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SMOOTH S^1 ACTIONS ON HOMOTOPY COMPLEX PROJECTIVE SPACES AND RELATED TOPICS[1]

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This paper is dedicated to Professors Leroy M. Kelly and Fritz Herzog who gave so enthusiastically of their time and talent in developing undergraduate mathematicians at Michigan State University. I was one of their beneficiaries.

0. **Introduction and motivation.** We begin by listing some questions and remarks which establish the theme of this paper.

1. Which cobordism classes of oriented manifolds admit nontrivial circle actions? Answer: Atiyah-Hirzebruch [**4**]: For a compact oriented manifold X of dim $4k$, its $\hat{\mathscr{A}}$ genus vanishes iff there is a multiple mX which is cobordant to Y, with $W_2(Y) = 0$, which admits a nontrivial circle action on each of its components. The $\hat{\mathscr{A}}$ genus is the genus belonging to the power series $(x/2)(\sinh x/2)^{-1}$.

2. Which manifolds in a given homotopy type admit nontrivial circle actions? More specifically, of those manifolds homotopy equivalent to complex projective n space, which admit nontrivial S^1 actions?

Strong conjecture. If $h: X \to CP^n$ is an orientation preserving homotopy equivalence and if X supports a nontrivial circle action then $h^*\hat{\mathscr{A}}(CP^n) = \hat{\mathscr{A}}(X)$ where

$$\hat{\mathscr{A}}(X) = \prod (x_i/2)(\sinh x_i/2)^{-1} \in H^*(X, Q)$$

and the elementary symmetric functions of the x_i^2 give the Pontrjagin classes of X. In other words, the homotopy equivalence must preserve the total $\hat{\mathscr{A}}$ cohomology class.

Weak conjecture. To the hypothesis of the strong conjecture add the condition that the fixed point set of the action consists of isolated fixed points. Then

$$h^*\hat{\mathscr{A}}(CP^n) = \hat{\mathscr{A}}(X).$$

A corollary of the strong conjecture is that most homotopy complex projective spaces do not admit S^1 actions. The weak conjecture is discussed in detail in Part II, §2.

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The validity of the weak conjecture is related to the representations of S^1 on the tangent space of X at the isolated fixed points. If X is homotopy equivalent to CP^n, there must be $n+1$ isolated fixed points p_j. To each we show how to associate an integer a_j and compare the eigenvalues of the S^1 action on the tangent space of X at p_j with the integers $\{\pm(a_k - a_j), k \neq j\}$.

A particularly good property of the homotopy type which is useful to exploit in connection with the second question is the existence of a spin^c structure. In the first few sections we discuss the properties of an equivariant spin^c structure.

Another idea we develop in connection with S^1 actions on manifolds in general, is the exploitation of a theorem of Stewart (Part I, 6.1) which is concerned with lifting an S^1 action on X to an S^1 action on a principle S^1 bundle over X. Using this theorem and assuming $H^1(X, Z) = 0$, we define a function F from the additive group $H^2(X, Z)$ to the multiplicative group of units of $K^*_{S^1}(X)$. Assuming X is a spin^c manifold and using Stewart's theorem we construct an "orientation class" $\delta_{S^1} \in K^*_{S^1}(TX)$ (TX = tangent bundle of X). This class generates $K^*_{S^1}(TX)$ as a free module over $K^*_{S^1}(X)$.

The index homomorphism $\mathrm{Id}^X_{S^1} : K^*_{S^1}(TX) \to R(S^1)$ is a homomorphism of $R(S^1)$ modules and is intimately connected to the representations of S^1 on the normal fibers of the components of the fixed point set. Suppose that $z_1, \ldots, z_s$ is a basis for $H^2(X, Z)$ and let $\Phi(y_1, \ldots, y_s)$ be any polynomial in indeterminants y_i with integer coefficients. Set $w_i = F(z_i) \in K^*_{S^1}(X)$. Then the condition that

$$\mathrm{Id}^X_{S^1}(\delta_{S^1}\Phi(w_1, w_2, \ldots, w_S)) \in R(S^1)$$

for every Φ imposes stringent restrictions on the representations of S^1 on the normal fibers of the components of the fixed point set. This idea is exploited in connection with Part II, Theorems 2.11 and 2.12.

The principle applications of the ideas developed here are in Part II, Theorems 2.8-2.12. They deal with the relationship between $\mathscr{A}(X)$, the integers $\{(a_k - a_j)\}$ and the integers $\{x_{jk}\}$ which are the roots of the S^1 action on TX at p_j.

Another interesting item, which was suggested by the above mentioned results, is an example of an exotic action of S^1 on CP^3. It is exotic in the sense that the eigenvalues of the S^1 action on TCP^3 at the four isolated fixed points are distinct from those of the linear case (Part I, 6.4). See Part II, §4 for more detail. Another significant feature of this example is the fact that the bilinear form $\langle\ \rangle$ of Part II, §3 is nondegenerate in this case, see §5 of Part II.

We have interspersed the ideas and theorems with numerous examples and conjectures. We hope the reader finds the former of sufficient interest to consider the latter.

This paper is divided into two parts and is organized as follows:

I. GENERALITIES CONCERNING SMOOTH ACTIONS OF COMPACT LIE GROUPS ON MANIFOLDS

1. **Properties of the index homomorphism** $\mathrm{Id}_G^X: K_G^*(TX) \to R(G)$.
2. **The group** $\mathrm{spin}^c(m)$.
   (a) The half spin representations Δ_+ and Δ_- as complex $\mathrm{spin}^c(m)$ modules.
   (b) The elliptic pairing of $\mathrm{spin}^c(m)$ modules:
$$R^m \times \Delta_\pm \to \Delta_\mp.$$
3. $\mathrm{Spin}^c(m)$ **bundles.**
4. K_G **orientation of** G **manifolds and Poincaré duality.**
   (a) Equivariant homology dual to K_G^*.
   (b) Examples of orientations constructed from equivariant $\mathrm{spin}^c(m)$ structures.
5. **Formula for** $\mathrm{Id}_G^X: K_G(TX) \to R(G)$ **in terms of**:
   (a) Orientation class of X.
   (b) Representations of G on normal fibers to fixed point set.
6. **Specialization to** S^1 **actions.**
   (a) Stewart's theorem.
   (b) The homomorphism from $H^2(X, Z)$ to the group of units of $K_{S^1}^*(X)$.
   (c) Standard example—Illustration of (a) and (b) for the case of "linear actions" of S^1 on CP^n.

II. APPLICATIONS TO S^1 ACTIONS ON A HOMOTOPY COMPLEX PROJECTION SPACE X AND SPECULATIONS

1. **Generalities.**
   (a) The equivariant "Hopf bundle" $\eta \in K_{S^1}^*(X)$.
   (b) The integers a_j associated to the component X_j of the fixed point set of the S^1 action by restricting η to a point $p_j \in X_j$.
   (c) Comparison of $K_{S^1}^*(X)$ with $K_{S^1}^*(X^{Z_{p^r}})$, $X^{Z_{p^r}}$ = fixed point set of $Z_{p^r} \subset S^1$.
2. S^1 **actions on** X **with isolated fixed points.**
   (a) Number theoretic properties of the eigenvalues of the representations of S^1 on the tangent space at the isolated fixed points.
   (b) Theorem 2.8; The relations between the eigenvalues of the representations of S^1 above and the integers a_j defined by the equivariant Hopf bundle η.
   (c) The class $\hat{\mathscr{A}}(X) \in H^*(X, Q)$.

3. **Speculation: The bilinear form $\langle\ \rangle$ on $K_G^*(X)$.**
   (a) Analogy with cup product pairing for ordinary cohomology theory.
   (b) When is $\langle\ \rangle$ nondegenerate over $R(G)$?
   (c) Examples where $\langle\ \rangle$ is nondegenerate.
4. **An exotic action of S^1 on CP^3.**
   (a) Exotic representations on TCP^3 at isolated fixed points.
   (b) Identification of differential structure.
5. **The bilinear form $\langle\ \rangle$ on $K_{S^1}^*(X)$, $X = CP^3$.**

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I. Generalities concerning smooth actions of compact Lie groups on manifolds

1. **Properties of the index homomorphism.** $\mathrm{Id}_G^X : K_G^*(TX) \to R(G)$. Here we review the relevant properties of the equivariant K theory of [**1**], [**5**] and [**6**]. Throughout, G is a compact Lie group acting smoothly on a manifold X. Denote by $K_G^*(X)$ the equivariant K theory of X. We note that $K_G^*(Y)$ is defined for any locally compact G space Y, in particular for $Y = TX$ the tangent space of X. In this case, $K_G^*(TX)$ is a module over $K_G^*(X)$ via π^* where $\pi : TX \to X$ is the projection.

If $i : Z \to X$ is the inclusion of a G invariant submanifold Z whose normal bundle v' is complex, there is a homomorphism

$$i_* : K_G^*(Z) \to K_G^*(X)$$

with the property

$$i^* i_*(x) = \lambda_{-1}(v') \cdot x \tag{1.1}$$

when $x \in K_G^*(Z)$ and $\lambda_{-1} : K_G^*(Z) \to K_G^*(Z)$ is the operation which sends a G vector bundle ξ to $\sum(-1)^i \lambda^i(\xi)$, $\lambda^i(\xi)$ is the ith exterior power of ξ.

We note that $TZ \subset TX$ always has a complex normal bundle namely $\pi^*(v \otimes C)$ where $\pi : TZ \to Z$ is the projection and v is the normal bundle of Z in X. Thus if Ti denotes the inclusion of TZ in TX, the homomorphism

$$Ti_* : K_G^*(TZ) \to K_G^*(TX)$$

satisfies

$$Ti^* Ti_* x = \lambda_{-1}(v \otimes C) \cdot X \tag{1.2}$$

for $x \in K_G^*(TZ)$. Recall $K_G^*(TZ)$ is a $K_G^*(X)$ module via π^*.

Note that $K_G(\text{point}) = R(G)$ is the complex representation ring of G and $K_G^*(X)$ is an $R(G)$ module. An important example is the case $G = S^1$, the circle group. Then $R(S^1) = Z[t, t^{-1}]$ is the ring of Laurent series $\sum_{i=-N}^{N} a_i t^i$. Here N is an arbitrary positive integer and all a_i are integers.

Of fundamental importance is the existence of a homomorphism of $R(G)$ modules:

$$\mathrm{Id}_G^X : K_G^*(TX) \to K_G^*(\mathrm{pt}) = R(G);$$

$R(G)$ is the complex representation ring of G (character ring of G). This homomorphism satisfies a few basic properties which makes it quite accessible to computation. Let $\omega : G^1 \to G$ be a homomorphism. Then there is a homomorphism $\omega^* : K_G^*(TX) \to K_G^*(TX)$ and a commutative diagram

(1.3)
$$\begin{tikzcd} K_G^*(TX) \arrow[r, "\omega^*"] \arrow[d, "\mathrm{Id}_G^X"] & K_{G^1}^*(TX) \arrow[d, "\mathrm{Id}_{G^1}^X"] \\ R(G) \arrow[r, "\tilde{\omega}"] & R(G^1) \end{tikzcd}$$
(Compatibility axiom).

Of course ω^* is defined for any G space Y. If $i : Z \to X$ is the inclusion of an invariant submanifold, then there is a commutative diagram

(1.4)
$$\begin{tikzcd} K_G^*(TZ) \arrow[r, "Ti_*"] \arrow[d, "\mathrm{Id}_G^X"] & K_G^*(TX) \arrow[d, "\mathrm{Id}_G^X"] \\ R(G) \arrow[r, "\text{identity}"] & R(G). \end{tikzcd}$$

(1.5) If X is a point, Id_X^G is the identity map of $R(G) = K_G^*(TX)$.

Let G be abelian and $g \in G$. Denote by $\mathfrak{p}$ the prime ideal of characters of $R(G)$ which vanish at g. The localized ring $R(G)_{\mathfrak{p}}$ consists of the fractions $\{\chi/\psi | \chi, \psi \in R(G), \psi(g) \neq 0\}$ with the relation $\chi_1/\psi_1 = \chi_2/\psi_2$ if there is an $\omega \in R(G)$ with $\omega(g) \neq 0$ and $\omega(\chi_1\psi_2 - \chi_2\psi_1) = 0$. If M is an $R(G)$ module $M_{\mathfrak{p}} = M \otimes_{R(G)} R(G)_{\mathfrak{p}}$.

If S is a subset of G, X^S denotes the set of points of X fixed by elements of S. Note that since G is abelian, X^g is a G invariant submanifold of X for $g \in G$. There is then this basic theorem of Atiyah-Segal [**5**].

(1.6) LOCALIZATION THEOREM. *The inclusion $i : X^g \to X$ induces isomorphisms $i_{\mathfrak{p}}^* : K_G^*(X)_{\mathfrak{p}} \to K_G^*(X^q)_{\mathfrak{p}}$ and $Ti_{\mathfrak{p}}^* : K_G^*(TX)_{\mathfrak{p}} \to K_G^*(TX^g)_{\mathfrak{p}}$. The latter has inverse $\lambda_{-1}(\nu \otimes C)^{-1}(Ti)_{\mathfrak{p}*}$ where ν is the normal bundle of X^g in X.*

Thus Id_G^X is completely determined by (1.4), (1.5) and (1.6) in the case X^g consists of isolated points.

When ν is a G vector bundle over X and $Z \subset X$ is an invariant submanifold, we denote by $\nu|_Z$ this bundle restricted to Z. If $Z = x$ is a fixed point, it is a complex G module and we let $\nu|_x(g)$ denote the trace of the element g acting on ν_x for $g \in G$, i.e., the value at g of the character of G defined by ν_x.

2. **The group** spin$^c(m)$. Let V be a real vector space of dimension $m = 2n$. We suppose V endowed with the standard inner product with respect to an orthonormal base $e_1, e_2, \ldots, e_m$. Let $A(V)$ denote the Clifford algebra of V [**2**], [**13**]. For $v \in V \subset A(V)$ we have

$$v^2 = -\|v\|^2 \cdot 1 \tag{2.1}$$

where $1 \in A(V)$ is the identity.

