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A concrete approach to division rings, by John Dauns, Research and Education in Mathematics, Vol. 2, Heldermann, Verlag, Berlin, 1982, xx + 417 pp., $78.00. ISBN 3-8853-8202-4

Skew fields, by P. X. Draxl, London Mathematical Society Lecture Note Series 81, Cambridge University Press, New York, 1982, 182 pp., $19.95. ISBN 0-5212-7274-2

In 1843 Hamilton set down the generators and relations for the algebra of quaternions. This was the first nontrivial division algebra or skew field. That is,

the quaternions are an associative, unitary, not necessarily commutative algebra for which each nonzero element has a multiplicative inverse. Since their introduction, division algebras have fascinated many mathematicians. Considering this, it is surprising that there are so few books on the subject. The books reviewed here are therefore welcome.

To begin with, let us further delineate the subject. F will always be a fixed field and every algebra will be an F algebra. If D is a division algebra, then the center $Z(D) = \{ a \in D \mid ab = ba$ for all $b \in D \}$ is a field. D is called finite dimensional if D is finite dimensional as a $Z(D)$ vector space. The theories of finite- and infinite-dimensional division algebras have very different flavors. The latter is more ring theoretic. In the former the properties of F are more closely tied to properties of the field $Z(D)$. This has meant that the study of finite-dimensional division algebras benefits from the number theoretic, Galois theoretic, geometric, and K-theoretic study of fields. Though half of Professor Dauns' book concerns the infinite-dimensional case, in this review we will confine ourselves to finite-dimensional division algebras. In fact, for us a division algebra will always be assumed to be finite dimensional.

In what follows, we intend to describe a bit of the theory of division algebras, and then show how these books fit in and what part they cover. Our tactic will be to use some solved and unsolved problems to organize the discussion. We have to apologize, at the outset, for the topics we were forced to omit.

To begin, we must define an integer, which is the most basic invariant of a division algebra. For a division algebra D, the dimension of D over $Z(D)$ is always square. The square root of this dimension is called the degree of D.

The modern study of finite-dimensional division algebras starts with the definition of the Brauer group. If D, D' are division algebras with the same center K, then $D \otimes_K D'$ is a simple algebra. By Wedderburn's theorem, $D \otimes_K D'$ is isomorphic to a matrix algebra $M_m(D'')$ for a unique division algebra D''. We use this observation to define an associative, commutative product on the isomorphism classes of division algebras with center K. That is, we set $D \cdot D' = D''$. K itself is the unit element. Alternatively, and usually, one proceeds as follows. Two simple algebras $M_m(D)$ and $M_s(D')$ are called Brauer equivalent if $D \cong D'$. We can now define an operation on the equivalence classes of simple algebras by setting $[A][B] = [A \otimes_K B]$. Since each equivalence class contains a unique division algebra, this definition yields the same structure as above. We have actually defined an abelian group because if D^o is the opposite algebra of D (same K structure but reversed multiplication), $D \otimes_K D^o \cong M_m(K)$. The group we have defined is called the Brauer group of K and is written $\mathrm{Br}(K)$. The group $\mathrm{Br}(K)$ is always torsion, since if D has degree n, then the class of D is annihilated by n. Finally, Br is a functor. If $K \to L$ is a homomorphism of fields, the map $D \to D \otimes_K L$ induces a natural group homomorphism $\mathrm{Br}(K) \to \mathrm{Br}(L)$.

The introduction of the Brauer group is extemely important because $\mathrm{Br}(K)$ has been shown to be describable using powerful mathematical machines: group cohomology, étale cohomology, and algebraic K-theory, among others. On the other hand, theorems about $\mathrm{Br}(K)$ do not necessarily answer questions

about the concrete structure of division algebras. In consequence, the theory of division algebras, at its best, involves the interplay of this Brauer group aspect and another side which can be roughly called ring theoretic. The difference between these two aspects is, perhaps, characterized by the two equivalence relations, Brauer equivalence and isomorphism. Theorems about $\text{Br}(K)$ sometimes only yield results about some simple algebra Brauer equivalent to D. The ring theoretic aspect tries to answer questions about the internal structure of D. It has greatly benefited in recent years from the theory of the universal division algebras and polynomial identities.

The first obvious question about division algebras can be roughly phrased as: what do they look alike? In this vein one asks: how do you construct division algebras? Beyond the quaternions, the first construction to consider is that of cyclic algebras. Let L/K be a cyclic Galois extension of degree n with group generated by g. Choose $0 \neq b \in L$. Form the L vector space $\oplus_{i=0}^{n-1} Lu_i$. Make this a K algebra by specifying that u_0 is the multiplicative unit, $L \cong Lu_0$, $u_i u_j = u_{i+j}$ if $i + j < n$, $(u_1)^n = b$, and $u_1 a = g(a)u_1$ if $a \in L$. The resulting algebra, called a cyclic algebra, is always simple with center K and is sometimes a division algebra. This algebra is written $\Delta(L/K, g, b)$.

