha} . \tag{7.31}$$

We know that l_ξ is finite and differs from zero only for a finite number of charges so that the sum over ξ in (7.31) contains only a finite number of non-vanishing terms. We have, by (7.31)

$$\omega_\alpha^{\text{ex}}(A) = \left(\boldsymbol{\Psi}_\alpha^{\text{ex}}, \varrho_\alpha(A) \sum_{\xi, i} \boldsymbol{T}_i^\xi \circ \boldsymbol{T}_i^{\xi *} \boldsymbol{\Psi}_\alpha^{\text{ex}}\right) = \sum_{\xi, i} (\boldsymbol{\Psi}_i^\xi, \varrho_\xi(A) \boldsymbol{\Psi}_i^\xi) , \tag{7.32}$$

where

$$\boldsymbol{\Psi}_i^\xi = \boldsymbol{T}_i^{\xi *} \boldsymbol{\Psi}_\alpha^{\text{ex}} \tag{7.33}$$

may be considered as a vector in (the ξ-component of) the representation π defined in (7.28).

7.6. **Lemma.** *Let the single particle states entering in α be mutually orthogonal. Then, with $\boldsymbol{\Psi}_i^\xi$ defined by* (7.33)

$$(\boldsymbol{\Psi}_i^\xi, \boldsymbol{\Psi}_j^\xi) = d(\xi)\, d(\varrho_\alpha)^{-1} \delta_{ij} . \tag{7.34}$$

Proof. We have

$$(\boldsymbol{\Psi}_i^\xi, \boldsymbol{\Psi}_j^\xi) = (\boldsymbol{\Psi}_\alpha^{\text{ex}}, \boldsymbol{T}_i^\xi \circ \boldsymbol{T}_j^{\xi *} \boldsymbol{\Psi}_\alpha^{\text{ex}}) .$$

We have computed expressions of this form in (7.26). If the single particle states in (7.26) are mutually orthogonal and $\boldsymbol{\Phi}_k = \boldsymbol{\Psi}_k$ then only the term where p is the identity permutation contributes. This holds equally well if the single particle states in (7.26) do not all have the same charge. Hence

$$(\boldsymbol{\Psi}_i^\xi, \boldsymbol{\Psi}_j^\xi) = \mathrm{Tr}_{\varrho_\alpha}(T_i^\xi T_j^{\xi *}) .$$

The part of the commutant of $\varrho_a(\mathfrak{A})$ corresponding to charge ξ is isomorphic to the algebra of all $l_\xi \times l_\xi$ matrices in such a way that $T_i^\xi T_j^{\xi *}$ corresponds to the matrix unit whose only non-vanishing entry is in the (j, i) position. Hence [1; Proposition 6.6] gives (7.34) as required.

7.7. **Theorem.** *Let the single particle states entering in the configuration α be mutually orthogonal. Then the state $\omega_\alpha^{\mathrm{ex}}$ corresponds to a density matrix W_α^{ex} in the representation π which is given by*

$$W_\alpha^{\mathrm{ex}} = \sum_\xi d(\xi)\, d(\varrho_\alpha)^{-1} P_\alpha^{\mathrm{ex}}(\xi) . \tag{7.35}$$

The summation extends over those charges which are contained in ϱ_α with multiplicity $l_\xi \neq 0$. $P_\alpha^{\mathrm{ex}}(\xi)$ is the projection on the l_ξ-dimensional space spanned by the vectors $\boldsymbol{\Psi}_i^\xi$ of Eq. (7.33). The support of $\omega_\alpha^{\mathrm{ex}}$ is accordingly represented by

$$E_\alpha^{\mathrm{ex}} = \sum_\xi P_\alpha^{\mathrm{ex}}(\xi) \tag{7.36}$$

and the transition probability is given by

$$w_{\beta\alpha} = \sum_\xi d(\xi)\, d(\varrho_\alpha)^{-1}\, \mathrm{Tr}(P_\alpha^{\mathrm{in}}(\xi)\, P_\alpha^{\mathrm{out}}(\xi)) = \sum_\xi d(\xi)\, d(\varrho_\alpha)^{-1} \sum_{i,j} |A_{\beta j;\alpha i}^\xi|^2 . \tag{7.37}$$