$A(V)$ is the direct sum $A^+ \oplus A^-$ where A^+ is spanned by the products $e_{i_1}e_{i_2} \cdots e_{i_k}$ with k even and A^- by the products with odd k. The multiplicative subgroup of $A(V)$ generated by elements of the unit sphere $S^{m-1} \subset V \subset A(V)$ is denoted by spin(m). The intersection spin$(m) \cap A^+$ is the group spin(m).

The group spin(m) acts in an obvious manner on $A^+ \otimes \boldsymbol{C}$ giving a linear representation of spin(m). This representation is reducible

$$A^+ \otimes \boldsymbol{C} = 2^m(\Delta_+ \oplus \Delta_-)$$

where Δ_+ is the + eigenspace of $(i)^n e_1 e_2 \cdots e_m = \tau$ and Δ_- is the negative eigenspace of τ.

Observe that $\tau^2 = 1$ and τ commutes with elements of A^+ and so with spin(m) and

$$\tau v = -v\tau \quad \text{for } v \in V.$$

Because of this, left multiplication by $v \in V$, denoted by $L(v)$, maps Δ_+ to Δ_- and vice versa. Let

$$\theta: V \times \Delta_\pm \to V \times \Delta_\mp$$

be the map defined by

$$\theta(v, \delta) = (v, L(v)\delta), \qquad v \in V, \delta \in \Delta_\pm. \tag{2.2}$$

Then θ is elliptic, i.e., for fixed $v \neq 0$ in V the linear map

$$\theta_v: v \times \Delta_\pm \to v \times \Delta_\mp$$

defined by restricting θ is an isomorphism. This follows from the fact that

$$\theta_v \circ \theta_v(v, \delta) = (v, L(v)L(v)\delta) = (v, -\|v\|^2\delta)$$

because $L(v) \circ L(v) = L(v^2) = -\|v\|^2 \cdot 1$ by (2.1).

The generator $\varepsilon = -1 \in A(v)$ of the double covering

$$\pi_1 : \mathrm{spin}(m) \to SO(m)$$

acts as multiplication by -1 on Δ_+ and Δ_-. This means that the action of spin(m) on these two representation spaces may be extended to the group

$$\mathrm{spin}^c(m) = \mathrm{spin}(m) \times_{Z_2} S^1.^2$$

Here $Z_2 \subset \mathrm{spin}(m)$ is the subgroup generated by $-1 \in \mathrm{spin}(m)$ and $Z_2 \subset S^1$ is the subgroup generated by $-1 \subset S^1$. Explicitly if $[g, t]$ denotes an equivalence class in spin$^c(m)$ determined by $g \in \mathrm{spin}(m)$ and $t \in S^1 \in \boldsymbol{C}$, then

$$[g, t]\delta = t \cdot (g \cdot \delta) \quad \text{for } \delta \in \Delta_\pm.$$

Of particular importance to us is the commutative diagram

(2.3)
$$\begin{tikzcd} U(m) \arrow[r, "\psi_0"] \arrow[dr, "\psi"'] & \mathrm{spin}^c(m) \arrow[d, "\pi"] \\ & SO(m). \end{tikzcd}$$

Here $\pi[g, t] = \pi_1(g)$;

$$\psi\, \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}) = \mathrm{diag}\begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix} \subset SO(m),$$

$$\psi_0\, \mathrm{diag}(e^{i\theta_1}, e^{i\theta^2}, \dots, e^{i\theta_n})$$

$$= \left[\prod_{j=1}^{n} (\cos\theta_j/2 - \sin\theta_j/2\, e_{2j-1}e_{2j}), \exp[-i(\textstyle\sum\theta_j/2)]\right].$$

Note that

$$\prod_{j=1}^{n} (\cos\theta_j/2 - \sin\theta_j/2\, e_{2j-1}e_{2j}) \in \mathrm{spin}(m) \subset A(V)$$

so ψ_0 makes sense and $\pi\psi_0 = \psi$.

Observe that spin$^c(m)$ has a central circle subgroup S^1 and the quotient is $SO(m)$. The orbit map is π.

Moreover there is an exact sequence of groups

(2.4) $\qquad 1 \to \mathrm{spin}(m) \xrightarrow{i} \mathrm{spin}^c(m) \xrightarrow{j} S^1 \to 1, \qquad j[g, t] = t^2,$

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[2] In general if X is a right G space and Y is a left G space $X \times_G Y$ denotes the space obtained from $X \times Y$ by identifying $(xg, g^{-1}y)$ with (x, y). $x \in X, y \in Y, g \in G$.

and a commutative diagram

$$(2.5)\qquad \begin{tikzcd} \mathrm{spin}^c(m) \times S^1 \arrow[r, "m_1"] \arrow[d, "j \times d"] & \mathrm{spin}^c(m) \arrow[d, "j"] \\ S^1 \times S^1 \arrow[r, "m_2"] & S^1. \end{tikzcd}$$

Here m_1 is multiplication in $\mathrm{spin}^c(m)$, m_2 multiplication in S^1 and d is the squaring map $d(t) = t^2$. Since S^1 is central in $\mathrm{spin}^c(m)$, m_1 is a homomorphism of groups.

3. Spin^c **bundles.** Here we collect some of the properties of $\mathrm{spin}^c(m)$ bundles which will be useful in our analysis of actions on spin^c manifolds.

The classifying space of a group G is denoted by B_G. From diagram (2.5) and the fact that m_1 and m_2 are homomorphisms of groups we obtain a commutative diagram

$$(3.1)\qquad \begin{tikzcd} B_{\mathrm{spin}^c(m)} \times B_{S^1} \arrow[r, "\bar{m}_1"] \arrow[d, "\bar{j} \times \bar{d}"] & B_{\mathrm{spin}^c(m)} \arrow[d, "\bar{j}"] \\ B_{S^1} \times B_{S^1} \arrow[r, "\bar{m}_2"] & B_{S^1} . \end{tikzcd}$$

The map $\bar{m}_1$ makes $B_{\mathrm{spin}^c(m)}$ the total space of a principle B_{S^1} bundle over $B_{SO(m)}$ and there is a commutative diagram of fiber spaces

$$(3.2)\qquad \begin{tikzcd} B_{S^1} \arrow[r] \arrow[d, "\bar{i}"] & B_{S^1} \arrow[d, "\bar{d}"] \\ B_{\mathrm{spin}^c(m)} \arrow[r, "\bar{j}"] \arrow[d, "\bar{\pi}"] & B_{S^1} \arrow[d, "\lambda"] \\ B_{SO(m)} \arrow[r, "W_2"] & K[Z_2, 2]. \end{tikzcd}$$

which shows that the principle bundle ξ defined by $\bar{\pi}$ is induced from the bundle over $K[Z_2, 2]$ via the map W_2. Of course, the bundle over $K[Z_2, 2]$ arises from the diagram of groups $1 \to Z_2 \to S^1 \xrightarrow{d} S^1 \to 1$.

Principle B_{S^1} bundles over $B_{SO(m)}$ induced from this principle B_{S^1} bundle over $K[Z_2, 2]$ are classified by $H^2(B_{SO(m)}, Z_2) \cong Z_2$. This group is generated by the universal second Stiefel-Whitney class W_2. Thus, to justify the notation W_2 for the map inducing the bundle ξ, it suffices to show that $\pi_2(B_{\mathrm{spin}^c(m)})$ is not $Z \oplus Z_2 = \pi_2(B_{S^1} \times B_{SO(m)})$, i.e., that W_2 is not the trivial map. But this follows from the fact that $\pi_1(\mathrm{spin}^c(m)) = Z$ which is a consequence of the fact that spin(m) is simply connected and the exact sequence (2.4). This gives

LEMMA 3.3. *$B_{\mathrm{spin}^c(m)}$ is the total space of a principle B_{S^1} bundle over $B_{SO(m)}$ induced from the nontrivial bundle over $K[Z_2, 2]$ by a map $W_2: B_{SO(m)} \to K[Z_2, 2]$ realizing the universal second Stiefel-Whitney class.*

Let δ be a principle $SO(m)$ bundle over a space X classified by a map $c: X \to B_{SO(m)}$. By definition a spin$^c(m)$ structure (briefly a spin^c structure) on δ is a homotopy class of maps $\tilde{c}: X \to B_{\mathrm{spin}^c(m)}$ such that $\bar{\pi}\tilde{c}$ is homotopic to c. Let $\sigma \in H^2(B_{S^1}, Z)$ be a generator of this group.

LEMMA 3.4. *The* mod 2 *reduction of $\tilde{c}^*\bar{j}^*(\sigma)$ is $W_2(\delta)$, the second Stiefel-Whitney class of δ.*

PROOF. Let σ_2 be the mod 2 reduction of σ. Then if $i \in H^2(K[Z_2, 2], Z_2)$ is the generator, $\lambda^*(i) = \sigma_2$ and

$$W_2(\delta) = c^* W_2^*(i) = \tilde{c}^*\bar{\pi}^* W_2^*(i) = \tilde{c}^*\bar{j}^*\sigma_2$$

which is the mod 2 reduction of $(\tilde{c}^*\bar{j}^*\sigma)$.

We remark that the multiplication $\bar{m}_2$ of (3.1) corresponds to the tensor product of complex line bundles. Since $B_{\mathrm{spin}^c(m)}$ is the total space of a principle B_{S^1} action, $H^2(X, Z)$ acts on $[X, B_{\mathrm{spin}^c(m)}]$, the set of homotopy classes of maps of X to $B_{\mathrm{spin}^c(m)}$, in the following manner. Let $f \in [X, B_{\mathrm{spin}^c(m)}]$ and $g \in [X, B_{S^1}] = H^2(X, Z)$. Then we obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f \times g"] & B_{\mathrm{spin}^c(m)} \times B_{S^1} \arrow[r, "\bar{m}_1"] \arrow[d, "\bar{j} \times \bar{d}"] & B_{\mathrm{spin}^c(m)} \arrow[d, "\bar{j}"] \\
& B_{S^1} \times B_{S^1} \arrow[r, "\bar{m}_2"] & B_{S^1}.
\end{tikzcd}$$

Denote the composition $\bar{m}_1 f \times g$ by $f \circ g$. This defines the action of $H^2(X, Z)$ on $[X, B_{\mathrm{spin}^c(m)}]$.

Let $\hat{f}$ denote the complex line bundle over X defined by $\bar{j}f$. If P_f denotes the principle spin$^c(m)$ bundle over X induced by f, then $\hat{f} = P_f \times_{\mathrm{spin}^c(m)} C$ where spin$^c(m)$ acts on C via the representation j of spin$^c(m)$ to S^1 given by (2.4). Let $\hat{g}$ denote the complex line bundle over X determined by g.

LEMMA 3.5. $(f \circ g)\hat{} = \hat{f} \cdot \hat{g}^2$.

PROOF. $(f\circ g)\hat{} = \bar{j}\bar{m}_1(f \times g) = \bar{m}_2(\bar{j} \times \bar{d})(f \times g) = \bar{m}_2(\bar{j}f \times \bar{d}g) = \hat{f} \cdot \hat{g}^2$
Since $B_{\mathrm{spin}^c(m)}$ is the fiber product of

$$B_{SO(m)} \xrightarrow{w_2} K[Z_2, 2] \quad \text{and} \quad B_{S^1} \xrightarrow{\lambda} K[Z_2, 2]$$

we have:

COROLLARY 3.6. *The* $\mathrm{spin}^c(m)$ *structures on a principle* $SO(m)$ *bundle* δ *are in* 1-1 *correspondence with elements* $d \in H^2(X, Z)$ *whose* mod 2 *reduction is* $W_2(\delta)$ *the second Stiefel-Whitney class of* δ. *An explicit correspondence is this: Let* P *be the total space of a principle* $\mathrm{spin}^c(m)$ *bundle such that the orbit space* $P/S^1 = Q$ *of* P *by* $S^1 \subset \mathrm{spin}^c(m)$ *is the total space of* δ. *(Since* $\mathrm{spin}^c(m)/S^1 = SO(m)$, Q *is the total space of a principle* $SO(m)$ *bundle.) Then the correspondence is given by* $P \to c_1(\xi)$. *Here* ξ *is the line bundle whose total space is* $P \times_{\mathrm{spin}^c(m)} C$ *and* $c_1(\xi)$ *is its first Chern class.*

Suppose that X is a smooth m dimensional manifold. By definition a $\mathrm{spin}^c(m)$ structure on X is a $\mathrm{spin}^c(m)$ structure on its tangent bundle TX.

LEMMA 3.7. *If* $H^3(X, Z_2) = 0$ *then* X *has a* $\mathrm{spin}^c(m)$ *structure.*

PROOF. $H^2(X, Z) \to H^2(X, Z_2)$ is onto.

REMARK. *We find it convenient at times to use the total space* P *of a principle* $\mathrm{spin}^c(m)$ *bundle to designate the* $\mathrm{spin}^c(m)$ *structure it defines.*

4. **Orientation of G manifolds and Poincaré duality.** Let X be a compact G manifold of dimension m. Let W be a (real) G vector bundle over X of dimension k. An orientation for W is a class $\omega_G \in K_G^k(W)$ such that $i^*\omega_G \in K_G^k(W|O)$ generates $K_G^*(W|O)$ freely over $K_G^*(O)$ for every orbit O. Here i is the inclusion of $W|O$ in W.

DEFINITION. *An orientation for* X *is an orientation* $\alpha_G \in K_G^m(TX)$ *of the tangent bundle of* X.

Observe that if X has a boundary ∂X, then ∂X is oriented by $j^*(\alpha_G)$, $j:\partial X \to X$ because

$$K_G^m(TX|_{\partial X}) = K_G^m(T\partial X \times R^1) = K_G^{m-1}(T\partial X).$$

An orientation class α_G provides a Thom homomorphism $\psi = \psi_G^X : K_G^*(X) \to K_G^*(TX)$, $\psi_G^X(\lambda) = \alpha_G \cdot \lambda$.

