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Calculus on Heisenberg manifolds, by Richard Beals and Peter Greiner. Annals of Mathematics Studies, vol. 119, Princeton University Press, Princeton, N.J., 1988, x+194 pp., $52.50 (cloth), $20.00 (paper). ISBN 0-691-08501-3

To study the Laplace-Beltrami operator on a Riemannian manifold, one approximates the manifold in a neighborhood of a given point by Euclidean space and the operator by a second order constant coefficient operator which, in terms of appropriate coordinates, is just minus the Euclidean Laplacian. One can then exploit the abelian group structure of $\mathbf{R}^n$ and the Euclidean dilations. The Laplacian on $\mathbf{R}^n$ is a second order translation invariant differential operator which is homogeneous of degree 2 with respect to Euclidean dilations:

$$\Delta f(\lambda x) = \lambda^2(\Delta f)(\lambda x). \tag{1}$$

The symbol of the Euclidean Laplacian is $-|\xi|^2$, which vanishes only at 0, so one can use classical pseudodifferential operator calculus to construct a parametrix for the Laplace-Beltrami operator. Certain other operators which arise naturally do not satisfy (1). For example, the heat operator $\partial/\partial t - \Delta$ on $\mathbf{R}^{n+1}$ is not homogeneous with respect to Euclidean dilations, but it is homogeneous of degree 2 with respect to the nonisotropic dilations

$$\delta_\lambda(x,t) = (\lambda x, \lambda^2 t), \qquad x \in \mathbf{R}^n,\ t \in \mathbf{R}. \tag{2}$$

The principal symbol of the heat operator is $|\xi|^2$ which vanishes on the line $\xi = 0$, but the full symbol $|\xi|^2 + i\tau$ vanishes only at 0. Thus, although classical pseudodifferential operator calculus does not apply, a modification does—one treats the dual variable τ of t as a second order symbol, since it is homogeneous of degree 2 under the dilations (2). On $\mathbf{R}^3$ with coordinates (x,y,t), the operator

$$X^2 + Y^2, \tag{3}$$

where $X = \partial/\partial x + 2y(\partial/\partial t)$ and $Y = \partial/\partial y - 2x(\partial/\partial t)$, is also homogeneous of degree 2 under the dilations (2). The vector fields X and Y are left invariant vector fields on the three dimensional Heisenberg group H_1, so (3) is also left invariant. It is not elliptic, but it is subelliptic. Unlike the heat operator, its full symbol at the origin of H_1 vanishes on the τ axis, where τ is dual to t.

The Heisenberg group can be identified with the boundary of the Siegel domain $\Omega = \{(z,w) \in \mathbf{C}^2 : \operatorname{Im} w > |z|^2\}$ via the map $(x,y,t) \to (x+iy, i|z|^2)$. Under this identification, if Ω is equipped with a certain invariant metric, the operator (3) is $-4\operatorname{Re}\Box_b$ acting on functions on the boundary of Ω. The operator $\Box_b$ is a second order differential operator associated with the tangential Cauchy-Riemann, or $\overline{\partial}_b$, complex on the boundary of a domain in $\mathbf{C}^n$ or, more generally, on a CR manifold. It arises naturally from the study of the boundary behavior of holomorphic functions and $\overline{\partial}$

closed forms in several complex variables. The operator $\Box_b$ is not elliptic, but under appropriate hypotheses on the domain, it is subelliptic.