When K has characteristic prime to n and contains ρ, a primitive nth root of one, the construction of cyclic algebras is even simpler. The field L above has the form $K(a^{1/n})$ where $g(a^{1/n}) = \rho a^{1/n}$. It follows that $\Delta(L/K, g, b)$ is generated by α, β satisfying $\alpha^n = a$, $\beta^n = b$, and $\alpha\beta = \rho\beta\alpha$. In this case we write the cyclic algebra as $(a, b)_{n,K}$. The quaternions are precisely the degree two cyclic algebra $(-1, 1)_{2,R}$, where R is the real field.

It is natural to ask whether every division algebra is cyclic. A highpoint in twentieth century algebra was the Brauer-Hasse-Noether proof that every division algebra with center a local or global field is cyclic. On the other hand, Albert constructed a noncyclic division algebra which is an example of the more general crossed product construction. Let L/K be a Galois extension of fields with Galois group G, and let $c: G \times G \to L^*$ be a two-cocycle. That is, the map c satisfies $g(c(h, k))c(g, hk) = c(g, h)c(gh, k)$ for all $g, h, k \in G$. Form the L vector space $\oplus_{g \in G} Lu(g)$ and make this a K algebra by setting $u(g)a = g(a)u(g)$ for $a \in L$, and $u(g)u(h) = c(g, h)u(gh)$. This algebra, written $\Delta(L/K, G, c)$, is always simple with center K and is sometimes a division algebra. If G is cyclic, $\Delta(L/K, G, c)$ can be shown to be isomorphic to a cyclic algebra as defined above. The crossed product construction is more general than cyclic algebras in another way. Not every division algebra is equivalent to a cyclic algebra, but every division algebra is Brauer equivalent to a crossed product. Even more, the crossed product construction allows one to prove that $\text{Br}(K)$ is isomorphic to the cohomology group $H^2(G, M^*)$, where M is the separable closure of K, M^* is the multiplicative group of M, and G is the Galois group of the extension M/K. For later use we note an important fact. Let D be a division algebra and G a group of order the degree of G. Then D is a crossed product with group G if and only if there is a subfield $L \subseteq D$ with L/K G-Galois.

The above ideas and facts were largely developed about 50 years ago. The study of division algebras was given new life around 15 years ago with some

work of Amitsur, which we will now describe. The facts in the above paragraph make it natural to ask whether every division algebra is isomorphic to a crossed product. A highpoint of the ring theoretic side of this theory was the proof by Amitsur that noncrossed products exist. The full theorem, as amplified by others, appears below. In this generality the theorem is not in print, but can be pieced together from [Ri, S1, and S2].

THEOREM 1. *Suppose n is an integer and p is the characteristic of F. There is a noncrossed product division F algebra of degree n if*:
  (i) *n is divisible by q^3 for any prime q, or*
  (ii) *n is divisible by q^2 for an odd prime $q \neq p$ and F does not contain a primitive qth root of one.*

Theorem 1 leaves open some obvious questions. Note first that division algebras of degrees 2 and 3 are known to be cyclic. The key remaining cases are contained in:

Question 1. If D is a division algebra of prime degree greater than 3, is D a crossed product (and therefore cyclic?)

That a division algebra D, of degree n, is not a crossed product says that D has no subfield Galois of degree n over $Z(D)$. In the case of n composite, it is natural to ask more elementary questions about the subfields of D. It is known that D must have subfields which are of degree n over $Z(D)$, and n is the greatest possible such degree. If q^m is the highest power of the prime q dividing n, D has a subfield of degree q^m over $Z(D)$. The following, however, is not known and is of interest.

Question 2. If D has degree q^m does D have a subfield of degree less than q^m over $Z(D)$?

One aspect of Amitsur's counterexample was the proof that certain crossed products with group G were not crossed products with respect to other groups H. On the other hand, there are some isolated known cases where the opposite is true. For example, certain crossed products with the dihedral group are also cyclic [RS]. The general question could be phrased:

Question 3. For which groups G and H is every G crossed product a crossed product with respect to H?

An example of the importance of Question 3 is its relation with Question 1. For instance, suppose G is a nonabelian group of order 20, and some G crossed product is not a crossed product with an abelian group. Then there is a noncyclic division algebra of degee 5.