LEMMA 4.1. ψ_G^X *is an isomorphism.*

PROOF. Let $\bar{X}$ denote the orbit space of X by G. There are two sheaves over $\bar{X}$, $\mathcal{S}_q$ and $\mathcal{T}_q$ whose stalks are respectively

$$\mathcal{S}_q(\bar{x}) = K_G^q(Gx), \qquad \mathcal{T}_q(\bar{x}) = K_G^q(TGx),$$

where $\bar{x} \in \bar{X}$ and $Gx \subset X$ is the orbit of $x \in X$ lying over $\bar{x}$.

Multiplication by α_G induces a map of the spectral sequence [**12**]

$$E_2^{p,q} = H^p(\bar{X}, \mathscr{S}_q) \Rightarrow K_G^*(X)$$

to the spectral sequence

$$E_2^{p,q} = H^p(\bar{X}, \mathscr{T}_q) \Rightarrow K_G^*(TX)$$

which is an isomorphism on the E_2 level.

COROLLARY 4.2. *$\psi_G^{\partial X}$ is an isomorphism.*

COROLLARY 4.3. *$\psi_G^{(X,\partial X)}: K_G^*(X, \partial X) \to K_G^*(TX, TX|_{\partial X})$ is an isomorphism.*

PROOF. Multiplication by α_G induces a map of the exact sequence of the pair $(X, \partial X)$ to $(TX, TX|_{\partial X})$ which is an isomorphism on two terms by the preceding. The result follows by the five lemma.

REMARK. *We could equally well have defined an orientation for X by means of a class $\beta_G \in K_G^*(NX)$ where NX is the normal bundle of X which is equivariantly imbedded in a complex representation space M for G. These are equivalent concepts.*

The significance of this remark is that $K_G^*(NX)$ is the equivariant homology of X dual to $K_G^*(X)$ if X is a closed manifold. To see this, note that $X \subset M \subset M^+ = S^{2n}$ where M^+ is the one point compactification of M which we assume has complex dimension n. Then by definition

$$K_i^G(X) = K_G^{2n-i}(S^{2n}, S^{2n} - X) \cong K_G^{2n-i}(NX, NX|_{\partial X}) \tag{4.4}$$

by excision.

The map C which collapses the exterior of the closure $\overline{NX}$ of NX in M^+ induces

$$C^*: K_G^*(\overline{NX}, \partial\overline{NX}) \to K_G^*(M, +) = R(G).$$

The composition of C^* with the map $m: K_G^*(X) \otimes K_G^*(NX, NX|_{\partial X}) \to K_G^*(NX, NX|_{\partial X})$ which exhibits $K_G^*(NX, NX|_{\partial X})$ as a module over $K_G^*(X)$ defines the duality pairing

$$\hat{d}: K_G^i(X) \otimes K_{m-i}^G(X) \to R(G).$$

There is a second duality pairing which is more appropriately related to our purpose when X is a closed manifold. It is the map

$$\hat{d}: K_G^*(X) \otimes K_G^*(TX) \to R(G)$$

which is defined by

$$\hat{d}(x \otimes y) = \mathrm{Id}_G^X(x \cdot y)$$

for $x \in K_G^*(X)$ and $y \in K_G^*(TX)$.

REMARK. *When X is a closed manifold, these two pairings are the same. The point is that the pairing d is more generally defined while the pairing $\hat{d}$ is more accessible to computation because of the properties of the index homomorphism. To extend the definition of d to spaces X which are not manifolds we must assume that X is imbedded in a complex representation space M of G with an equivariant regular neighborhood $\overline{NX} \subset M^+$. Then $K_i^G(X) = K_G^{2n-i}(\overline{NX}, \partial \overline{NX})$ and d is defined for such G spaces X as above.*

To justify the above remark we offer

PROPOSITION 4.5. *If X is a closed G manifold there is an isomorphism $\phi: K_G^*(TX) \to K_G^*(NX)$ which takes $\hat{d}$ to d.*

PROOF. There is a commutative diagram of vector bundles

$$\begin{array}{ccc} TNX & \to & TX \\ \downarrow & & \downarrow \pi \\ NX & \to & X . \end{array}$$

Since $NX \subset M$ is an open subset, $TNX \cong NX \times M$ as a G vector bundle over NX. However, $TNX \cong \pi^*(NX \otimes C)$. Since TNX is a complex G bundle over NX as well as over TX, we have Thom isomorphisms

$$\psi_1: K_G^*(NX) \to K_G^*(TNX),$$

$$\psi_2: K_G^*(TX) \to K_G^*(TNX),$$

and a commutative diagram

$$\begin{array}{ccccc} K_G^*(NX) \otimes K_G^*(X) & \xrightarrow{m} & K_G^*(NX) & \xrightarrow{C^*} & K_G^*(M) \\ \downarrow \psi_1 \otimes 1 & & \downarrow \psi_1 & & \downarrow \\ K^*(TNX) \otimes K_G^*(X) & \xrightarrow{m} & K_G^*(TNX) & \longrightarrow & K_G^*(TM) \\ \uparrow \psi_2 \otimes 1 & & \uparrow \psi_2 & & \uparrow \\ K_G^*(TX) \otimes K_G^*(X) & \xrightarrow{m} & K_G^*(TX) & \xrightarrow{\mathrm{Id}_G^*} & R(G) \end{array}$$

in which all vertical maps are isomorphisms. Since d is defined by the top row and $\hat{d}$ by the bottom, the demonstration is complete.

With the equivariant homology of a manifold X defined by (4.4) we obtain Poincaré duality free if an orientation is given.

PROPOSITION 4.6. *If X is a compact oriented G manifold of dimension m, then X satisfies Poincaré duality: $K_i^G(X) \cong K_G^{m-i}(X, \partial X)$.*

PROOF. If $\alpha_G \in K_G^m(TX)$ is an orientation then

$$\psi^X: K_G^*(X, \partial X) \to K_G^*(TX, TX|_{\partial X})$$

is an isomorphism. But

$$K_G^*(TX, TX|_{\partial X}) \cong K_G^*(NX, NX|_{\partial X}) = K_*^G(X).$$

In view of Proposition 4.6, we expect the difficulties of studying G actions on manifolds by using Poincaré duality to be intimately connected with the existence of an orientation. There is a very general situation in which it is often possible to construct an orientation. This occurs when X is a spin$^c(m)$ manifold. Again this means there is a principle spin$^c(m)$ bundle P over X such that

$$P \times_{\text{spin}^c(m)} V = TX \quad \text{the tangent bundle of } X.$$

(4.7) We assume: that G acts on the left on P, commutes with the right action of spin$^c(m)$ and is compatible with the natural left action on Q, the principle $SO(m)$ bundle associated to TX, (in other words, the frame bundle of TX), obtained by sending $(g, [v_1, v_2, \ldots, v_m]) \to [dgv_1, dgv_2, \ldots, dgv_m]$ for $g \in G$, $[v_1, \ldots, v_m]$ a frame in Q and dg the differential of G.

The elliptic pairing (2.2) is the basic property for constructing a class δ_G in $K_G^m(TX)$ from the equivariant spin$^c(m)$ structure on X.

For brevity, set $H = \text{spin}^c(m)$. We define a $G \times H$ complex of vector bundles [6, p. 489] over $P \times V$:

$$P \times V \times \Delta_+ \xrightarrow{\Phi} P \times V \times \Delta_-,$$

$$\Phi(p, v, \delta) = (p, \theta(v, \delta)).$$

The $G \times H$ action on $P \times V$ is given by

$$(g, h)(p, v) = (gph^{-1}, hv);$$

the action on $P \times V \times \Delta_\pm$ is given by

$$(g, h)(p, v, b) = (gph^{-1}, hv, hb).$$

Since θ is an elliptic pairing, this complex defines an element

$$\delta_G \in K_{G \times H}^*(P \times V) = K_G^m(P \times_H V) = K_G^m(TX).$$

Here are some examples in which δ_G or a close variant defines an orientation.

EXAMPLE 1. X is a point, m is even, $G = U(m/2)$ is the unitary group of isometries of $C^{m/2}$ and $\hat{V} = C^{m/2}$ denotes the standard $U(m/2)$ module. It is a $U(m/2)$ bundle over a point. We define an orientation class Δ_U for $\hat{V}$.

The elliptic pairing $\theta : V \times \Delta_+ \to V \times \Delta_-$ of (2.2) gives an elliptic $U(m/2)$ complex

$$\hat{\theta} : \hat{V} \times \Delta_+ \to \hat{V} \times \Delta_-$$

over a point. Here Δ_+ and Δ_- are $U(m/2)$ modules via the homomorphism ψ_0 of (2.3). This complex defines an element $\Delta_U \in K_U^*(\hat{V})$.

PROPOSITION 4.8. *$K_U^*(\hat{V})$ is a free module over $R(U) = K_U^*(\mathrm{pt})$ generated by Δ_U.*

PROOF. The symbol of the de Rham complex of $\hat{V}$, $\lambda_{\hat{V}} \in K_U^*(\hat{V})$ is a generator [**6**], so

$$\Delta_U = a \cdot \lambda_{\hat{V}} \in K_U^*(\hat{V})$$

for some $a \in R(U)$. Let $j: \mathfrak{T} \to U$ denote the inclusion of the maximal torus and $\hat{j}$ the composition

$$K_U^*(\hat{V}) \xrightarrow{i^*} K_U^*(0) \xrightarrow{j^*} K_T^*(0) = R(\mathfrak{T}).$$

Here i^* is the restriction defined by the inclusion of the origin 0. Now

$$\hat{j}\lambda_{\hat{V}} = \lambda_{-1}(\hat{V}) = \prod_{i=1}^{m/2} (1 - t_i) \in R(\mathfrak{T}),$$

$$R(\mathfrak{T}) = Z[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_{m/2}, t_{m/2}^{-1}].$$

But $\hat{j}\Delta_U = (\Delta_+ - \Delta_-)|_T = \prod(1 - t_i^{-1})t_i$. This follows from the definition of ψ_0 and the fact that the trace of $t = (t_1, t_2, \ldots, t_{m/2})$ acting on Δ_+ minus the trace of t on Δ_- is $\prod(1 - t_i^{-1})t_i$. Since $R(U) \to R(\mathfrak{T})$ is injective, $a = (-1)^{m/2}$.

If $G \xrightarrow{\rho} U(m/2)$ is a representation then $\Delta_G \in K_G^*(\hat{V})$, is defined to be $\rho^*\Delta_U$. It generates $K_G^*(\hat{V})$ freely over $R(G)$.

EXAMPLE 2. X is a simply connected spin$^c(m)$ manifold with an S^1 action which satisfies (4.7).

PROPOSITION 4.9. *δ_{S^1} is an orientation for X.*

PROOF. There are two kinds of orbits, a point and S^1. If p is a fixed point and $i: TX|_p \to TX$ the inclusion, $TX|_p = \hat{V}$ as an S^1 bundle over p and

$$i^*\delta_{S^1} = \Delta_{S^1} \in K_{S^1}^*(\hat{V}).$$

If S^1 is an orbit, then because X is simply connected

$$P|_{S^1} \cong S^1 \times \mathrm{spin}^c(m)$$

is the restriction of the principle spin$^c(m)$ bundle over X to the orbit S^1. If x denotes the point S^1/S^1, then $K_{S^1}^*(S^1) = K^*(x)$ and

$$K_{S^1}^*(TX|_{S^1}) = K_{S^1}^*(S^1 \times \hat{V}) \cong K^*(\hat{V})$$

and $i^*(\delta_{S^1}) = \Delta_1 \in K^*(\hat{V})$ is an orientation for $\hat{V}$ over x. Here 1 denotes the trivial group.

EXAMPLE 3. $X = U(Z)$ is the group of isometries of a complex G module Z of real dimension m. So as a space $X = U(m/2)$. The G module structure on Z gives a representation ρ of G in $U(Z)$ and G acts on $X = U(Z)$ via inner automorphisms. Then

$$TU(Z) = U(Z) \times M \tag{4.10}$$

where M is the real G module ρ^*M^1 and M^1 is the tangent space of $U(Z)$ at the identity. It is a real $U(Z)$ module via the adjoint representation of $U(Z)$ on M^1. We emphasize the fact that (4.10) is the expression of $TU(Z)$ as a G vector bundle over $U(Z)$.

Either M^1 or $M^1 \times R^1$, depending on the dimension of M^1, is a complex $U(Z)$ module $\hat{V}$. Then restricting to G we obtain a complex of G vector bundles

$$U(Z) \times \hat{V} \times \Delta_+ \xrightarrow{1 \times \hat{\theta}} U(Z) \times \hat{V} \times \Delta_-$$

whose symbol $\alpha_G \in K_G^0(U(Z) \times \hat{V})$ is

$$1 \otimes \Delta_G \in K_G^*(U(Z)) \otimes_{R(G)} K_G^*(\hat{V}) = K_G^*(U(Z) \times \hat{V})$$

by the Künneth theorem of [**10**]. Since $U(Z) \times \hat{V}$ is either $TU(Z)$ or $TU(Z) \times R^1$ as a G space, $\alpha_G \in K_G^*(TU(Z))$ and

PROPOSITION 4.11. *α_G is an orientation.*

PROOF. This follows from the fact that $\alpha_G = 1 \otimes \Delta_G$ and Proposition 4.8 together with the Künneth theorem.

EXAMPLE 4. $X = G/H$ where H and G are compact connected Lie groups, H has maximal rank in G and G/H has a spin^c structure. G acts on G/H by left translation. Then the G action on X satisfies the condition of (4.7) and the class α_G constructed from the spin^c structure is an orientation for X. We omit details of the proof.

5. **The homomorphism** $K_G^*(X) \xrightarrow{\psi_G} K_G^*(TX) \xrightarrow{\mathrm{Id}_G^X} R(G)$. When X is oriented we can consider the composition $\mathrm{Id}_G^X \psi_G = \tau_G^X$. It provides powerful invariants for the G action on X. The most important case for our purpose occurs when X has a spin$^c(m)$ structure (m even) and is oriented by δ_G. We require an explicit formula for $\mathrm{Id}_G^X(\delta_G)$ in terms of the representations of G in the normal fibers to the fixed point sets.

We make these assumptions:

(5.0) G is topologically cyclic, i.e., has a dense generator g and for each component X_i^G of the fixed point set of G, X^G, we have $H^3(X_i^G, Z_2) = 0$. In addition we assume that the action of G on X satisfies (4.7).

