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# THE INNER PRODUCT OF PATH SPACE MEASURES CORRESPONDING TO RANDOM PROCESSES WITH INDEPENDENT INCREMENTS

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Let $X_1(t)$ and $X_2(t)$ be any two stochastically continuous, homogeneous random processes on $[0, T]$ with independent increments. It follows that $E(\exp(irX_k(t))) = \exp(tD_k(r))$, where

$$D_k(r) = i\alpha_k r - \beta_k \frac{r^2}{2} + \int_R \left(e^{iru} - 1 - \frac{iru}{1 + u^2}\right) d\sigma_k(u) \tag{1}$$

for some $\alpha_k \in R$, $\beta_k \geqq 0$, and Borel measure $\sigma_k$ with $\int_R (u^2/(1 + u^2))\, d\sigma_k(u) < \infty$ (and with $\sigma_k(\{0\}) = 0$). We denote by $\rho_k$ (resp. $\rho_k^{*t}$) the probability measure on $R$ with characteristic function, $\exp(D_k(r))$ (resp. $\exp(tD_k(r))$), and by $\tilde{\rho}_k$ the probability measure on path space corresponding to $X_k$. $\tilde{\rho}_k$ is a Borel measure (with respect to the Skorokhod topology) on $D = D[0, T]$, the space of real valued functions on $[0, T]$ which are right-continuous and have left-hand limits, and may be defined in terms of $\rho_k$ in the usual way.

If $\mu_1$ and $\mu_2$ are two measures on $R$ (or $D$), we define $\sqrt{\mu_1}\sqrt{\mu_2}$ as the unique measure satisfying

$$\frac{d(\sqrt{\mu_1}\sqrt{\mu_2})}{d\nu} = \sqrt{\frac{d\mu_1}{d\nu}}\sqrt{\frac{d\mu_2}{d\nu}}$$

for any $\nu \gg \mu_1, \mu_2$; $(\sqrt{\mu_1} - \sqrt{\mu_2})^2$ thus denotes the (positive) measure, $(\mu_1 + \mu_2) - 2\sqrt{\mu_1}\sqrt{\mu_2}$. Given $\rho_1$ and $\rho_2$ as above, we define $N = N(\rho_1, \rho_2) = \int_R d(\sqrt{\sigma_1} - \sqrt{\sigma_2})^2$; $N$ may be finite or infinite. If $N < \infty$, it is easily shown that $\int_R (|u|/(1 + u^2))\, d|\sigma_1 - \sigma_2| < \infty$ and we then define

$$\gamma = \gamma(\rho_1, \rho_2) = \frac{1}{2}\left(\alpha_1 - \alpha_2 - \int_R \frac{u}{1 + u^2}\, d(\sigma_1 - \sigma_2)\right).$$

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We define $K = K(\rho_1, \rho_2)$ as follows:

$$K = \frac{\gamma^2}{2\beta} + \frac{N}{2}, \quad \text{if } N < \infty \text{ and } \beta_1 = \beta_2 = \beta > 0,$$

$$= \frac{N}{2}, \quad \text{if } N < \infty \text{ and } \beta_1 = \beta_2 = 0 \text{ and } \gamma = 0,$$

$$= +\infty, \quad \text{otherwise.}$$

When $K < \infty$, we define $\rho_3$ as the measure on $R$ with characteristic function $\exp(D_3(r))$, where $D_3$ is given by (1) with

$$\alpha_3 = \frac{\alpha_1 + \alpha_2}{2} - \frac{1}{2}\int_R \frac{u}{1+u^2}\, d(\sqrt{\sigma_1} - \sqrt{\sigma_2})^2(u),$$

$\beta_3 = \beta$ (where $\beta = \beta_1 = \beta_2$), and $\sigma_3 = \sqrt{\sigma_1}\sqrt{\sigma_2}$. We let $\tilde{\rho}_3$ denote the related measure on $D$ corresponding to a third homogeneous random process with independent increments, $X_3$, in the obvious way.