The transition amplitudes may be written as

$$A_{\beta j;\alpha i}^\xi = d(\varrho_\beta)^{1/2}\, d(\varrho_\alpha)^{1/2}\, d(\xi)^{-1}\, (\boldsymbol{\Psi}_\beta^{\mathrm{out}}, \boldsymbol{T}_{ji}^\xi \boldsymbol{\Psi}_\alpha^{\mathrm{in}}) , \tag{7.38}$$

where

$$\boldsymbol{T}_{ji}^\xi = (\varrho_\beta | T_j'^{\xi} | \varrho_\xi) \circ (\varrho_\xi | T_i^{\xi *} | \varrho_\alpha) . \tag{7.39}$$

Thus the amplitudes (7.38) play the rôle of S-matrix elements. However, one has to bear in mind that $|A_{\beta j;\alpha i}^\xi|^2$ cannot be measured in any experiment if either l_ξ or l'_ξ is greater than one[20].

[20] Note that some l_ξ will be >1 unless $d(\xi)=1$ for all $\xi \in \Delta_s/\mathscr{I}$. In fact, let $\varrho \in \Delta_s$ appear only once in the decomposition of $\varrho\bar{\varrho}\varrho$ then the subspace of intertwiners from ϱ to $\varrho\bar{\varrho}\varrho$ is one-dimensional. Since the intertwiners $\bar{R}$ and $\varrho(R)$ (where $\bar{R}$ and R are defined as in Section 3) both belong to this space, they must be proportional and by (3.6), (3.8) we see that $d(\varrho)=1$.

Appendix

The question of whether a sector can have infinite statistics ($\lambda=0$) is still open. However if any such sectors exist, their properties would be very different from those of the sectors with finite statistics discussed in this paper. We shall show here that a covariant sector with positive energy having a conjugate sector in the sense of Theorem 3.1 must have $\lambda \neq 0$.

We shall use the following lemma.

A.1. **Lemma.** *Let ϱ be irreducible with infinite statistics and let $\boldsymbol{W}=(\varrho\varrho_1|W|\iota)$ and $\boldsymbol{V}=(\varrho_2\varrho|V|\iota)$ be intertwiners then $\varrho_2(W)^* V=0$.*

Proof. It suffices to consider the case $V \neq 0$; let ϕ be the left inverse of ϱ defined by $V^* V\phi(A)=V^*\varrho_2(A)V$, $A \in \mathfrak{A}$, and set $\boldsymbol{S}=(\varrho_2|S|\varrho_1)$ $=(\boldsymbol{I}_{\varrho_2}\times \boldsymbol{W}^*)\circ(\boldsymbol{V}\times \boldsymbol{I}_{\varrho_1})$. Then $S^*S=S^*\varrho_2(W)^*V=\varrho_1(W)^*S^*V$, hence

$$S^*S=\varrho_1(W)^*V^*V\phi(W). \tag{A.1}$$

However by [1; Theorems 4.2 and 4.3],

$$W=\varepsilon(\varrho,\varrho\varrho_1)\,\varrho(W)=\varrho(\varepsilon(\varrho,\varrho_1))\,\varepsilon_\varrho\varrho(W)$$

so $\phi(W)=\varepsilon(\varrho,\varrho_1)\,\phi(\varepsilon_\varrho)\,W=0$ since ϱ has infinite statistics. Thus from (A.1), $S^*S=0$ which implies $S=\varrho_2(W)^*V=0$ as required.