We now investigate the behavior of the class δ_G when restricted to $K_G^*(TX^G)$. Over X_j^G we have a diagram of bundles

$$\begin{array}{ccc} P_0 \subset P|_{X_j^G} & = & \overline{P} \\ \big\downarrow & & \big\downarrow \\ Q_0 \subset Q|_{X_j^G} & = & \overline{Q} \end{array}$$

where $\overline{Q}$ is the principle $SO(m)$ bundle associated to $TX|_{X^G}$ and Q_0 is the principle $SO(k_j) \times SO(m - k^j)$ associated to $TX_j^G \oplus NX_j^{G^j}$. Q_0 is a reduction of $\overline{Q}$. Here we assume X_j^G is orientable so its dimension k_j is even as is m. The group of the bundle P_0 is $\text{spin}^c(k_j) \times_{S^1} \text{spin}^c(m - k_j) = H_0$. For convenience of exposition, we concentrate on a particular component of X^G and drop the subscript j. We are interested in the class $Ti^*\delta_G \in K_G(TX^G)$; $Ti: TX^G \to TX$ is the inclusion. Note that Ti is the composition

$$TX^G \xrightarrow{S} TX^G \oplus NX^G \xrightarrow{j} TX$$

where S is the zero section of $TX^G \oplus NX^G$ as a bundle over TX^G and j is the inclusion.

Let $d \in H^2(X)$ be the class associated to the principle $\text{spin}^c(m)$ bundle P over X. See Corollary 3.6. Since $H^3(X^G, Z_2) = 0$, every element of $H^2(X^G, Z_2)$ is the reduction of an element of $H^2(X^G, Z)$. Choose $d_1 \in H^2(X^G, Z)$ whose reduction is $W_2(X^G)$ and let $d_2 = f^*d - d_1$; $f: X^G \to X$ the inclusion. The principle $\text{spin}^c(k)$ and $\text{spin}^c(m - k)$ bundles determined by d_1 and d_2 are denoted by P_1 and P_2. Then $P_1 \times P_2$ is the total space of a principle $\hat{H}_0 = \text{spin}^c(k) \times \text{spin}^c(m - k)$ bundle over $X \times X$ whose orbit space $P_1 \times_{S^1} P_2$ by the diagonal action $g(x, y) = (g^{-1}x, gy)$, $g \in S^1, x \in P_1$, $y \in P_2$, is the total space of a principle H_0 bundle over $X^G \times X^G$. If $D: X^G \to X^G \times X^G$ denotes the diagonal map then $D^*P_1 \times_{S^1} P_2 = P_0$.

Now we remark that

$$\overline{P} \times_{\text{spin}^c(m)} V \times \Delta_\pm = P_0 \times_{H_0} V \times \Delta_\pm.$$

The homomorphism $\text{spin}^c(k) \times \text{spin}^c(m - k) \to \text{spin}^c(m)$ makes $\Delta_\pm$ and V $\hat{H}_0$ modules and these decompose as

$$V = V^1 \oplus V^2,$$

$$\Delta_+ = \Delta_+^1 \cdot \Delta_+^2 \oplus \Delta_-^1 \cdot \Delta_-^2, \qquad \Delta_- = \Delta_+^1 \cdot \Delta_-^2 \oplus \Delta_-^1 \cdot \Delta_+^2,$$

where $\cdot$ denotes tensor product. Here the superscript 1 denotes a $\text{spin}^c(k)$ module and a superscript 2 a $\text{spin}^c(m - k)$ module.

Next we observe that

$$P_0 \times_{H_0} V \times \Delta_\pm = \hat{P}_0 \times_{\hat{H}_0} V \times \Delta_\pm$$

where $\hat{P}_0 = D^*P_1 \times P_2$. From these observations we have

$$j^*\delta_G = \tilde{D}^*\gamma_G \hat{\otimes} \theta_G = \gamma_G \cdot \theta_G \in K_G^*(TX^G \oplus NX^G).$$

Here $\gamma_G \hat{\otimes} \theta_G \in K_G^*(TX^G \times NX^G)$ is the external tensor product of the class $\gamma_G \in K_G^*(TX^G)$ defined by the complex

$$P_1 \times_{\text{spin}^c(k)} V^1 \times \Delta_+^1 \to P_1 \times_{\text{spin}^c(k)} V^1 \times \Delta_-^1,$$

$\theta_G \in K_G^*(NX^G)$ is the class defined by the complex

$$P_2 \times_{\text{spin}^c(m-k)} V^2 \times \Delta_+^2 \to P_2 \times_{\text{spin}^c(m-k)} V^2 \times \Delta_-^2$$

and $\tilde{D}: TX^G \oplus NX^G \to TX^G \times NX^G$ is the bundle map covering the diagonal map D. Now $P_2 \times_{\text{spin}^c(m-k)} V^2 \times \Delta_\pm^2 \times \omega_\pm$ are vector bundles over X^G and $S^*\gamma_G\theta_G = \gamma_G \cdot (\omega_+ - \omega_-)$ hence

$$Ti^*\delta_G = S^*j^*\alpha_G = \gamma_G(\omega_+ - \omega_-) \in K_G^*(TX^G).$$

We record this as

LEMMA 5.1. *Let G be cyclic and $H^3(X^G, Z_2) = 0$. Let $d \in H^2(X, Z)$ determine a* spin$^c(m)$ *structure P on X. Choose $d_1 \in H^2(X^G, Z)$ whose* mod 2 *reduction is $W_2(X^G)$ and let $d_2 = f^*(d) - d_1$, $f: X^G \to X$ the inclusion. Then d_1 and d_2 determine* spin^c *structures P_1 and P_2 on TX^G and NX^G and we obtain G vector bundles $\omega_+, \omega_- \in K_G^*(X)$ coming from P_2 and a class $\gamma_G \in K_G^*(TX^G)$ coming from P_1 such that*

$$Ti^*\delta_G = \gamma_G(\omega_+ - \omega_-) \in K_G^*(TX^G)$$

where δ_G is the orientation class associated to the principle spin$^c(m)$ *bundle P.*

Now the action of G on NX^G gives a representation $\rho: G \to SO(m-k)$. Namely if $x \in X^G$ then $gx = x$; so, for $q \in Q_0$,

$$gq = q\rho(g),$$

$$\rho(g) = \text{diag}\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix} \in SO(m-k),$$

$i = 1, 2, \ldots, (m-k)/2$. The numbers θ_i are well defined modulo $\pi = 3.1416\ldots$.

The action of G on P_2 gives a homomorphism $\tilde{\rho}: G \to \text{spin}^c(m-k)$ and a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\tilde{\rho}"] \arrow[dr, "\rho"'] & \text{spin}^c(m-k) \arrow[d, "\pi"] \\ & SO(m-k). \end{tikzcd}$$

Moreover,

$$\tilde{\rho}(g) = \left[\prod_{j=1}^{(m-k)/2} (\cos\theta_j/2 - \sin\theta_j/2\, e_{2j-1}e_{2j}),\ \exp[-i(\textstyle\sum\theta_j/2 + \lambda)]\right]$$

for some $\lambda = \lambda(g)$, $g \to e^{-i\lambda}$ is a homomorphism from G to S^1 and $gp = p\tilde{\rho}(g)$ if p lies over x; thus the action of g on the fibers Δ_+^2 and Δ_-^2 of ω_+ and ω_- is given by the representation $\tilde{\rho}$.

REMARK. *Observe that if X^G and hence NX^G has n components, we obtain n representations $\tilde{\rho}_i$, $i = 1, 2, \ldots, n$, which completely describe the action of G on the fibers of ω_+ and ω_-.*

The normal bundle NX^G has a decomposition invariant under G,

$$NX^G = NX^G(-1) + \sum_u NX^G(u),$$

where the u are complex numbers of absolute value 1 with positive imaginary part and $NX^G(u)$ has a complex structure in which $g \in G$ operates by multiplication by u. This gives a lifting $\hat{\rho}$ of the representation $\rho: G \to SO(m - k)$ to $U((m - k)/2)$ and

$$\hat{\rho}(g) = \operatorname{diag}\{vI_v\} \subset U((m - k)/2).$$

I_v is the $n_v \times n_v$ identity matrix where $n_v = \dim_C N(v)$, $v = e^{i\theta}, 0 \leqq \theta < \pi$.

For a complex vector bundle L of dimension s and complex number z, set

$$\mathscr{F}(L, z) = z^{-s/2} \prod_{j=1}^{s} [z^{-1/2}e^{-y_j/2} - z^{1/2}e^{y_j/2}]^{-1}$$

$$= \prod_{j=1}^{s} [e^{-y_j/2} - ze^{y_j/2}]^{-1}$$

where the y_j are the formal roots of the total Chern class of L. Observe that

$$\mathscr{F}(L, -1) = \prod [e^{-y_j/2} + e^{y_j/2}]^{-1}$$

also makes sense because this is a symmetric function of the y_j^2 and so is a function of the Pontrjagin classes of $N(-1)$.

Another cohomology class is defined by

$$\hat{\mathscr{A}}(X^G) = \prod_j x_j(e^{x_j/2} - e^{-x_j/2})^{-1} \in H^*(X^G, \boldsymbol{C}).$$

The elementary symmetric functions of the variables x_j^2 are the Pontrjagin classes of X^G.

We are now in position to give an explicit formula for $\operatorname{Id}_G^X[\delta_G](g) \in \boldsymbol{C}$, i.e., the value of the character $\operatorname{Id}_G^X[\delta_G]$ at $g \in G$.

PROPOSITION 5.2

$$\mathrm{Id}_G^X[\delta_G](g) = \sum_j \left(e^{f_j^*(d)/2} \mathscr{A}(X_j^G) \prod_{u_j} \mathscr{F}(NX_j^G(u_j), u_j) e^{-i\lambda_j} \right) [X_j^G].$$

The sum is over the components X_j^G of X^G. The numbers u_j are determined by the representation $\rho_j: G \to SO(m - k_j)$. Explicitly if

$$\rho_j(g) = \mathrm{diag}\left\{ \begin{pmatrix} \cos\theta_l & \sin\theta_l \\ -\sin\theta_l & \cos\theta_l \end{pmatrix} \right\} \subset SO(m - k_j),$$

then the u_j's are the complex numbers $\{e^{i\theta_l}\}$ and $\dim_C(NX^G(u))$ = number of l with $e^{i\theta_l} = u$. The number $\lambda_j = \lambda_j(g)$ associated to the jth component is determined by the lifting $\tilde{\rho}_j: G \to \mathrm{spin}^c(m - k)$ via

$$\tilde{\rho}_j(g) = \left[\prod_{l=1}^{(m-k)/2} \cos\theta_l/2 - \sin\theta_l/2\, e_{2l-1}e_{2l}, \exp[-i(\textstyle\sum\theta_l/2 + \lambda_j)] \right]$$

and $g \to e^{i\lambda_j(g)}$ defines a homomorphism of G to S^1. The inclusion of X_j^G in X is denoted by f_j and the cohomology class in the sum is evaluated on the orientation class $[X_j^G]$. The class $d \in H^2(X, Z)$ determines the spin^c *structure on X used to define δ_G.*

PROOF. $K_G^*(X^G) = R(G) \otimes K^*(X^G)$.

Let $ch_g: K_G^*(X^G) \to K^*(X^G, C)$ be defined by $ch_g \chi \otimes u = \chi(g) \cdot ch(u)$ for $\chi \in R(G)$ and $u \in H^*(X^G)$. Let $e \otimes \mathrm{Id}_1^{X^G} R(G) \otimes K^*(X^G) \to C \otimes_Z Z = C$ be defined by $e \otimes \mathrm{Id}_1^{X^G}(\chi \otimes u) = \chi(g) \cdot \mathrm{Id}_1^{X^G}(u)$. Let $Ti: TX^G \to TX$ be the inclusion. Then

$$\mathrm{Id}_G^X \delta_G(g) = e \otimes \mathrm{Id}_1^{X^G}(Ti^* \delta_G / \lambda_{-1}(NX^G \otimes C)) \quad \text{by [5] or Part I, (1.6),}$$

$$= e \otimes \mathrm{Id}_1^{X^G} \left(\gamma_G \left\{ \frac{\omega_+ | X_j^G - \omega_- | X_j^G}{\lambda_{-1}(NX^G \otimes C)} \right\} \right), \quad \text{by Lemma 5.1,}$$

$$= \sum_j \left[\frac{ch_1 \gamma_G|_{X_j^G} \mathscr{I}(X_j^G)}{\prod w_{jl}} \right] ch_g \left(\frac{\omega_+ | X_j^G - \omega_- | X_j^G}{\lambda_{-1}(NX_j^G \otimes C)} \right) [X_j^G],$$

where the w_{jl}, $l = 1, 2, \ldots$, are the formal roots of the Pontrjagin classes of X_j. This formula follows from [6, p. 559, Proposition 2.17]. Let d_{1j} and d_{2j} be the jth components in $H^2(X^G, Z) = \prod_j H^2(X_j^G, Z)$ of the classes d_1 and d_2 discussed in Lemma 5.1. Then

$$ch_1 \gamma_G|_{X_j^G} = e^{d_{1j}/2} \prod_l e^{w_{jl}/2} - e^{-w_{jl}/2},$$

$$\mathscr{I}(X_j^G) = \prod \frac{w_{jl}}{1 - e^{w_{jl}}} \cdot \frac{-w_{jl}}{1 - e^{-w_{jl}}}$$

and

$$ch_g\left(\frac{\omega_+|X_j^G - \omega_-|X_j^G}{\lambda_{-1}(NX_j^G \otimes C)}\right) = e^{d_{2j}/2}e^{-i\lambda_j}\prod_{u_j}\mathscr{F}(NX^G(u_j), u_j).$$

Since $d_{1j} + d_{2j} = f_j^*(d)$, the result follows.