A second order differential operator P is called *hypoelliptic with loss of one derivative* if each first derivative of u is locally in L^2 whenever Pu is locally in L^2. The loss of one derivative is by comparison with the Laplacian (or any second order elliptic differential operator): if $\Delta u \in L^2_{\text{loc}}(U)$ for some open set U then u and its first and second derivatives are in $L^2_{\text{loc}}(U)$. The heat operator, the operator (3) and $\Box_b$ on $(0, 1)$ forms on the boundary of a strictly convex domain in $\mathbf{C}^n$, $n \geq 3$, are examples of operators which are hypoelliptic with loss of one derivative. Let P be a second order differential operator which is hypoelliptic with loss of one derivative on an $n+1$-dimensional manifold M. The goal of the book under review is to develop a complete, explicitly computable pseudodifferential operator calculus, called the calculus of *$\mathscr{V}$-pseudodifferential operators*, which contains both P and its parametrix. Instead of approximating by Euclidean space in a neighborhood of a point, the authors approximate by a two-step nilpotent group of the form $H_m \times \mathbf{R}^{n-2m}$, where H_m is a Heisenberg group of dimension $2m+1$. The approximating group is constructed from the operator P and may vary from point to point on M. The operator P is then approximated by a left invariant operator on the group, a *model operator*, and this operator has a left invariant inverse which gives a good first approximation to the parametrix for P. In the classical pseudodifferential operator calculus, the principal symbol of the composition of two operators is given by the product of the two principal symbols. This is just the Fourier transform of the convolution of the two approximating translation invariant operators. In the Beals-Greiner calculus, the principal symbol of the composition is the Fourier transform of the convolution of the approximating left invariant operators. In general, this does not correspond to an algebraic operation on the symbols, so the resulting calculus is much more complicated than the classical one.

In principle, this pseudodifferential operator calculus allows one to give an explicit construction of a parametrix for $\Box_b$ on $(0, 1)$ forms on the boundary of a bounded strictly convex domain in $\mathbf{C}^n$, $n \geq 3$, and, more generally, on (p, q) forms on a compact CR manifold satisfying condition $Y(q)$. Thus, the calculus provides a powerful tool for studying partial differential equations in several complex variables. A modification of the calculus has been used to study the heat equation for $\Box_b$ and to obtain detailed results about the heat kernel. For example, the trace of the heat kernel for $\Box_b$ has an asymptotic expansion in powers of the time t for small positive t whose coefficients are integrals of locally computable functions on M. This is an analogue of the Minakshisundaram-Pleijel **[MP]** asymptotic expansion in Riemannian geometry. In the case of a strictly pseudoconvex CR manifold with a Levi metric, there is an analogue of the McKean-Singer [**MS**] result in Riemannian geometry—the coefficients are integrals of local geometric invariants of the metric and CR structure. Larger algebras of pseudodifferential operators, containing both the classical operators and modifications of the $\mathscr{V}$-pseudodifferential operators, have been used to study the heat equation and to prove regularity results for the $\bar{\partial}$-Neumann problem. The introduction to the book gives references for these topics.

Until the publication of *Calculus on Heisenberg manifolds*, the only treatment of the Beals-Greiner calculus was their article [**BG**]. The book is very well written and it is self-contained. It makes an important topic accessible to nonexperts for the first time. The Beals-Greiner calculus should serve as a model for the development of algebras of pseudodifferential operators, with a complete calculus, suitable for handling other nonelliptic problems. In particular, it might serve as a model for a constructive method of finding parametrices for second order subelliptic differential operators which lose more than one derivative. Such operators arise naturally in a number of contexts, including the study of sums of squares of vector fields on homogeneous nilpotent Lie groups and general sums of squares of vector fields satisfying Hörmander's condition and the study of $\Box_b$ on the boundaries of weakly pseudoconvex domains. As a first step in this direction, Cummins [**C**] has developed a calculus for operators modeled on sums of squares of vector fields on three step nilpotent Lie groups.

Calculus on Heisenberg manifolds is an excellent book. In addition to giving a thorough development of the Beals-Greiner calculus, the book includes a history of symbols, kernels and $\Box_b$ in the introduction. Chapter 3 begins with a review of standard pseudodifferential operators. This review provides a very good sketch of the important features of the standard calculus (along with references for details).

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Nancy K. Stanton
University of Notre Dame

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An introduction to independence for analysts, by H. G. Dales and W. H. Woodin. London Mathematical Society Lecture Notes, vol. 115, Cambridge University Press, Cambridge, New York and Melbourne, 1987, xiii + 241 pp., $29.95. ISBN 0-521-33996-0

About sixty years ago, K. Gödel proved his famous results on the incompleteness of formal theories. The fact that under very general hypotheses, a formal mathematical theory, if consistent, cannot answer to all statements