Knowing that some division algebras are not cyclic, one can then ask whether all division algebras may be in some way "as good as" cyclic. The algebras $(a, b)_{n,K}$ seems particularly nice because their structure is determined by two independent parameters, "a" and "b". This notion of being determined by independent parameters can be made precise, but is easier to consider the equivalent lifting property, which we now describe. If T is a local ring , $M \subseteq T$ is the maximal ideal, and $K = T/M$, then $D = (a, b)_{n,K}$ "lifts" to T. That is, there is a rank n T algebra, A, with $A/MA \cong D$. The idea is that to lift D one merely has to choose preimages for a and b. If D were arbitrary of prime degree, then D always lifts in this way [S3]. One is led to ask:

Question 4. Does every division algebra have this lifting property?

There is another aspect of Theorem 1 too technical to fully explain here. The actual noncrossed product division algebras constructed in Theorem 1 are the so called generic division algebras $UD(F, n, r)$. These algebras and their centers have been the subject of much recent research. We refer the reader to, for example, [**J**, **P**, **F**, and **S3**].

On the Brauer group side of things, Merkuriev and Suslin have recently proved [**MS**] the truly remarkable result we are about to describe. In [**Mi**] Milnor defined the functor K_2. In the case of K_2 of a field K, $K_2(K)$ is generated by all "symbols" $\{a, b\}$ where $0 \neq a, b \in K$, subject to the relations that $\{a, b\}$ is bimultiplicative and $\{a, 1 - a\} = 1$. Assume K contains a primitive nth root of one. It is then shown in [**Mi**] that the map $K_2(K) \to Br(K)$, defined by $\{a, b\} \to (a, b)_{n, K}$, is well defined. The result of Merkuriev and Suslin is:

THEOREM 2. *Suppose K contains a primitive nth root of one. Then the above map induces an isomorphism $K_2(K)/nK_2(K) \cong Br(K)_n$, where $Br(K)_n$ is the set of elements of order dividing n. In particular, every division algebra of exponent n and center K is Brauer equivalent to a product of cyclic algebras of degree n.*

As with Amitsur's result, Theorem 2 has spawned a series of important questions, and more should arise with time. The first, and obvious one, concerns the restriction placed on K.

Question 5. Is there a version or versions of Theorem 2 true for all fields?

Several possible answers come to mind. Of course, one can ask whether any division algebra is Brauer equivalent to a product of cyclics. A favorite possible answer of this reviewer is one related to the lifting question mentioned above. The Brauer group has been generalized to a functor with domain the class of all commutative rings (even all schemes). Suppose F contains all possible roots of one, and T is a local F algebra with maximal ideal M. An easy consequence of Theorem 2 is that the natural map $Br(T) \to Br(T/M)$ is a surjection. One can now ask whether this is a surjection for all F. Very recently, Merkuriev showed that any odd order element of $Br(T/M)$ is in the image of $Br(T)$ [**M1**].

Another outstanding issue left open by Theorem 2 concerns the number of the cyclics mentioned there.

Question 6. If D is a division algebra, what is the minimal number of cyclic algebras needed in order to write the class of D as a product of cyclics?

Little is known about Question 6. The best results to date consider the question of whether division algebras are isomorphic to products of cyclics (see e.g. [**R1**]).

Finally, we consider the group structure of $Br(K)$. If K contains all possible roots of one, then Theorem 2 can be used to show that $Br(K)$ is a divisible group. For general K one can ask:

Question 7. What is the abelian group structure of $Br(K)$?

Examples are known (e.g. [**FS**]) which show that the group $Br(K)$ can be far from divisible. On the other hand, Merkuriev has shown that not all abelian groups can be realized as the Brauer group of some field K. In part, Merkuriev

showed that a 3-group, which is the Brauer group of a field, must contain a nonzero divisible group [M2].

We have yet to explicitly mention a question that some will think should have come first. Namely, can we compute Br(K)? More generally can we say interesting things about division algebras with special centers? Relatively complete knowledge is available concerning division algebas over local and global fields. The Brauer groups of rational function fields over local and global fields have been computed (see [FS] and the references there). However, little else is known about division algebras over these fields or other relatively "nice" fields. For example, assume K has transcendence degree 2 over F and F is algebraically closed. Suppose D is a division algebra with center K, and the order in the Brauer group of the class of D has the form $2^m 3^n$. Then D has degree $2^m 3^n$ [A]. Also, if D has degree 2, 3, or 4, then D is cyclic. Naturally, one asks:

Question 8. Does every division algebra D with center K as above have order equal to its degree? Is every such D cyclic?

A great deal of the progress on the Brauer group side, including Theorem 2, is based on the corestriction map. If $K \subseteq L$ is a finite extension of fields, there is a natural homomorphism Br(L) → Br(K) which is called the corestriction or transfer. Intuitively, the corestriction is the map induced by the usual norm map from L to K. The precise definition of the corestriction is fairly technical, and in some ways this map is not well understood. Though it is hard to phrase an explicit question, it seems that future work in Brauer groups will continue to study and use the corestriction.