We can give a better geometric interpretation of the numbers λ_j as follows. Let

$$\omega = P \times_{\mathrm{spin}^c(m)} \boldsymbol{C}$$

be the line bundle over X determined by the representation $\mathrm{spin}^c(m) \to S^1$ which sends $[g, t]$ to t^2. Then ω is a G bundle over X. Here G acts on ω via

$$g[p, c] = [gp, c], \qquad p \in P, c \in \boldsymbol{C}.$$

The restriction of ω to the component X_j^G defines a one dimensional complex representation of G, namely the representation of G in the fibers of ω. So if $x \in X_j^G$ and ω_x is the fiber over x, then g operates on ω_x by multiplication by $e^{i\omega_j}$.

PROPOSITION 5.3: *The numbers ω_j defined by the restriction of the line bundle ω to the component X_j^G by*

$$\omega|_{p_j}(g) = e^{i\omega_j}, \qquad p_j \in X_j^G,$$

and the numbers λ_j and θ_{jl} defined by the representation of G on NX_j^G by

$$\tilde{\rho}_j : G \to \mathrm{spin}^c(m - k_j),$$

$$\tilde{\rho}_j(g) = \left[\prod_l (\cos\theta_{jl}/2 - \sin\theta_{jl}/2\, e_{2l-1}e_{2l}, \exp\left[-i\left(\sum_l \theta_{jl}/2 + \lambda_j\right)\right]\right],$$

are related by

$$-\left(\sum_l \theta_{jl} + 2\lambda_j\right) = \omega_j.$$

PROOF. If $x \in X_j^G$, then $gx = x$; so if $p \in P$ lies over x, then

$$gp = p\tilde{\rho}_j(g)$$

and

$$\begin{aligned} g[p, c] &= [p\rho_j(g), c] = [p, \rho_j(g)c] \\ &= [p, \exp[-i(\textstyle\sum \theta_{jl} + 2\lambda_j)]c] \end{aligned}$$

and this shows that the representation of G on ω_x has character value, at $g \in G$, $\exp[-i(\sum \theta_{jl} + 2\lambda_j)]$.

REMARK. *More generally for* $u \in K_G^*(X)$,

$$\mathrm{Id}_G^X(\delta_G \cdot u)(g) = \sum_j ch_g(u|X_j^G) \cdot A_j[X_j^G], \qquad g \in G$$

where A_j *is the expression in parentheses occurring in Proposition* 5.2. *In particular when* $g = 1 \in G$,

$$\mathrm{Id}_G^X(\delta_G \cdot u)(1) = ch(u)e^{d/2}\hat{\mathscr{A}}(X)[X].$$

An important special case occurs when $G = S^1$ and the fixed point set X^G consists of isolated points p_j, $j = 1, 2, \ldots, l$. Let $t = e^{i\theta} \in S^1$ be a generic point. Then

$$\rho_j(t) = \operatorname{diag}\left\{\begin{pmatrix} \cos x_{jk}\theta & \sin x_{jk}\theta \\ -\sin x_{jk}\theta & \cos x_{jk}\theta \end{pmatrix}\right\}, \tag{5.4}$$

$k = 1, 2, \ldots, m/2$, where each x_{jk} is an integer well defined up to sign.

$$\tilde{\rho}_j(t) = \left[\prod \cos x_{jk}\theta/2 - \sin x_{jk}\theta/2\, e_{2k-1}e_{2k}, e^{-i\alpha_j}\right] \tag{5.5}$$

where $\alpha_j = \sum_k x_{jk}\theta/2 + \lambda_j\theta$ and λ_j is an integer.

COROLLARY 5.6. *If* S^1 *acts on* X *with* l *isolated fixed points* $p_j, j = 1, 2, \ldots, l$, *then it is possible to fix the signs of the integers* x_{jk} *such that the* $l \times m/2$ *matrix* $((x_{jk}))$ *whose jth row gives the representation of* S^1 *on* $TX|p_j$ *via* (5.4) *and the* l *integers* λ_j *determined by* (5.5) *determine the homomorphism* $\mathrm{Id}_{S^1}^X : K_{S^1}^*(TX) \to R(S^1)$ *by*

$$\mathrm{Id}_{S^1}^X(\delta_{S^1}u) = \sum_{j=1}^{l} u_j(t)t^{-\lambda_j} \prod_{l=1}^{m/2} \frac{1}{1 - t^{x_{jl}}}$$

where $u_j(t) \in R(S^1) = Z[t, t^{-1}]$ *denotes the character obtained by restricting* u *to* p_j. *Moreover if* P *is the principle* spin$^c(m)$ *bundle used to define* δ_{S^1} *and* $\omega = P \times_{\mathrm{spin}^c(m)} C$ *the line bundle determined by* P, *then* ω *is an* S^1 *bundle over* X *and the integers* β_j *defined by*

$$\omega|_{p_j}(t) = t^{\beta_j} \in Z[t, t^{-1}]$$

and the integers λ_j *are related by*

$$-\left(\sum_{k=1}^{m/2} x_{jk} + 2\lambda_j\right) = \beta_j.$$

PROOF. One of the fundamental properties of the homomorphism $\mathrm{Id}_G^Y : K_G^*(TY) \to R(G)$ is that for $Y =$ point, it is the identity. By hypothesis

$$X^{S^1} = \bigcup_{j=1}^{l} p_j.$$

If $Ti_j: Tp_j \to TX$ is the inclusion,

$$Ti_j^*(\delta_{S^1}u) = u_j(t) \cdot \prod_{k=1}^{m/2} (t^{x_{jk}/2} - t^{-x_{jk}/2}) t^{-x_{jk}/2 - \lambda_j},$$

$$\lambda_{-1}(Np_j \otimes C) = \prod_{k=1}^{m/2} (1 - t^{x_{jk}})(1 - t^{-x_{jk}}).$$

The signs of the x_{jk}, $k = 1, 2, \ldots, m/2$, are chosen as follows: Let

$$TX|_{p_j} = e_1 \oplus e_2 \oplus \cdots \oplus e_{m/2}$$

be the splitting of this representation space of S^1 into oriented 2 planes invariant under S^1. The orientation of $TX|_{p_j}$ is to be the one given by this direct sum representation. Give e_i a complex structure compatible with the given orientation on e_i. Then for $z_k \in e_k$, $t \in S^1$ operates on z_k by multiplication by $t^{x_{jk}}$ and the sign of the integer x_{jk} is determined. Thus

$$\begin{aligned} \mathrm{Id}_{S^1}^X(\delta_{S^1}u) &= \sum_j \frac{Ti_j^*(\delta_{S^1}u)}{\lambda_{-1}(Np_j \otimes C)} \quad \text{by (1.4) and (1.6)}, \\ &= \sum_{j=1}^{l} u_j(t) t^{-\lambda_j} \prod_{k=1}^{m/2} \left(\frac{1}{1 - t^{x_{jk}}} \right). \end{aligned}$$

The rest of Corollary 5.6 follows from Proposition 5.3 by replacing the data $(\omega_j, \theta_{jl}, \lambda_j)$ by $(\beta_j\theta, x_{jl}\theta, \lambda_j\theta)$ and g by $t = e^{i\theta}$.

6. **Specialization to S^1 actions.** There is a theorem of Stewart [**15**] improved by Su [**16**] which to my knowledge has found little use until now. It is fundamental to the rest of our discussion. The situation is this: X is a paracompact space supporting a left action of a torus group $\mathfrak{T}_1$; P is a principle $\mathfrak{T}_2$ bundle over X. The torus $\mathfrak{T}_2$ acts on the right of P.

THEOREM 6.1 (STEWART [**15**] AND SU [**16**]). *If $H^1(X, Z) = 0$, the left action of $\mathfrak{T}_1$ on X lifts to a left action of $\mathfrak{T}_1$ on P which commutes with the principle right action of $\mathfrak{T}_2$ on P. If $(t, p) \to t \cdot p$ and $(t, p) \to t \circ p$ denote two liftings of $\mathfrak{T}_1$ to P then there is a homomorphism $\theta: \mathfrak{T}_1 \to \mathfrak{T}_2$ such that*

$$t \circ p = t \cdot p \cdot \theta(t).$$

We shall restrict application of this theorem to the case $\mathfrak{T}_1 = \mathfrak{T}_2 = S^1$. Suppose, in addition to the hypothesis of 6.1, that X is a smooth manifold of dimension m and $H^3(X, Z_2) = 0$. Then X has a spin$^c(m)$ structure by Lemma 3.9. Let P be a principle spin$^c(m)$ bundle over X associated to the tangent bundle of X.

THEOREM 6.2. *The left S^1 action on X lifts to a left S^1 action on P which commutes with the right action of* $\mathrm{spin}^c(m)$ *on P. Thus the action of X satisfies the hypothesis* (4.7).

PROOF. Let Q denote the principle $SO(m)$ bundle associated to the tangent bundle of X. From the exact sequence of groups

$$S^1 \to \mathrm{spin}^c(m) \to SO(m)$$

we see that P is a principle S^1 bundle over Q. If $m > 2$, then $H^1(Q, Z) = 0$. There is a natural left S^1 action on Q commuting with the principle right action of $SO(m)$. By 6.1 this action lifts to P and commutes with the principle right action of S^1.

The lifted action may not commute with the principle $\mathrm{spin}^c(m)$ action on P so a modified lifting may be necessary. We have satisfied these.

Hypothesis. P is the total space of a principle right $\mathrm{spin}^c(n)$ bundle over X. There is a left action of S^1 on P which commutes with the right action of $S^1 \subset \mathrm{spin}^c(n)$. Moreover, the induced left action of S^1 on the orbit space $Q = P/S^1$ commutes with the right action of $SO(n)$ on Q.

We now show that this hypothesis implies that the left action of S^1 on P can be modified so as to commute with the principle right action of $\mathrm{spin}^c(n)$ and so that the induced left action on Q is left unaltered.

Let $s \in S^1$, $h \in \mathrm{spin}^c(n)$. Then for $x \in P$ we have

(i) $(sx)h = s(xh)\hat{\psi}(s, x, h)$ where $\hat{\psi}(s, x, h) \in S^1$, the center of $\mathrm{spin}^c(n)$. This is a consequence of the fact that the S^1 action on Q commutes with the right action of $SO(n)$.

(ii) $\hat{\psi}(s, xt, h) = \hat{\psi}(s, x, h)$ for $t \in S^1$. This is a consequence of the fact that $t \in$ center $\mathrm{spin}^c(n)$ and the *Hypothesis.*

As a result of (ii), $\hat{\psi}(s, x, h) = (s, \pi x, h)$ where $\psi: S^1 \times Q \times SO(n) \to S^1$. Here π is the projection of P on Q. The function ψ has these properties:

(iii) $\psi(1, z, h) = \psi(s, z, 1) = 1$, $z \in Q$ and 1 the identity of the appropriate group.

(iv) $\psi(s_1 s_2, z, h) = \psi(s_1, s_2 z, h)\psi(s_2, z, h)$.

(v) $\psi(s, z, h_1 h_2) = \psi(s, z h_1, h_2)\psi(s, z, h_1)$.

Because of (iii), ψ is null homotopic and there is a unique lifting $\tilde{\psi}$ to R^1 which satisfies $\tilde{\psi}(1, z_0, 1) = 1$ for a fixed $z_0 \in Q$. Moreover $\tilde{\psi}$ will satisfy (iii), (iv) and (v) except we change from a multiplicative to an additive notation. Observe that

(vi) $\psi(s, zh', h'^{-1}h) - \psi(s, zh', h'^{-1}) = \tilde{\psi}(s, z, h)$ by (v).

Define, for $s \in S^1$, $z \in Q$,

$$\gamma(s, z) = \int_H \tilde{\psi}(s, zh, h^{-1})dh$$

where dh denotes the normalized Haar measure on $\mathrm{spin}^c(n) = H$. Then

$$\begin{aligned}\gamma(s, zh) - \gamma(s, h) &= \int_H \tilde{\psi}(s, zhh', h'^{-1})dh' - \int_H \tilde{\psi}(s, zh', h'^{-1})dh' \\ \text{(vii)} \qquad &= \int_H [\tilde{\psi}(s, zh'', h''^{-1}h) - \tilde{\psi}(s, zh'', h''^{-1})]dh'' \\ &= \int_H \tilde{\psi}(s, z, h)dh'' = \tilde{\psi}(s, z, h) \quad \text{by (vi).}\end{aligned}$$

(viii) $\gamma(s_1 s_2, z) = \gamma(s_1, s_2 z) + \gamma(s_2, z)$ by (iv).

Let $\bar{\gamma}(s, z) \in S^1$ denote the image in S^1 of $\gamma(s, z)$ under the covering map and define a new left action of S^1 on P by

$$s \circ x = sx(\bar{\gamma}(s, z(x)))$$

where $z: P \to Q$ is the projection. Then

$$(s \circ x)h = (sx\bar{\gamma}(s, z(x))h) = s(xh)\psi(s, z(x), h) \cdot \bar{\gamma}(s, z(x)),$$

$$s \circ (xh) = s(xh)\bar{\gamma}(s, z(x), h).$$

Since $\bar{\gamma}(s, z(x)h)\bar{\gamma}(s, z(x))^{-1} = \psi(s, z(x), h)$ by (vii) we have

(ix) $(s \circ x)h = s \circ (xh)$;

(x) $(s_1 s_2) \circ x = s_1 \circ (s_2 \circ x)$ by (viii).

Property (x) shows that $\circ$ is an action and property (ix) shows that this left action of S^1 commutes with the right action of $\mathrm{spin}^c(n)$. It follows from the definition of $\circ$ that the induced left action of S^1 on Q agrees with the original action.

COROLLARY 6.3. *Let X be a smooth manifold with $H^1(X, Z) = 0$ and which supports a smooth S^1 action. Then the class $\delta_{S^1} \in K^*_{S^1}(TX)$ is defined and is an orientation for X.*

PROOF. This follows from Theorem 6.2 and Proposition 4.9.

