BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, March 1972

THE CONSTRUCTION OF AN ASYMPTOTIC CENTER WITH A FIXED-POINT PROPERTY[1]

BY MICHAEL EDELSTEIN

Communicated by W. Fuchs, August 16, 1971

ABSTRACT. Given a bounded sequence $\{u_n: n = 1, 2, \ldots\}$ of points in a closed convex subset C of a uniformly convex Banach space, c_m denotes the point in C with the property that among all closed balls centered at points of C and containing $\{u_m, u_{m+1}, \ldots\}$ the one centered at c_m is of smallest radius. It is shown that the sequence $\{c_m: m = 1, 2, \ldots\}$ converges (strongly) to a point $c \in C$ called the asymptotic center of $\{u_n\}$ with respect to C. Further, for a class of mappings f of C into itself, which contains all nonexpansive mappings, $f(c) = c$ whenever an $x \in C$ exists such that $f^n(x) = u_n$, $n = 1, 2, \ldots$.

1. **Introduction.** Let C be a closed convex set in a uniformly convex Banach space X. (Recall that X is called uniformly convex if the modulus of convexity

$$\delta(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|x + y\| : \|x\|, \|y\| \leqq 1, \|x - y\| \geqq \varepsilon\}$$

is positive in its domain of definition $\{\varepsilon : 0 < \varepsilon \leqq 2\}$.) Given a bounded sequence $\{u_n : n = 1, 2, \ldots\}$ in the set C, define

$$r_m(y) = \sup\{\|u_k - y\| : k \geqq m\} \quad (y \in X). \tag{1}$$

It is well known, and easily proved, that a unique point $c_m \in C$ exists such that

$$r_m(c_m) = \inf\{r_m(y) : y \in C\} = r_m. \tag{2}$$

Clearly $r_m \geqq r_{m+1}$ and $r_m \geqq 0$ for all $m = 1, 2, \ldots$ so that $\{r_m : m = 1, 2, \ldots\}$ converges to $r = \inf\{r_m : m = 1, 2, \ldots\}$. We note that if $r = 0$ then, as can be readily verified, the sequence $\{u_n\}$ converges.

2. **The asymptotic center.**

DEFINITION. If $\{c_m\}$ converges then $c = \lim c_n$ is called the asymptotic center of $\{u_n\}$ (with respect to C).

THEOREM 1. *With X, C and $\{u_n\}$ as above, the sequence $\{c_m\}$ converges. (Thus the asymptotic center c exists.)*

PROOF. If $r = 0$ then, as can be readily seen, $\{u_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} u_n = \lim_{m\to\infty} c_m \,(= c)$. We may then assume that $r > 0$. Suppose now, for a contradiction, that $\{c_m\}$ fails to converge. Then an $\varepsilon > 0$ exists such that for any natural number N there are integers

AMS 1969 *subject classifications*. Primary 4785; Secondary 5485.

Key words and phrases. Asymptotic center, fixed-point theorem, nonexpansive mappings.

[1] This research was supported by the National Research Council of Canada, Grant A-3999.

$n > m \geqq N$ with $\|c_m - c_n\| \geqq \varepsilon$. From the uniform convexity of X and the fact that

$$\|u_k - c_n\| \leqq r_n \leqq r_m \qquad (k \geqq n),$$
$$\|u_k - c_m\| \leqq r_m \qquad (k \geqq m),$$

it follows that

$$(3) \qquad \left\|u_k - \frac{c_m + c_n}{2}\right\| \leqq r_m\left(1 - \delta\left(\frac{\|c_m - c_n\|}{r_m}\right)\right)$$
$$\leqq r_m(1 - \delta(\varepsilon/D)) \qquad (k \geqq n),$$

where D is the diameter of $\{u_n\}$. On the other hand, since $\frac{1}{2}(c_m + c_n) \neq c_n$, there is a $k \geqq n$ such that

$$(4) \qquad r_n < \left\|u_k - \frac{c_m + c_n}{2}\right\|.$$

For such a k, (3) and (4) hold simultaneously so that $r_m - r_n \geqq r_m\delta(\varepsilon/D) \geqq r\delta(\varepsilon/D)$. This, however, is impossible since $\{r_k\}$ converges.