THEOREM. (i) *If* $K = \infty$, *then* $\sqrt{\tilde{\rho}_1}\sqrt{\tilde{\rho}_2} = 0$.
(ii) *If* $K < \infty$, *then* $\sqrt{\tilde{\rho}_1}\sqrt{\tilde{\rho}_2} = e^{-TK}\tilde{\rho}_3$.

Since $\tilde{\rho}_1 \perp \tilde{\rho}_2 \Leftrightarrow \sqrt{\tilde{\rho}_1}\sqrt{\tilde{\rho}_2} = 0$, we immediately obtain

COROLLARY 1. $\tilde{\rho}_1 \perp \tilde{\rho}_2$ *if and only if either*
(i) $N = \infty$, *or*
(ii) $N < \infty$, *but* $\beta_1 \neq \beta_2$, *or*
(iii) $N < \infty$, *and* $\beta_1 = \beta_2 = 0$, *but* $\gamma \neq 0$.

When $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are not mutually singular, we wish to define a quantitative description of their "overlap." Accordingly, we consider for two measures, $\mu_1$ and $\mu_2$, on $D$ (resp. $R$) the decomposition of $D$ (resp. $R$) into a disjoint union of three sets ($S_1$, $S_2$, and $S_{12}$) with the properties that $\mu_1(S_2) = 0 = \mu_2(S_1)$ and that $\mu_1 \approx \mu_2$ on $S_{12}$. Although these properties do not completely determine the three sets, we may uniquely define $\mu_1(\operatorname{supp}\mu_2)$ as $\mu_1(S_{12})$, and $\mu_1((\operatorname{supp}\mu_2)^c)$ as $\mu_1(S_1)$. If $\mu_1$ and $\mu_2$ are probability measures, it follows that $0 \leqq \mu_1(\operatorname{supp}\mu_2) \leqq 1$, and that $\mu_1 \ll \mu_2 \Leftrightarrow \mu_1(\operatorname{supp}\mu_2) = 1$, while $\mu_1 \perp \mu_2 \Leftrightarrow \mu_1(\operatorname{supp}\mu_2) = 0$.

LEMMA. *Suppose* $\mu_1$ *and* $\mu_2$ *are finite Borel measures on* $S$ $(= D$ *or* $R)$. *If we let* $\nu_1 = \sqrt{\mu_1}\sqrt{\mu_2}$ *and* $\nu_n = \sqrt{\mu_1}\sqrt{\nu_{n-1}}$, *then* $\lim_{n\to\infty} \nu_n(S) = \mu_1(\operatorname{supp}\mu_2)$.

This lemma together with a countable number of applications of the theorem yields

COROLLARY 2. *If $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are not mutually singular, then* $\sigma_1((\text{supp}\,\sigma_2)^c) < \infty$ *and* $\tilde{\rho}_1(\text{supp}\,\tilde{\rho}_2) = \exp(-T\sigma_1((\text{supp}\,\sigma_2)^c))$. *Thus* $\tilde{\rho}_1 \ll \tilde{\rho}_2 \Leftrightarrow K < \infty$ *and* $\sigma_1 \ll \sigma_2$; *and* $\tilde{\rho}_1 \approx \tilde{\rho}_2 \Leftrightarrow K < \infty$ *and* $\sigma_1 \approx \sigma_2$.