6.4. *Standard example.* Here is an important example which illustrates the foregoing remarks: Let $X = CP^n = U(n+1)/U(1) \times U(n)$. X is a $\mathrm{spin}^c(2n)$ manifold. A principle $\mathrm{spin}^c(2n)$ bundle associated to TCP^n has total space

$$U(n+1) \times_H \mathrm{spin}^c(2n) = P$$

where $H = U(1) \times U(n)$ acts on $\mathrm{spin}^c(2n)$ through the composition (see (2.3))

$$H \xrightarrow{\mathrm{Ad}} U(n) \xrightarrow{\psi_0} \mathrm{spin}^c(2n).$$

For $x \in U(1) = S^1$, let $d(x) \in U(n)$ be the diagonal matrix with $d_{ii}(x) = x$. Then $\mathrm{Ad}(x, y) = d(x^{-1})y$ for $y \in U(n)$. Moreover, the complex line bundle

$$\omega = P \times_{\mathrm{spin}^c(2n)} C = U(n+1) \times_H C$$

is $\mathscr{H}^{n+1}$ where $\mathscr{H}$ is the Hopf bundle over X.

Left action of $U(n+1)$ on the coset space $U(n+1)/H$ makes X into a left $U(n+1)$ space. Any representation $\phi: S^1 \to U(n+1)$ will then define an action of S^1 on X. Such a representation is given by

$$\phi(t) = \begin{bmatrix} t^{a_0} & & & \\ & t^{a_1} & & \\ & & \ddots & \\ & & & t^{a_n} \end{bmatrix} \subset U(n+1)$$

for $t \in S^1$ and integers a_i. In terms of homogenous coordinates $(z_0 : z_1 : \ldots : z_n)$ on CP^n, the action takes the form

$$t(z_0 : z_1 : \ldots : z_n) = (t^{a_0} z_0 : t^{a_n} z_1 : \ldots : t^{a_n} z_n)$$

and if the integers a_j are distinct, there are $n + 1$ isolated fixed points

$$p_i = (0 : 0 : \ldots : 1 : \ldots : 0)$$

all of whose coordinates are zero except the ith which is 1. The representation ρ_j of S^1 on $TX|_{p_j}$ is given by

$$\rho_j(e^{i\theta}) = \mathrm{diag}\left\{\begin{pmatrix} \cos(a_k - a_j)\theta & \sin(a_k - a_j)\theta \\ -\sin(a_k - a_j)\theta & \cos(a_k - a_j)\theta \end{pmatrix}\right\} \subset SO(2n), \qquad k \neq j.$$

The S^1 action is already lifted to the principle $\mathrm{spin}^c(2n)$ bundle P. Explicitly

$$\text{(i)} \qquad t[u, s] = [\phi(t)u, s]$$

for $t \in S^1$, $u \in U(n+1)$ and $s \in \mathrm{spin}^c(2n)$. Moreover

$$\text{(ii)} \qquad \begin{aligned} \tilde{\rho}_j(e^{i\theta}) = \prod_{k \neq j} [&\cos(a_k - a_j)\theta/2 \\ &- \sin(a_k - a_j)\theta/2\, e_{2k-1}e_{2k}, \exp[-i\textstyle\sum(a_k - a_j)\theta/2]]. \end{aligned}$$

The Hopf bundle $\mathscr{H} = U(n+1) \times_H C$, where H acts on C via the representation $U(1) \times U(n) \to U(1)$ defined by $(x, y) \to x$, becomes an S^1 bundle over X by setting

$$\text{(iii)} \qquad t \circ [u, c] = [\phi(t)u, c], \qquad u \in U(n+1), c \in C.$$

In the same manner the bundle $\mathscr{H}^{n+1}$ becomes an S^1 bundle over X and, by (iii),

$$\mathscr{H}|_{p_j}(t) = t^{a_j}, \qquad \mathscr{H}^{n+1}|_{p_j}(t) = t^{(n+1)a_j} \in Z[t, t^{-1}].$$

Since ω and $\mathscr{H}^{n+1}$ are isomorphic as vector bundles, their associated principle S^1 bundles are isomorphic and since we have two liftings of S^1 to this principle bundle $P(\omega) = P(\mathscr{H}^{n+1})$, by 6.1 there is an integer θ such that

$$t \circ p = t \cdot p \cdot t^{\theta}, \qquad t \in S^1, p \in P(\omega) = P(\mathscr{H}^{n+1}).$$

This integer is determined by the restriction of ω and $\mathscr{H}^{n+1}$ to any fixed point p_j. But

$$\omega|_{p_j}(t) = t^{-\Sigma_{k \neq j}(a_k - a_j)} \quad \text{by (i) and (ii),}$$

while

$$\mathscr{H}^{n+1}|_{p_j}(t) = t^{(n+1)a_j}$$

so

$$\text{(iv)} \qquad \theta - \sum_{k \neq j} (a_k - a_j) = (n+1)a_j$$

and

$$\theta = \sum_{k=0}^{n} a_k .$$

The character $\mathrm{Id}_{S^1}^{CP^n}(\mathscr{H}^k \delta_{S^1})$ is given by

$$\text{(v)} \qquad \sum_{j=0}^{n} t^{ka_j} \prod_{l \neq j} (1 - t^{(a_l - a_j)})^{-1}$$

by Corollary 5.6 ($\lambda_j = 0$ by (5.5) and (ii)).

As a second application of the Stewart theorem we define a homomorphism F from the additive group $H^2(X, Z)$ to the multiplicative group of units of $K_{S^1}^*(X)$. We assume $H^2(X, Z)$ is free abelian. Let $z_1, \ldots, z_s$ be a basis for this group. The z_i's determine complex line bundles over X which we also denote by z_i. Let P_i denote the principle S^1 bundle over X associated to z_i. Then

$$z_i = P_i \times_{S^1} C.$$

By 6.1, the left S^1 action on X lifts to a left S^1 action on P_i which commutes with the right principle S^1 action on P_i. Define a left S^1 action on z_i by

$$t[p, c] = [tp, c].$$

Then z_i with this action of S^1 becomes an element $F(z_i) = W_i \in K_{S^1}^*(X)$.

Define $F(\sum \lambda_j z_j) = \prod W_j^{\lambda_j}$, $\lambda_j \in Z$.

Let $L(z_1, \ldots, z_s) = \sum n_j z_j = L$, $n_j \in Z$, be an element whose mod 2 reduction is $W_2(X)$ and P_L the principle spin$^c(m)$ bundle over X defined by L, i.e., $P_L \times_{\text{spin}^c(m)} C$ is the line bundle with the first Chern class L. Use Theorem 6.1 to lift the S^1 action to P_L. Suppose for simplicity that X^{S^1} consists of n isolated fixed points $\{p_i\}$ and define integers z_{ij}, $i = 1, \ldots, s$, $j = 1, 2, \ldots, n$, by

$$W_i|_{p_j}(t) = t^{z_{ij}} \in Z[t, t^{-1}]$$

and integers x_{ij} (defined up to sign) by

$$\rho_i(t) = \operatorname{diag}\begin{pmatrix} \cos x_{ij}\theta & \sin x_{ij}\theta \\ -\sin x_{ij}\theta & \cos x_{ij}\theta \end{pmatrix}, \qquad j = 1, 2, \ldots, m/2, i = 1, 2, \ldots, n.$$

Then set $L_i = L(z_{1i}, z_{2i}, \ldots, z_{si}) = \sum n_k z_{ki} \in Z$.

THEOREM 6.5. *There is an integer N such that for any polynomial $\Phi(y_1, y_2, \ldots, y_s)$ with integer coefficients,*

$$E_\Phi = t^{N/2} \sum_{i=1}^{n} \Phi(t^{z_{1i}}, t^{z_{2i}}, \ldots, t^{z_{si}}) \cdot t^{L_i/2} \prod_{j=1}^{m/2} (t^{-x_{ij}/2} - t^{x_{ij}/2})^{-1} \in Z[t, t^{-1}] = R(S^1).$$

PROOF. Since $\prod W_i^{n_i}$ and $P_L \times_{\text{spin}^c(m)} C = \omega$ are isomorphic as vector bundles there is an integer N such that

$$\omega|_{p_j}(t) = t^N \cdot \prod_i W_i^{n_i}|_{p_j}(t) \quad \text{(by 6.1)}$$
$$= t^N \cdot \prod_i t^{n_i z_{ij}} = t^{N+L_j}.$$

The result follows from Corollary 5.6 by setting $\beta_j = N + L_j$ and $\mu_i = \Phi(t^{z_{1i}}, t^{z_{2i}}, \ldots, t^{z_{si}})$ because $E_\Phi = \mathrm{Id}^X_{S^1}(\Phi(W_1, W_2, \ldots, W_s)\delta_{S^1})$ and

$$t^{(N+L_i)/2} \cdot \prod_{j=1}^{m/2} (t^{-x_{ij}/2} - t^{x^{ij}/2})^{-1} = t^{-\lambda_i} \prod_{j=1}^{m/2} (1 - t^{x_{ij}})^{-1},$$
$$\lambda_j = (\textstyle\sum x_{ij}/2 + \beta_j/2).$$

REMARK. Because of the occurrence of square roots in the terms in the expression for E_Φ, these terms are only defined up to sign. We will have more to say later.

II. Applications to S^1 actions on homotopy complex projective spaces

Part II of the paper is devoted to studying this situation: X is a smooth manifold homotopy equivalent to CP^n and supports a smooth S^1 action. What properties of the example of Part I, (6.4) persist here?

1. **Generalities.** The most important property of this situation is

PROPOSITION 1.1. *Let η' be the pull back of the Hopf bundle over CP^n via a homotopy equivalence from X to CP^n. Then η' admits an S^1 action making it an S^1 bundle η over X.*

PROOF. Let $F: H^2(X, Z) \to K^*_{S^1}(X)$ be the function defined in §6. Then $\eta = F[c_1(\eta')]$ where $c_1(\eta')$ is the first Chern class of η'. It generates $H^2(X, Z)$.

Let X^{S^1} denote the fixed point set of S^1 acting on X. Then

$$X^{S^1} = X_0 \cup X_1 \cup \cdots \cup X_{l-1}$$

is the disjoint union of l connected components X_i and each X_i is a cohomology CP^{k_i-1} where $\sum_{i=0}^{l-1} k_i = n+1$. Moreover, the natural map $H^*(X) \to H^*(X_i)$ is an isomorphism for $* \leqq 2k_i + 1$. This is a result of [9], [11].

Choose a point $p_j \in X_j$.

DEFINITION 1.2. *Define l integers a_j by $\eta|_{p_j}(t) = t^{a_j}$.*

REMARK. *The integers a_j are not uniquely defined by the S^1 action on X; they depend on the lifting of the action to the principle S^1 bundle associated to η. However a second lifting will give rise to a new set of integers a'_j which are related to the old set by*

$$a'_j = a_j + \theta.$$

This is a consequence of Part I, 6.1. *Thus the integers a_j are well defined up to translation.*

Having this set of integers, we can now compare the given action with the S^1 action on CP^n of 6.3 which is determined by the representation $\phi: S^1 \to U(n+1)$ defined by

$$\phi(t) = \operatorname{diag}\{t^{-a_j} I_j\}_{j=0,1,\ldots,l-1}.$$

I_j is the identity matrix of dimension k_j.

REMARK. *The complex K theory of X and of the components X_i of X^{S^1} is given by*

$$K^*(X) \cong Z[\eta']/((\eta'-1)^{n+1}),$$

$$K^*(X_i) \cong Z[\eta'_i]/((\eta'_i-1)^{k_i}),$$

where η'_i is the restriction of η' to X_i.

The homomorphism $\tau: 1 \to S^1$ of the trivial group to S^1 induces a homomorphism $\tau^*: K^*_{S^1}(X) \to K^*(X)$.

LEMMA 1.3. *τ^* is surjective.*

PROOF. Since $\tau^*\eta = \eta'$, τ^* is surjective by the above Remark.

LEMMA 1.4. *There is an exact sequence*

$$0 \to K^*_{S^1}(X) \xrightarrow{\Delta} K^*_{S^1}(X) \xrightarrow{\tau^*} K^*(X) \to 0$$

where Δ is multiplication by $(t - 1) \in Z[t, t^{-1}]$.

PROOF. Let S^1 act on $(X \times D^2, X \times S^1)$ by $t(x, d) = (tx, td)$ for $x \in X$, $d \in D^2$ and $t \in S^1$. Then on $X \times S^1$ this action is equivalent to the action defined by

$$t(x, d) = (x, td), \qquad \|d\| = 1.$$

An equivalence of actions is provided by

$$(x, d) \to (d^{-1}x, d), \qquad \|d\| = 1.$$

In view of this,

$$K^*_{S^1}(X \times S^1) \cong K^*(X \times S^1/S^1) \cong K^*(X)$$

because S^1 acts freely on $X \times S^1$. By the Thom isomorphism theorem [**1**],

$$K^*_{S^1}(X) \cong K^*_{S^1}(X \times D^2, X \times S^1).$$

This isomorphism composed with the restriction

$$K^*_{S^1}(X \times D^2, X \times S^1) \to K^*_{S^1}(X \times D^2) \cong K^*_{S^1}(X)$$

is multiplication by $(t - 1)$ [**1**]. Making these identifications, the exact triangle for the pair $(X \times D^2, X \times S^1)$ becomes

$$\begin{tikzcd}
K^*_{S^1}(X) \arrow[rr] & & K^*_{S^1}(X) \arrow[dl, "\tau^*"] \\
& K^*(X) \arrow[ul] &
\end{tikzcd}$$

Since τ^* is surjective, the result follows.

Let $\mathfrak{T} \subset K^*_{S^1}(X)$ be the $\Lambda = Z[t, t^{-1}]$ torsion subgroup of $K^*_{S^1}(X)$.

LEMMA 1.5. *τ^* maps $\mathfrak{T}$ to zero.*

PROOF. By Lemma 1.4, $\mathfrak{T}$ has no $t - 1$ torsion. Let $x \in \mathfrak{T}$ and $\lambda \in \Lambda$ such that $\lambda x = 0$. We may suppose that λ is prime to $t - 1$. Then in $\Lambda \otimes Q$ there are elements a and b such that

$$1 = a\lambda + b(t - 1).$$

Let d be an integer such that $da = a'$ and $db = b'$ are in Λ. Then $d = a'\lambda + b'(t-1)$ and $dx = b'(t-1)x$ so

$$d\tau^*(x) = \tau^*(dx) = \tau^*(b'(t-1)x) = 0$$

by Lemma 1.4. Since $K^*(X)$ is free abelian, $\tau^*(x) = 0$.