REMARK. If X is a Hilbert space then c belongs to the closed convex hull of $\{u_n\}$. (This follows immediately from the fact that the same assertion holds for each c_m, $m = 1, 2, \ldots$.)

3. **A fixed-point theorem.**

THEOREM 2. *Let C, $\{u_n\}$ and c $(=\lim_{m\to\infty} c_m)$ be as in Theorem 1 and $f: C \to C$ be a mapping of C into itself satisfying the following conditions:*

(1) *$u_n = f^n(x)$ for some $x \in C$ and all $n = 1, 2, \ldots$;*

(2) *there exists a positive integer n_0 and neighborhood V of c in C such that*

$$(5) \qquad \|f^k(x) - f(v)\| \leqq \|f^{k-1}(x) - v\| \qquad (k \geqq n_0, v \in V).$$

Then $f(c) = c$.

PROOF. If $r = 0$ then $c = \lim_{n\to\infty} u_n = \lim_{n\to\infty} f^n(x)$ and $f(c) = c$. Let then r be positive and suppose that $f(c) \neq c$. Set $\eta = \|c - f(c)\|$ and choose $N \geqq n_0$ large enough so that $c_n \in V$, $\|c - c_n\| \leqq \eta/3$ and $\|f(c) - f(c_n)\| \leqq \eta/3$ for $n \geqq N$; and, therefore, $\|c_n - f(c_{n-1})\| \geqq \eta/3$ for all $n \geqq N + 1$. Now, for all k and n with $k \geqq n \geqq N + 1$ we have

$$\|f^k(x) - c_n\| \leqq r_n \leqq r_{n-1},$$
$$\|f^k(x) - f(c_{n-1})\| \leqq \|f^{k-1}(x) - c_{n-1}\| \leqq r_{n-1}.$$

By uniform convexity then

$$\left\| f^k(x) - \frac{c_n + f(c_{n-1})}{2} \right\| \leqq r_{n-1}\left(1 - \delta\left(\frac{\eta}{3r_{n-1}}\right)\right)$$

$$\leqq r_{n-1}\left(1 - \delta\left(\frac{\eta}{3D}\right)\right) \quad (k \geqq n \geqq N + 1),$$

where again D denotes the diameter of $\{u_n\} = \{f^n(x)\}$. On the other hand, since $(c_n + f(c_{n-1}))/2 \neq c_n$, there is a $k \geqq n$ such that

$$r_n < \left\| f^k(x) - \frac{c_n + f(c_{n-1})}{2} \right\|.$$

Thus $r_n < r_{n-1}(1 - \delta(\eta/3D))$ and $r_{n-1} - r_n > r_{n-1}\delta(\eta/3D) \geqq r\delta(\eta/3D)$. This, however, is impossible as $\{r_n\}$ converges.

4. **An immediate consequence of Theorem 2 is the following.**

COROLLARY. *Let C be a closed and bounded convex set in a uniformly convex Banach space and suppose that f is a continuous mapping of C into itself such that for each $x \in C$ there is a positive integer $N = N(x)$ such that, for all integers $n \geqq N$ and all $y \in C$,*

$$\|f^n(x) - f^n(y)\| \leqq \|f^{n-1}(x) - f^{n-1}(y)\|.$$

Then $f(\xi) = \xi$ for some $\xi \in C$.

REMARK. The well-known theorem of Browder [**1**] and Göhde [**2**], asserting that each nonexpansive mapping of a closed and bounded convex subset of a uniformly convex Banach space into itself has a fixed point, follows from the above corollary upon setting $N = 1$ (for all $x \in C$).

REFERENCES

1. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. **54** (1966), 1041–1044. MR **32** #4574.

2. D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr. **30** (1965), 251–258. MR **32** #8129.

DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA

Current address: Department of Mathematics, University of California, Berkeley, California 94720