The theorem itself is proved in two parts. It is first shown that when $K < \infty$, $\sqrt{\tilde{\rho}_1}\sqrt{\tilde{\rho}_2} \geqq e^{-TK}\tilde{\rho}_3$. The basic ingredient in this demonstration is the proposition that for finite Borel measures on $D$, $\sqrt{\mu_1 * \nu_1}\sqrt{\mu_2 * \nu_2} \geqq (\sqrt{\mu_1}\sqrt{\mu_2}) * (\sqrt{\nu_1}\sqrt{\nu_2})$, where $*$ denotes convolution. In the second part of the proof, it is first noted that

$$\int_D d(\sqrt{\tilde{\rho}_1}\sqrt{\tilde{\rho}_2}) \leqq \left(\int_R d(\sqrt{\rho_1^{*T/n}}\sqrt{\rho_2^{*T/n}})\right)^n$$

for all integers $n$, and then a lengthy determination of the fact that

$$\lim_{a\to 0} \frac{\int_R d(\sqrt{\rho_1^{*a}}\sqrt{\rho_2^{*a}}) - 1}{a} = -K(\rho_1, \rho_2) \tag{2}$$

allows us to conclude that $\int_D d(\sqrt{\tilde{\rho}_1}\sqrt{\tilde{\rho}_2}) \leqq e^{-TK}$. The conclusions of the theorem then follow immediately.

Our results in the purely Gaussian case ($\sigma_1 = \sigma_2 = 0$) that $\tilde{\rho}_1 \approx \tilde{\rho}_2$ if $\beta_1 = \beta_2$ (and $\alpha_1 = \alpha_2$ when $\beta_1 = \beta_2 = 0$) and that otherwise $\tilde{\rho}_1 \perp \tilde{\rho}_2$ are well known [**1**], [**2**]. In the general case, Skorokhod [**3**, Chapter 4] has previously obtained a (somewhat complicated) set of sufficient conditions for the equivalence of $\tilde{\rho}_1$ and $\tilde{\rho}_2$ and has calculated the Radon-Nikodym derivative, $d\tilde{\rho}_1/d\tilde{\rho}_2$, under those conditions. It can readily be shown that the conditions for equivalence of Corollary 2 actually imply Skorokhod's conditions which are thus seen to be in fact necessary as well as sufficient.

REMARK 1. When $\tilde{\rho}_1 \approx \tilde{\rho}_2$, the results of the theorem, as stated in terms of $\sqrt{\tilde{\rho}_1}\sqrt{\tilde{\rho}_2}$, can be regarded as a kind of symmetric substitute for a determination of $d\tilde{\rho}_1/d\tilde{\rho}_2$ or $d\tilde{\rho}_2/d\tilde{\rho}_1$.

REMARK 2. The calculation of (2) for infinitely divisible measures on $R^s$ has also been carried out, leading to the expected results. The requirement that $\gamma = 0$ when $\beta_1 = \beta_2 = \beta = 0$ in order to have $K$ finite is replaced by the condition that $\vec{\gamma}$ be orthogonal to the null space of $B$, and $\gamma^2/2\beta$ in the definition of $K$ is replaced by $(\vec{\gamma}, (2B)^{-1}\vec{\gamma})$. Here, $\vec{\gamma}$ and the positive semi-definite matrix $B$ are respectively the $R^s$-analogues for $\gamma$ and $\beta$. The $s$-dimensional analogues to our main results then follow.

REMARK 3. It is clear that our results can be extended to the measures associated with *nonhomogeneous* processes with independent increments (on finite or infinite time intervals). Within this more general context, our present results will play a "local" role.

REMARK 4. The measures $\rho_k^{*t}$, acting by convolution on the bounded continuous functions, define contraction semigroups, $\exp(tA_k)$; while the measure $\sqrt{\rho_1^{*t}}\sqrt{\rho_2^{*t}}$ similarly defines a contraction operator which we denote by $F(t)$. $F(t)$ is in general not a semigroup, but when $K < \infty$, the proof of (2) yields the very strong results that

$$\lim_{t\to 0} \frac{F(t) - I}{t} = A_3 - KI \tag{3}$$

and

$$\lim_{n\to\infty} \left(F\left(\frac{t}{n}\right)\right)^n = \exp(t(A_3 - KI)), \tag{4}$$

where $I$ is the identity operator. In (3) the limit is taken strongly (on the domain of $A_3$), while in (4) the limit is taken in the *uniform* operator topology.

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