BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 2, March 1983

0273-0979/82/0000-0978/$01.75

The structure of locally compact Abelian groups, by David L. Armacost, Pure and Applied Mathematics: A Series of Monographs and Textbooks, vol. 68, Dekker, New York, 1981, vii + 168 pp., $23.75.

Fourier analysis is undoubtedly one of the most important branches of mathematical analysis. It began as a separate area of study in the 19th century with the investigation of convergence of Fourier series; it has developed widely in the past 150 years; at least since the publication of Weil's book [**7**], mathematicians have realized that some of the main results in the subject extend naturally to locally compact Abelian (LCA) groups. It is reasonable, then, for harmonic analysts to ask what an LCA group can look like.

Suppose that we want to classify LCA groups. What likely paths are open?

1. The fundamental theorem on the structure of LCA groups says that any such group G can be written as $\mathbf{R}^n \times G_0$, where G_0 has a compact open subgroup. Moreover, both factors are unique up to isomorphism. Since we presumably understand $\mathbf{R}^n$, this means that we need only classify those LCA groups with compact open subgroups—discrete groups, for instance. A glance at any book on infinite Abelian groups ([**3** or **4**], for example) will show that classifying such groups is a hopeless task.

2. It might still be possible to classify the compact Abelian groups, thus making a dent in the general classification. Unfortunately, the Pontryagin Duality Theorem gives a bijective correspondence between compact and discrete Abelian groups. Thus this classification problem is also intractable.

3. The structure theorem says that (modulo factors of $\mathbf{R}^n$) every LCA group G_0 is the extension of a compact group K by a discrete group D. We could simply hope to classify such extensions, thus reducing the general classification problem to the two special problems mentioned above plus one problem in group cohomology. Indeed, we might use the Pontryagin Duality Theorem to describe the extension in terms of the discrete groups D and $K^\wedge$ ($K^\wedge$ is the dual of K). Thus the extension problem becomes a purely algebraic one (no topology involved). The formal computations are not hard; they are given in [**2**] (where they are used for a somewhat specialized purpose). However, this approach introduces two related problems. First of all, the compact open subgroup K of G_0 is generally not uniquely determined; thus the above approach gives a large number of different descriptions of the same group. Secondly, inequivalent extensions of K by D can give rise to isomorphic copies of G_0. (For example, there are two isomorphism classes of groups of order p^2, but $H^2(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z}) \cong \mathbf{Z}/p\mathbf{Z}$.) For the moment, then, this approach seems unsatisfactory.

4. Since the general classification problem seems hopeless, we might examine LCA groups with special properties. For instance, we might look at connected LCA groups. By the structure theorem, we need only consider compact

connected groups; it is known that these are the groups whose duals are torsion-free. Since there is no good classification of torsion-free groups, the analysis again stops. To go farther, we might look at a subclass of the connected LCA groups. For instance, we might consider arcwise connected LCA groups. Again, we may restrict attention to compact groups K; it turns that K is arcwise connected iff $\mathrm{Ext}(K^{\wedge}, \mathbf{Z}) = 0$. Such groups $K^{\wedge}$ are called Whitehead groups; it is known that all free Abelian groups are Whitehead groups, and Shelah [**5**] proved that the existence of other Whitehead groups is an undecidable question in Zermelo-Fraenkel set theory. So much for that attempt.

5. It is known, however, that countable Whitehead groups are free Abelian. Moreover, countable discrete groups correspond to second countable compact groups under Pontryagin duality. Thus we may be able to obtain classification results by restricting attention to second countable groups. A further cause for hope in this direction is Ulm's Theorem, which gives a (rather complicated) classification of countable torsion groups. (Under Pontryagin duality, discrete torsion groups correspond to totally disconnected compact groups.) Various authors have given theorems about the structure of totally disconnected second countable groups satisfying other hypotheses (see, e.g., [**1** and **6**]), but I do not know of any paper which classifies such groups by dualizing Ulm's Theorem. It may be that the dualized theorem is too complicated to be worth stating, of course. In any case, one approach to the structure problem is to examine totally disconnected groups closely.

6. Another possibility is to mimic the standard results in Abelian group theory, with appropriate modifications to account for the topology. For instance, any torsion group is an (algebraic) direct sum of its p-subgroups. Define an LCA group G to be a topological p-group if $p^n x \to 0$ as $n \to \infty$ for all $x \in G$. To what extent is a topologicial torsion group a (topological) direct sum of topological p-groups? The answer is quite satisfactory: topological torsion groups are restricted direct products of their topological p-groups. There are also a number of structure theorems about topological p-groups.

7. Finally, one may decide to ignore the question of classification and agree as follows: the only LCA groups worth worrying about are those that arise in other branches of mathematics, and we know what they are: $\mathbf{R}$, $\mathbf{T}$, $\mathbf{Z}$, the p-adic integers $\mathbf{Z}_p$, the p-adic fields $\mathbf{Q}_p$, and finite products of these groups. What we ought to do is to find properties these groups enjoy as topological groups that make them special. It is easy to find holes in the premise to this argument (number theorists will note the absence of adele groups, and analysts often use countable products of the 2-element group with itself), but there still may be interest in the problem of characterizing groups by their special properties.

The book under review touches on many (but not all) of these approaches, and also investigates some related topics. An LCA group G is called *monothetic* if there is an element $\Phi \in \mathrm{Hom}(\mathbf{Z}, G)$ such that $\mathrm{Im}\,\Phi$ is dense. A characterization of monothetic groups has been known for some time. Let $\mathbf{T}$ be the circle group with the ordinary topology, let $\mathbf{T}_d$ be the circle group with the discrete topology, and let $\wedge$ denote the Pontryagin dual of a group. Then G is monothetic iff G is isomorphic to a quotient of $\mathbf{Z} \cong \mathbf{T}^{\wedge}$ or $\mathbf{T}_d^{\wedge}$. A similar result is known for groups G containing a dense image of $\mathbf{R}$ (such groups are called *solenoidal*). The book gives corresponding results for $\mathbf{Z}_p$ and $\mathbf{Q}_p$. A related problem is the following: Given an LCA group H, for what groups G are the continuous images of H dense in G? Armacost investigates this question for various H. He also gives a brief discussion of the question of putting different topologies on a given Abelian group.

This book is one of the best organized that I have ever read. It begins with a careful selection of results assumed known to the reader (with references); the author refers to these results when needed, and he is unusually thorough in providing both these references and references to results previously proved. As a result, the book is a pleasure to read. There are only a few misprints and errors, and they should not cause any trouble; for instance in §4.27, p. 52, the reference should be to 4.25(a) rather than 4.26(a).

While the book does take up a great variety of topics, it can hardly be described as encyclopedic; after all, it is only 154 pages long. The author extends his coverage by concluding each chapter with a list of additional results, together with either a reference or a sketch of a proof. Still, many topics are slighted (as the author would no doubt admit); in particular, I wish that he had given more space to cohomological questions.

It would be unreasonable to expect mathematicians to buy this book in droves; the price is enough to inhibit most people. But many mathematicians may enjoy browsing through their library's copy of this pleasant and well-written book about an interesting and accessible subject.

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AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 2, March 1983

0273-0979/82/0000-0979/$01.50

Between nilpotent and solvable, edited by Michael Weinstein, Polygonal Publishing House, Passaic, N.J., 1982, 240 pp., $22.00.

In this book we find, in one volume, descriptions of many classes of finite solvable groups which include the nilpotent groups; supersolvable groups, M-groups, CLT-groups and related classes, linear groups, seminilpotent and