LEMMA 1.6. *Let $i: X^{S^1} \to X$ denote the inclusion of the fixed point set. Then i^* maps $\mathfrak{I}$ to zero.*

PROOF. $K^*_{S^1}(X^{S^1}) = \prod_{i=0}^{l-1} K^*(X_i) \otimes_Z \Lambda$ is a free Λ module by the Remark above.

Let $\hat{K}_{S^1}(X) = K^*_{S^1}(X)/\mathfrak{I}$. Then the above two lemmas imply that we may regard the domain of τ^* and i^* as $\hat{K}_{S^1}(X)$.

Define $f(x) \in \Lambda[x]$ by

$$f(x) = \prod_{i=0}^{l=1} (x - t^{a_i})^{k_i} \tag{1.7}$$

and define a map of algebras A' from $\Lambda[x]$ to $\hat{K}_{S^1}(X)$ by $A'(x) = \eta$.

LEMMA 1.8. $A'(f(x)) = 0$.

PROOF. Let $\mathfrak{p} \in \Lambda$ be the prime ideal of characters which vanish at a generic $t \in S^1$. Then $X^t = \{x \in X | tx = x\} = X^{S^1}$. Let $\Lambda_{\mathfrak{p}}$ be Λ localized at $\mathfrak{p}$. By the Atiyah-Segal localization theorem Part I, (1.6), $i^*_{\mathfrak{p}}: \hat{K}_{S^1}(X)_{\mathfrak{p}} \to K^*_{S^1}(X^t)_{\mathfrak{p}}$ is an isomorphism. Since $\hat{K}_{S^1}(X)$ is a torsion free Λ module and $K^*_{S^1}(X^t)$ is a free Λ module, this means that i^* is a monomorphism. But $i^*A'(f(x)) = 0$ so $A'(f(x)) = 0$.

Thus there is an induced map of algebras

$$A: \Lambda[x]/(f(x)) \to \hat{K}_{S^1}(X).$$

Let $p \in Z$ be a prime and ξ a primitive $p^{r\text{th}}$ root of unity, $Q(\xi) = k$ the field of primitive $p^{r\text{th}}$ roots of unity and $\mathcal{O}$ the integers in k. Then $\mathcal{O}$ is a Λ module via $g(t) \to g(\xi)$ when $g(t) \in \Lambda$. Clearly if $\Gamma = \Lambda[x]/(f(x))$, then

$$\Gamma \otimes_\Lambda \mathcal{O} = \mathcal{O}[x]/(\bar{f}(x)) \quad \text{where } \bar{f}(x) = \prod_{a \in Z_{p^r}} (x - \xi^a)^{d_a};$$

d_a is the sum of the k_j with $a_j \equiv a(p^r)$. Moreover

$$\Gamma \otimes_\Lambda k \cong \prod_{a \in Z_{p^r}} k[x]/((x - \xi^a)^{d_a})$$

as an algebra.

We shall see that $\hat{K}_{S^1}(X) \otimes_\Lambda k$ is a useful tool for studying the fixed point set $X^{Z_{p^r}}$.

LEMMA 1.9. *$A \otimes_\Lambda 1_k : \Gamma \otimes_\Lambda k \to \hat{K}_{S^1}(X) \otimes_\Lambda k$ is an isomorphism of algebras.*

PROOF. The Λ rank of $\hat{K}_{S^1}(X)$ is $n + 1$ because $K^*_{S^1}(X^{S^1}) = R(S^1) \otimes K^*(X^{S^1})$ is a free Λ module of rank $n + 1$ and $K^*_{S^1}(X)$ has the same Λ rank as $K^*_{S^1}(X^{S^1})$ by Part I, (1.6). It suffices to show that A induces a monomorphism

$$\tilde{A} = A \otimes_\Lambda 1_{\mathcal{O}} : \Gamma \otimes_\Lambda \mathcal{O} \to \hat{K}_{S^1}(X) \otimes_\Lambda \mathcal{O}.$$

It is easy to check that the kernel of the composition

$$\Gamma \otimes_\Lambda \mathcal{O} \to \hat{K}_{S^1}(X) \otimes_\Lambda \mathcal{O} \xrightarrow{\tau^* \otimes_\Lambda 1} K^*(X) \otimes_\Lambda \mathcal{O} \cong K(X) \otimes \mathcal{O}/(1 - \xi)\mathcal{O}$$

is $\Gamma \otimes_\Lambda (1 - \xi)\mathcal{O}$. This means that if x is in the kernel of $\tilde{A}$, $x = (1 - \xi)x_1$. Since $\hat{K}_{S^1}(X)$ is torsion free, $\tilde{A}(x_1) = 0$ and inductively $x = (1 - \xi)^n x_n$. Since $\Gamma \otimes_\Lambda \mathcal{O}$ is a free $\mathcal{O}$ module and $(1 - \xi)$ is not a unit of $\mathcal{O}$, this can only happen if x is zero.

Let $\mathfrak{T}_{p^r}$ be the Λ torsion subgroup of $K^*_{S^1}(X^{Z_{p^r}})$ and let j denote the inclusion of $X^{Z_{p^r}}$ in X.

LEMMA 1.10. *The composition*

$$\Gamma \otimes_\Lambda k \xrightarrow{A \otimes_\Lambda 1} \hat{K}_{S^1}(X) \otimes_\Lambda k \xrightarrow{j^* \otimes_\Lambda 1} \hat{K}_{S^1}(X^{Z_{p^r}}) \otimes_\Lambda k$$

is an isomorphism.

PROOF. Let $\mathfrak{p}$ be the prime ideal of characters vanishing at $\xi \in S^1$. Then $j^*_\mathfrak{p} : K_{S^1}(X)_\mathfrak{p} \to K_{S^1}(X^{Z_{p^r}})_\mathfrak{p}$ is an isomorphism. This implies that $j^*_\mathfrak{p} : \mathfrak{T}_\mathfrak{p} \to (\mathfrak{T}_{p^r})_\mathfrak{p}$ is an isomorphism; hence, $j^*_\mathfrak{p}$ induces an isomorphism $\hat{K}_{S^1}(X)_\mathfrak{p} \to \hat{K}_{S^1}(X^{Z_{p^r}})_\mathfrak{p}$. Tensoring these groups over $\Lambda_\mathfrak{p}$ with k and observing that

$$\hat{K}_{S^1}(X) \otimes_{\Lambda_\mathfrak{p}} k = \hat{K}_{S^1}(X) \otimes_\Lambda k,$$
$$\hat{K}_{S^1}(X^{Z_{p^r}}) \otimes_{\Lambda_\mathfrak{p}} k = \hat{K}_{S^1}(X^{Z_{p^r}}) \otimes_\Lambda k,$$

we obtain the desired result.

Let Z_∞ be the union of those components of $X^{Z_{p^r}}$ which miss X^{S^1} and let Z_a, $a \in Z_{p^r}$, be the union of those components which contain an X_i with $a_i \equiv a(p^r)$.

LEMMA 1.11. *Z_a is connected and empty if there is no $a_i \equiv a(p^r)$.*

PROOF. Since there are no points of Z_∞ fixed by S^1 by definition, it follows from the Atiyah-Segal localization theorem that $K^*_{S^1}(Z_\infty)$ is a Λ torsion module. Moreover

$$K^*_{S^1}(X^{Z_{p^r}}) = \prod_{a \in Z_{p^r}} K^*_{S^1}(Z_a) \times K^*_{S^1}(Z_\infty)$$

and the torsion subgroup $\mathfrak{T}_{p^r}$ splits as

$$\mathfrak{T}_{p^r} = \prod_a \mathfrak{T}_a \times K^*_{S^1}(Z_\infty)$$

where $\mathfrak{T}_a$ is the torsion subgroup of $K^*_{S^1}(Z_a)$. Thus

$$\hat{K}_{S^1}(X^{Z_{p^r}}) \cong \prod_a \hat{K}_{S^1}(Z_a).$$

Suppose $Z_a \neq \varnothing$. Then for some i there are S^1 equivariant maps (see Definition 1.2)

$$p_i \to Z_a \to p_i$$

which imply that $\Lambda = K^*_{S^1}(p_i)$ is a direct factor of $\hat{K}_{S^1}(Z_a)$ so $\hat{K}_{S^1}(Z_a)$ is nonzero. By Lemma 1.10 we have

$$\prod_a k[x]/((x - \xi^a)^{d_a}) \cong \prod_a \hat{K}_{S^1}(Z_a) \otimes k. \tag{1.12}$$

This is an isomorphism of $k[x]$ modules; so by the unique decomposition theorem for torsion modules over $k[x]$, [**14**],

$$\hat{K}_{S^1}(Z_a) \otimes_\Lambda k \cong \prod_{a \in S_a} k[x]/((x - \xi^a)^{d_a})$$

where S_a is some subset of Z_{p^r} which is nonzero if $Z_a \neq \varnothing$.

Since the number of factors on the left of (1.12) is precisely the number of distinct residue classes appearing among the $\{a_i\}$ and since the number of factors occurring on the right side of (1.12) is at least this number, it follows that S_a is zero if there is no $a_i \equiv a(p^r)$ and just a if there is an $a_i \equiv a(p^r)$.

Suppose some $Z_a \neq \varnothing$ is not connected. Then $Z_a = Y_1 \cup Y_2$ where Y_1 and Y_2 contain points p_i and p_j with $a_i \equiv a \equiv a_j$ so $\hat{K}_{S^1}(Y_i) \neq 0$, $i = 1, 2$, and

$$\hat{K}_{S^1}(Z_a) \cong \hat{K}_{S^1}(Y_1) \times \hat{K}_{S^1}(Y_2).$$

But this implies that there are more nonzero summands on the right side of (1.12) than on the left.

COROLLARY 1.13.

$$\hat{K}_{S^1}(X^{Z_{p^r}}) \otimes_\Lambda k \cong \prod_{a \in Z_{p^r}} k[x]/((1 - \xi^a)^{d_a})$$

as an algebra.

COROLLARY 1.14. *A sufficient condition that the fixed point set of Z_{p^r} strictly contain the fixed set of S^1, X^{S^1}, is that there are a pair of distinct integers i and j with $a_i \equiv a_j(p^r)$.*

PROOF. We have seen in the proof of Lemma 1.11, that the number of connected components of $X^{Z_{p^r}}$ intersecting X^{S^1} is the number of distinct residue classes among the l integers a_i, while the number of components of X^{S^1} is l by assumption. If the number of distinct residue classes among the a_j is less than l, then two components of X^{S^1} are in the same component of $X^{Z_{p^r}}$ showing $X^{Z_{p^r}}$ strictly contains X^{S^1}.

COROLLARY 1.15. *The l integers a_i are distinct.*

PROOF. For r large, $X^{Z_{p^r}} = X^{S^1}$ for any prime p. If some pair (a_i, a_j) were equal, then the collection of l integers $\{a_j\}$ would contain fewer than l residue classes mod p^r so $X^{Z_{p^r}}$ would strictly contain X^{S^1} by the preceding corollary.

2. **Isolated fixed points.** Now we restrict our attention to the case in which X^{S^1} consists of isolated fixed points. Then there must be $n + 1$ such points p_j, $j = 0, 1, 2, \ldots, n$. To each is attached an integer a_j as discussed in §1 and the $n + 1$ integers a_j are distinct by Corollary 1.15. Our aim is to compare the given action of S^1 on X with the linear action of S^1 on CP^n defined by the integers a_i of Part I, 6.4.

Introduce an $(n + 1) \times n$ matrix of integers x_{ij}, $i = 0, 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$ (defined up to sign), whose ith row gives the representation of S^1 on $TX|_{p_i}$ by

$$\rho_i(t) = \operatorname{diag}\left\{\begin{pmatrix} \cos x_{ij}\theta & \sin x_{ij}\theta \\ -\sin x_{ij}\theta & \cos x_{ij}\theta \end{pmatrix}\right\}, \qquad j = 1, 2, \ldots, n,\ t = e^{i\theta}.$$

LEMMA 2.1.

$$\psi_i(t) = \prod_{j \neq i} (1 - t^{a_j - a_i}) \cdot \prod_{j=1}^{n} (1 - t^{x_{ij}})^{-1} \in \Lambda.$$

PROOF. Let $e_i = \prod_{j \neq i} (\eta - t^{a_j}) \in \hat{K}_{S^1}(X)$. Then $\delta_{S^1} e_i \in \hat{K}_{S^1}(TX)$ and

$$\operatorname{Id}^X_{S^1}(\delta_{S^1} e_i) \in \Lambda. \tag{2.2}$$

Since the character of $\eta|_{p_j}$ is t^{a_j}, $e_i|_{p_j} = 0$ unless $j = i$ and

$$e_i|_{p_i}(t) = \prod_{j \neq i} (t^{a_i} - t^{a_j}).$$

By Corollary 5.6, with $u = e_i$, we have

$$\operatorname{Id}^X_{S^1}(\delta_{S^1} e_i)(t) = t^{-\lambda_i} \prod_{j \neq i} (t^{a_i} - t^{a_j}) \cdot \prod_{j=1}^{n} (1 - t^{x_{ij}})^{-1}$$

and the result follows from (2.2).

LEMMA 2.3. *$\psi_k(1) = \varepsilon$ where $\varepsilon = \pm 1$ is independent of k.*

PROOF. Since the normal bundle ν of X^{S^1} in X is trivially a complex bundle, there is a homomorphism $i_*: K^*_{S^1}(X^{S^1}) \to K^*_{S^1}(X)$ induced by the inclusion $i: X^{S^1} \to X$ and having the property that

$$i^* i_*(x) = \lambda_{-1}(v) \cdot x \quad \text{for } x \in K^*_{S^1}(X^{S^1}).$$

See Part I, (1.1).