A.2. **Proposition.** *Let ϱ be an irreducible, covariant, localized morphism with positive energy and infinite statistics, then there is no covariant localized morphism $\bar{\varrho}$ such that $\varrho\bar{\varrho}$ contains the vacuum representation.*

Proof. Let $L \to \mathscr{U}_\varrho(L)$ be a continuous, unitary representation of the Poincaré group with positive energy and satisfying (1.1) and let $L \to X_L(\varrho)$ denote the corresponding cocycle defined in (2.3). Let $\boldsymbol{W}=(\varrho\bar{\varrho}|W|\iota)$; it will suffice to show that $W=0$. Now $\alpha_L(W)$ intertwines ι and $\varrho_L\bar{\varrho}_L$ and $X_L(\varrho)$ intertwines ϱ and ϱ_L. Hence $\varepsilon(\varrho,\bar{\varrho}_L)\,X_L(\varrho)^{-1}\alpha_L(W)$ intertwines from ι to $\bar{\varrho}_L\varrho$. Thus applying Lemma A.1 with $\varrho_1=\bar{\varrho}$ and $\varrho_2=\bar{\varrho}_L$ we have

$$\bar{\varrho}_L(W)^*\varepsilon(\varrho,\bar{\varrho}_L)\,X_L(\varrho)^{-1}\alpha_L(W)=0\,.$$

However by [1; Theorems 4.2 and 4.3]

$$\bar{\varrho}_L(W)=\varepsilon(\varrho\bar{\varrho},\bar{\varrho}_L)\,W=\varepsilon(\varrho,\bar{\varrho}_L)\,\varrho(\varepsilon(\bar{\varrho},\bar{\varrho}_L))\,W.$$

Hence

$$W^*\varrho(\varepsilon(\bar{\varrho}_L,\bar{\varrho}))\,X_L(\varrho)^{-1}\alpha_L(W)=0\,,\qquad L\in\mathscr{P}. \tag{A.2}$$

Now by (2.3),

$$(W\Omega,\mathscr{U}_\varrho(x)\,W\Omega)=(W\Omega,X_x(\varrho)^{-1}\alpha_x(W)\,\Omega)\,.$$

However if $\bar{\varrho}$ is localized in $\mathcal{O}$ and $\mathcal{O}+x\subset\mathcal{O}'$ so that $\bar{\varrho}$ and $\bar{\varrho}_x$ are spacelike, we see from (A.2) that $(W\Omega,\mathcal{U}_\varrho(x)\,W\Omega)=0$. Since the function

$$x\to(W\Omega,\mathcal{U}_\varrho(x)\,W\Omega)$$

is the boundary value of a function analytic in $\mathbb{R}^4+i\mathrm{V}^+$ it must vanish identically. Thus $W\Omega=0$ which implies $W=0$ as required.

We have in fact proved rather more than stated in the proposition because we do not need the full Poincaré covariance of ϱ and $\bar{\varrho}$. It would suffice if ϱ is translation covariant with positive energy and if $\bar{\varrho}\in\Delta_t$ in the sense of [1].

Of course the method of constructing a conjugate given in Section III breaks down in the case of infinite statistics. On the other hand it is less clear why one cannot define a conjugate representation using the construction given in [1; § III] but the basic trouble is that the left inverse is not unique in the case of infinite statistics. Here too the behaviour of sectors with finite statistics differs markedly from those with infinite statistics. In fact if ϱ has finite statistics

$$\omega_0(X_x(\varrho)\,A X_x(\varrho)^{-1})\to\omega_0\circ\phi(A),\qquad A\in\mathfrak{A} \tag{A.3}$$

as x tends spacelike to infinity (see [1; Theorem 3.9 b]) whereas one can show that if ϱ is irreducible and covariant with positive energy and has infinite statistics then the states $A\to\omega_0(X_x(\varrho)\,AX_x(\varrho)^{-1})$ do not converge as x tends spacelike to infinity and no limit point of these states is locally normal.

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