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A NOTE ON CALDERÓN-ZYGMUND SINGULAR INTEGRAL CONVOLUTION OPERATORS

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The purpose of this note is to show that the notion of weak maximal function introduced in [**1**] (see also [**4**], where a similar notion is considered) can be used to obtain some new information on the Calderón-Zygmund singular integral convolution operator.

We will follow the notations of [**3**]. Let K be a kernel in $\mathbf{R}^n$ of class C^1 outside the origin satisfying

$$|K(x)| \leq C|x|^{-n}, \tag{1}$$

$$|\nabla K(x)| \leq C|x|^{-n-1}. \tag{2}$$

For $\varepsilon > 0$ and $f \in L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, set

$$T_\varepsilon(f)(x) = \int_{|y|\geq\varepsilon} f(x-y)K(y)\,dy$$

and

$$T(f)(x) = \lim_{\varepsilon\to 0} T_\varepsilon(f)(x), \qquad T^*(f)(x) = \sup_{\varepsilon>0} |T_\varepsilon(f)(x)|.$$

We will assume that K satisfies the usual properties ensuring that the mapping $f \mapsto T^*(f)$ is of weak type $(1,1)$ and that $T(f)(x)$ makes sense for a.e. x.

The notation $L^{1,\infty}$ will stand for the space of weak L^1 functions, and if $\varphi \in L^{1,\infty}$ and B is a ball we write

$$||\varphi||_{1,\infty}^B = \sup_{\delta>0} \delta m(\{x \in B\colon\ |\varphi(x)| > \delta\})$$

for the weak L^1 "norm" of φ on B. If $B = \mathbf{R}^n$, we simply write $||\varphi||_{1,\infty}$.

The *weak maximal function* introduced in [1] is defined for $\varphi \in L^{1,\infty}$ by

$$M_w\varphi(x) = \sup_{x\in B} \frac{||\varphi||_{1,\infty}^B}{m(B)},$$

the supremum being taken over all balls centered at x. The notation $M_w^m\varphi$ stands for the function obtained by applying m times the operator M_w, whenever this makes sense. In [**1**] it was already pointed out that for any m there is a $\varphi \in L^{1,\infty}$ such that $M_w^j\varphi \in L^{1,\infty}$ for $j = 1, \ldots, m$ but $M_w^{m+1}\varphi \notin L^{1,\infty}$. However, for $\varphi = Tf$, $f \in L^1$, the following holds:

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THEOREM. *If T is as above, $M_w^m T^* f \in L^{1,\infty}$ for all $f \in L^1$ and all $m \in N$, and $||M_w^m T^* f||_{1,\infty}$ grows as a geometric progression. Hence the same is true for Tf.*

As in [**1**], the motivation for this research is the following

QUESTION. What is the necessary and sufficient condition on a nonnegative function φ for the existence of $f \in L^1$ such that

$$\varphi \leq |Tf| \quad \text{a.e.?}$$

In other words, what is the precise description of the *magnitude* of Tf? The theorem gives a necessary condition stronger than $\varphi \in L^{1,\infty}$, namely,

$$||M_w^m \varphi||_{1,\infty} \leq C_1 C_2^m$$

but, as shown in the last section of [**1**], this condition is not sufficient (see §5 of [1] for other remarks concerning this question).

PROOF OF THE THEOREM. Let us first remark that the corresponding result with Tf replaced by the Hardy-Littlewood maximal function Mf is also true. In fact in this case something more precise is true, namely

$$M_w Mf(x) \leq CMf(x) \tag{3}$$

for some constant $C = C(n)$. This is shown in [**1**, pp. 9–10], and it is also implicit in [**2**].

In fact, our proof of the theorem will be a consequence of something similar to (3). We will show that

$$M_w T^* f \leq C\{T^* f + Mf\}. \tag{4}$$

Together with (3) this will give

$$M_w^m T^* f \leq C^m \{T^* f + Mf\}$$

(note that $M_w(\varphi + \psi) \leq 2(M_w\varphi + M_w\psi)$), which clearly implies the theorem.

In order to prove (4), fix x and let B be a ball centered at x. Let $2B$ denote the ball having the same center as x and twice the radius and set

$$f_1 = f\chi_{2B}, \qquad f_2 = f - f_1.$$

Then $T^* f \leq T^* f_1 + T^* f_2$ and

$$||T^* f||_{1,\infty}^B \leq 2(||T^* f_1||_{1,\infty}^B + ||T^* f_2||_{1,\infty}^B).$$

Since T^* satisfies a weak $(1,1)$ estimate, we have

$$\begin{aligned} ||T^* f_1||_{1,\infty}^B &\leq ||T^* f_1||_{1,\infty} \\ &\leq C||f_1||_1 = C \int_{2B} |f(y)|\, dy \leq Cm(B)Mf(x). \end{aligned} \tag{5}$$

Now we will prove that for $z \in B$

$$||T^* f_2(z)| \leq C(T^* f(x) + Mf(x)). \tag{6}$$

This implies

$$||T^* f_2||_{1,\infty}^B \leq Cm(B)(T^* f(x) + Mf(x)),$$

and together with (5) this gives

$$||T^*f||_{1,\infty}^B \le Cm(B)(T^*f(x) + Mf(x)),$$

which is (4).

For (6) we have to prove that for any $\varepsilon > r$, r being the radius of B,

$$|T_\varepsilon f_2(z)| \le C(T^*f(x) + Mf(x)). \tag{7}$$

Now

$$T_\varepsilon f_2(z) = \int_{\substack{\mathbf{R}^n \setminus 2B \\ |y-z| \ge \varepsilon}} f(y)K(z-y)\,dy.$$

If $\delta = \varepsilon + r$, it is clear that the contribution in this integral of $B_0 = \delta B/r$, the ball centered at x of radius δ, is dominated (using (1)) by

$$C\varepsilon^{-n} \int_{B_0} |f(y)|\,dy \le CMf(x).$$

It remains to estimate

$$I \overset{\text{def}}{=} \int_{|y-x|>\delta} f(y)K(z-y)\,dy.$$

This is compared with $T_\delta f(x)$ in the usual way:

$$I - T_\delta f(x) = \int_{|y-x|>\delta} f(y)\{K(z-y) - K(x-y)\}\,dy.$$

Using (2) we obtain

$$|I - T_\delta f(z)| \le C|z-x| \int_{|y-x|>\delta} |f(y)| \frac{dy}{|y-x|^{n+1}},$$

and it is well known that this is in turn dominated by $Mf(x)$. Therefore we have proved that

$$|T_\varepsilon f_2(z)| \le C(|T_\delta f(x)| + Mf(x)),$$

which yields (7) and finishes the proof of the theorem.

REMARK. S. Drury (private communication) has independently generalized some results of [**1**] proving that $M_w Tf \in L^{1,\infty}$, i.e. the case $m = 1$ of the Theorem. He replaces the condition (1) and (2) by the so-called Hörmander condition (see [**3**, p. 34, condition (2′)]).

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