There are various special pieces of the theory of division algebras which have their own peculiar rich subtleties. A p-algebra, for p a prime, is a simple algebra of p power degree and characteristic p. There is a theory of p-algebras, started by Albert, in which the Frobenius map and purely inseparable extensions play a big role. A peculiar feature of p-algebras is that the tensor product of cyclic p-algebras is cyclic again. This has meant that it is harder to construct noncyclic p-algebras. Such a construction was first made in [AS] (this reference in Draxl's book is mistaken).

An involution of a division algebra is a linear endomorphism τ which is product reversing and of order 2. If τ fixes the center it is called of the first kind. If not, it is of the second kind. Albert showed that D has an involution of the first kind if and only if D represents an element of the Brauer group of order dividing 2. Involutions of the second kind are similarly related to the corestriction (see [Sc]). A recent highlight in the theory of involutory division algebras was the construction of such an algebra of the first kind, which was not the tensor product of algebras of degree 2 [ART]. Theorem 2 implies that such an algebra is Brauer equivalent to a product of algebras of degree 2. In another direction there is a curious connection between involutions and Question 2. If D has even degree n and has an involution of the first kind, then D has a special such involution which is called of skew type. If τ is such an involution, there must be a noncentral $d \in D$ such that $\tau(d) = d$. Such an element d, moreover, must have degree no more than $n/2$ over $Z(D)$. Thus involutions can be used to give a very small part of the answer to Question 2.

I hope it is clear that the full subject of division algebras is difficult to write about because of the depth and diversity of the methods that the theory uses. (This reviewer is trying his hand at it himself.) It is inevitable that any book will seem very incomplete. Professor Draxl's book emphasizes the Brauer group side of the subject. The corestriction map is carefully defined and properly highlighted. Chapters are devoted to the Brauer group aspects of p-algebras and involutions. A particularly attractive feature of this book is the elegant treatment of a subject not touched on here, namely, the multiplicative group of a division algebra. These lecture notes begin very elementarily, so the author is merely able to state the Merkuriev-Suslin Theorem (the proof is truly daunting). This book is a very good introduction to the Brauer group of a field. I find it especially appropriate because it leaves the reader aware of how much more he must learn, and gives the reader some idea of where to go to learn it.

The first half of Professor Dauns' book is also on finite-dimensional division algebras. The orientation of this book is decidedly ring theoretic. It covers cyclic algebras, crossed products, and Amitsur's theorem. This book is extremely elementary and is expansively written. It therefore covers relatively less material. The author explicitly decided to incorporate a good deal of repetition. For example, results are sometimes first proved for division algebras and then independently proved for simple algebras. The author is also very interested in examples, many of which are worked out in detail. Sometimes the author computes a special case and refers the reader elsewhere for the general proof. The result of this is that a mature mathematician may find the book slow and incomplete. A student, however, may well benefit from this approach. I would be happier with the book if this same student received more of a feeling of where the field is now, how much more there was to learn in it, and what he or she needs to know in order to make further progress.

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Theory of charges, a study of finitely additive measures, by K. P. S. Bhaskara Rao and M. Bhaskara Rao, Academic Press, London, 1983, x + 315 pp., $55.00. ISBN 0-1209-5780-9

A charge is a finitely additive, extended real-valued set function defined on a field of sets. The notion is thus a familiar one even to those who may not have used the term. But why should we study finitely additive measures? Haven't Borel and Lebesgue made them obsolete? We have become so accustomed to countable additivity that most of us take it for granted and feel we would be lost without it. Nevertheless, no less an authority than S. Bochner is quoted as having remarked that finitely additive measures are more interesting, more difficult to handle, and perhaps more important than countably additive ones.

Everyone knows that density is a natural measure in the set of positive integers, and that it has proved very useful in number theory despite the fact that it is only finitely additive. Sometimes density is linked to a countably additive measure. For example, under an ergodic transformation of a normalized measure space, almost all points generate sequences of images that fall in any given measurable set with a frequency (that is, density) equal to the measure of the set. The law of large numbers establishes a similar link between a countably additive probability and densities on almost all sample sequences.

If countable additivity were really indispensable one might wonder how mathematicians managed to get along without it for so long. Of course, length, area and volume are actually countably additive, although this fact was not fully appreciated or exploited until the end of the last century. There are other circumstances in which countable additivity comes as a bonus; for example, when the domain is the field of closed open subsets of a compact space. As a consequence, any charge can be represented by a countably additive charge on a corresponding Stone space, but this representation is too esoteric to be of much use except for special purposes.

It is a remarkable fact that countable additivity is sometimes forced by an invariance requirement. D. Sullivan and G. A. Margulis have recently shown that, for $n \geqslant 3$, Lebesgue measure is the only finitely additive measure on the bounded measurable subsets of R^n that normalizes the unit cube and is isometry-invariant, thus settling a very old and classical problem of Ruziewicz.