There is a commutative diagram (Part I, (1.3))

$$\begin{array}{ccc} K_{S^1}(X) & \xrightarrow{\tau^*} & K^*(X) \\ \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} \\ K^*_{S^1}(X^{S^1}) & \xrightarrow{\tau^*} & K^*(X^{S^1}). \end{array}$$

Since $X^{S^1} = \bigcup_{i=0}^{n} p_j$,

$$K_{S^1}(X^{S^1}) = \prod K^*_{S^1}(p_j).$$

Let $f_k \in K^*_{S^1}(p_k)$ be the identity and $x = (0, 0, \ldots, f_k, \ldots, 0) \in K^*_{S^1}(X^{S^1})$. Then $\tau^* f_k$ is the identity of $K^*(p_k)$ and $i_* \tau^*(x) = \varepsilon(\eta - 1)^n$; ε depends on the orientation of X. This follows from the definition of i_*.

Let $\mathfrak{p}$ denote the prime ideal of characters which vanish at $t \in S^1$ and consider the element

$$d_k = \prod_{j \neq k} (\eta - t^{a_j})(t^{a_k} - t^{a_j})^{-1} \prod_{j=1}^{n} (1 - t^{x_{ij}})$$

of $\hat{K}_{S^1}(X)_{\mathfrak{p}}$. By the localization theorem $i^*_{\mathfrak{p}} : \hat{K}_{S^1}(X)_{\mathfrak{p}} \to K^*_{S^1}(X^{S^1})_{\mathfrak{p}}$ is an isomorphism. But

$$i^*_{\mathfrak{p}} d_k = \prod_{j=1}^{n} (1 - t^{x_{ij}}) x = \lambda_{-1}(v) \cdot x$$

and

$$i^*_{\mathfrak{p}} i_*(x) = \lambda_{-1}(v) \cdot x.$$

Since $i^*_{\mathfrak{p}}$ is a monomorphism, $i_*(x) = d_k$. Thus $d_k \in \hat{K}_{S^1}(X) \subset \hat{K}_{S^1}(X)_{\mathfrak{p}}$ so $\tau^* d_k$ is defined and

$$\tau^*(d_k) = \psi_k(1)(\eta - 1)^n$$

and

$$\psi_k(1)(\eta - 1)^n = \tau^*(d_k) = \tau^* i_*(x) = i_* \tau_*(x) = \varepsilon(\eta - 1)^n$$

so $\psi_k(1) = \varepsilon$.

COROLLARY 2.4.

$$\prod_{j \neq i} (a_j - a_i) = \varepsilon \prod_{j=1} x_{ij}.$$

PROOF.

$$\psi_i(1) = \lim_{t \to 1} \prod_{j \neq i} (1 - t^{a_j - a_i}) \prod_{j=1}^{n} (1 - t^{x_{ij}})^{-1}$$
$$= \prod_{j \neq i} (a_j - a_i) \Big/ \prod_{j=1}^{n} x_{ij} = \varepsilon.$$

DEFINITION 2.5. *For each integer m and each $i = 0, 1, 2, \ldots, n$, set*

$n_i(m)$ = number of $j \neq i$ such that m divides $a_j - a_i$,
$d_i(m)$ = number of $j = 1, 2, \ldots, n$ such that m divides x_{ij},
$\delta_i(m) = n_i(m) - \delta_i(m)$.

THEOREM 2.6. *$\delta_i(m) \geqq 0$ and $\delta_i(p^r) = 0$ if p is prime.*

PROOF. *Let $\phi_d(t) = \prod_{\xi \in S_d} (t - \xi)$* where S_d consists of the primitive dth roots of unity, i.e., $\phi_d(t)$ is the dth cyclotomic polynomial. Then

$$\tau^l - 1 = \prod_{d|l} \phi_d(t)$$

and

$$\psi_i(t) = \pm t^N \prod_m \phi_m(t)^{n_i(m)} \Big/ \prod_m \phi_m(t)^{d_i(m)}$$
$$= \pm t^N \prod_m \phi_m(t)^{\delta_i(m)},$$

where N is an integer. By Lemma 2.1, $\psi_i(t) \in \Lambda$. But each $\phi_m(t)$ defines a prime ideal in $\Lambda \otimes Q = Q[t, t^{-1}]$. Since $\psi_i(t) \in \Lambda \otimes Q$, $\delta_i(m) \geqq 0$. Next observe [7]

$$\phi_m(1) = p, \qquad m = p^r \; p \text{ prime},$$
$$= 1, \qquad m \text{ composite}.$$

Thus

$$\psi_i(1) = \pm \prod_{p,r} p^{\delta_i(p^r)} = \pm 1,$$

so $\delta_i(p^r) = 0$.

REMARK. *If $\delta_i(m) = 0$ for all m not just prime powers, then the two collections of integers*

$$\{|a_j - a_i| \,|\, j \neq i\} = \mathscr{S}_i \quad \text{and} \quad T_i = \{|x_{ij}| \,|\, j = 1, 2, \ldots, n\}$$

are equal and this would establish the truth of

Statement 2.7. If S^1 acts smoothly on a manifold X homotopy equivalent to CP^n with $n + 1$ isolated fixed points p_i, then the $n + 1$ integers a_i defined *by* $\psi|_{p_i}(t) = t^{a_i}$, $t = e^{i\theta} \in S^1$ *(Part* II, *Definition* 1.2), *and the integers* x_{ij}, $j = 1, 2, \dots, n$, *defining the representation* ρ_i *of* S^1 *on* $TX|_{p_i}$ *by*

$$\rho_i(e^{i\theta}) = \operatorname{diag}\begin{pmatrix} \cos x_{ij}\theta & \sin x_{ij}\theta \\ -\sin x_{ij}\theta & \cos x_{ij}\theta \end{pmatrix}, \qquad j = 1, 2, \dots, n,$$

are related by $\{|x_{ij}| \mid j = 1, 2, \dots, n\} = \{|a_j - a_i| \mid j \neq i\}$.

In fact, 2.7 is false, as we shall see in §4; however, here are some additional relations which must hold between the x_{ij} and the $(a_j - a_i)$.

THEOREM 2.8. *The* $n + 1 \times n$ *matrix of integers* $((x_{ij}))$, $i = 0, 1, \dots, n$, $j = 1, 2, \dots, n$, *whose ith row gives the real representation of* S^1 *on* $TX|_p$; *and the* $n + 1$ *integers* $\{a_i\}$ *defined by* $\eta|_{p_i}(t) = t^{a_i}$ *must satisfy these relations: There is an integer* N *such that, for every integer* k,

(i) $$t^{N/2} \cdot \sum_{i=0}^{n} t^{a_i(k+(n+1)/2)} \cdot \prod_{j=1}^{n} (t^{-x_{ij}/2} - t^{x_{ij}/2})^{-1} \in Z[t, t^{-1}].$$

(ii) $$\prod_{j \neq i} (a_j - a_i) = \varepsilon \prod_{j=1} x_{ij} \quad \textit{where } \varepsilon \textit{ is } + \textit{ or } -1 \textit{ independent of } i.$$

(iii) $$\sum_{i=0}^{n} \prod_{j=1}^{n} (t^{x_{ij}} + 1)(t^{x_{ij}} - 1)^{-1} \begin{array}{l} = 0, \textit{ if } n \textit{ is odd}, \\ = 1, \textit{ if } n \textit{ is even}. \end{array}$$

PROOF. Let $L = (n + 1)c_1(\eta')$ where again η' is the pull back of the Hopf bundle over CP^n. See Part II, Proposition 1.1. Since Stiefel-Whitney classes are preserved by homotopy equivalences, the mod 2 reduction of L is the second Stiefel-Whitney class of X. By Part I, Corollary 3.8, there is a principle spin$^c(2n)$ bundle P_L over X with

$$P_L \times_{\mathrm{spin}^c(2n)} C = (\eta')^{n+1}.$$

Part (i) now follows from Part I, Theorem 6.5, by setting $W_1 = \eta$ (Part II, Proposition 1.1), $z_1 = c_1(\eta')$, $z_{1j} = a_j$, $L_j = (n + 1)a_j$ and $\Phi(y_1) = y_1^k$. Part (ii) is a restatement of Corollary 2.4. Part (iii) is a consequence of the fact that the S^1 signature of X [**6**, p. 578] as a function on S^1 is the constant 0 or 1 depending on n. The left-hand side of (iii) is the expression for the symbol of the index operator on X.

We return to the Remark following Part I, Theorem 6.5, as we want an explicit formula for $\mathrm{Id}^X_{S^1}(\delta_{S^1}, \Phi)(t)$ when $\Phi \in K^*_{S^1}(X)$ is a polynomial $\Phi(\eta)$ in η involving only: the integers a_i, the integers $\{x_{jk}\}$ and the first Chern class of the line bundle $\omega = P \times_{\mathrm{spin}^c(2n)} C$. That is, we want to remove the ambiguity in signs which occur in Part I, Theorem 6.5 (at least for the case at hand) due to the presence of square roots. To that end we introduce the expressions

(*) $$\xi_i(t) = \prod_{j \neq i} (t^{-(a_j - a_i)/2} - t^{(a_j - a_i)/2}) \prod (t^{-x_{ij}/2} - t^{x_{ij}/2})^{-1}.$$

Of course, $\xi_i(t)$ is only defined up to sign; however, we impose the additional restriction: There is an integer M such that

$$(**)\qquad t^{M/2}\cdot\sum_{i=0}^{n}\Phi_i(t)\xi_i(t)\prod_{j\neq i}(1-t^{a_j-a_i})^{-1}\in Z[t,t^{-1}]$$

for every $\Phi=\Phi(\eta)\in K^*_{S^1}(X)$. Here $\Phi_i(t)=\Phi(t^{a_i})$.

LEMMA 2.9. *The vector $v=(\xi_0(t),\xi_1(t),\dots,\xi_n(t))$ is well defined up to sign. That is, if the $\xi_i(t)$ satisfy (*) and (**) and if $\xi_0(t)$ is fixed, the ambiguity in sign in the remaining $\xi_i(t)$ disappears.*

PROOF. Let $v_i=(\xi^i_0(t),\xi^i_1(t),\dots,\xi^i_n(t))$, $i=0,1$, be two distinct solutions of (*) and (**). Suppose $v_0\neq -v_0$. Then there must be a pair k and j with $\xi^0_j(t)=\xi^1_j(t)$ and $\xi^1_k(t)=-\xi^0_k(t)$. Let

$$\Phi(\eta)=\prod_{i\neq k;i\neq j}(\eta-t^{a_i})\in K^*_{S^1}(X).$$

Then $\Phi_i(t)=0$, $i\neq k,j$. Apply (**) to v_0 and v_1 and add the two expressions obtaining the condition

$$(***)\qquad 2t^{M/2}\cdot\Phi_j(t)\xi_j(t)\prod_{l\neq j}(1-t^{a_l-a_j})^{-1}\in Z[t,t^{-1}].$$

But

$$\Phi_j(t)\prod_{l\neq j}(1-t^{a_l-a_j})^{-1}=t^{\alpha}/(1-t^{a_k-a_j})$$

and $\xi_j(t)=t^{\beta/2}\cdot\psi_j(t)$ where α and β are integers. By Lemma 2.3, $\psi_j(t)$ has no zero at $t=1$. Thus

$$\Phi_j(t)\xi_j(t)\cdot\prod_{l\neq j}(1-t^{a_l-a_j})^{-1}$$

has a pole at $t=1$ contradicting (***).

The relationship between (**) and $\mathrm{Id}^X_{S^1}(\delta_{S^1},\Phi)(t)$ is this

THEOREM 2.10. *Let P be the principle* $\mathrm{spin}^c(2n)$ *bundle associated to TX with $c_1(\omega)=(n+1)b$ where $\omega=P\times_{\mathrm{spin}^c(n)}C$ and $b=c_1(\eta')$. Let δ_{S^1} be the orientation class constructed from P. Then there is an integer N such that, for every $\Phi=\Phi(\eta)\in K^*_{S^1}(X)$,*

$$\mathrm{Id}^X_{S^1}(\delta_{S^1}\Phi)(t)=t^{(N+\Sigma a_k)/2}\sum_{i=0}^{n}\Phi_i(t)\xi_i(t)\prod_{j\neq i}(1-t^{a_j-a_i})^{-1}.$$

PROOF. By Part I, Theorem 6.5, with $L_i = (n+1)a_i$ and $\Phi_i(t) = \Phi(t^{a_i})$ we have

$$\begin{aligned} E_\Phi &= \mathrm{Id}^X_{S^1}(\delta_{S^1}\Phi)(t) \\ &= t^{N/2}\sum_{i=0}^{n}\Phi_i(t)t^{(n+1)a_i/2}\prod_{j=1}^{n}(t^{-x_{ij}/2} - t^{x_{ij}/2})^{-1}. \end{aligned}$$

The result follows by arithmetic.

THEOREM 2.11. *Let S^1 act on a manifold homotopy equivalent to CP^n. If X^{S^1} consists of isolated fixed points, the cohomology class $\hat{\mathscr{A}}(X) \in H^*(X, Q)$ is determined by the equivariant "Hopf bundle" η and the integers x_{jk} describing the representations of S^1 on TX at the isolated fixed points.*

PROOF. Let

$$E_k(t) = t^{(N+\Sigma a_i)/2}\sum_{i=0}^{n}\frac{t^{a_i k}\xi_i(t)}{\prod_{j\neq i}(1 - t^{a_j - a_i})}.$$

Since the a_i are defined by $\eta|_{p_i}(t) = t^{a_i}$, it follows from Lemma 2.10 that $E_k(t)$ is determined up to sign by η and the x_{jk}'s.

Let $b = c_1(\eta')$ be the first Chern class of η', $[X] \in H_{2n}(X)$ the orientation class and $\langle y, [X]\rangle$ the value of the cohomology class $y \in H^*(X, Q)$ evaluated on the orientation class. Then by the Remark following Part I, Proposition 5.3,

$$\langle e^{(k+(n+1)/2)b}\hat{\mathscr{A}}(X), [X]\rangle = \mathrm{Id}^X_{S^1}(\eta^k\delta_{S^1})(1) = \lim_{t\to 1} E_k(t) = E_k$$

and E_k is determined up to a sign independent of k by the given data.

We observe that: $e^{((n+1)/2)b}$ is a unit of $H^*(X, Q)$ because its degree zero term is 1, $ch: K^*(X) \otimes Q \to H^*(X, Q)$ is an isomorphism so $\{e^{kb} = ch\eta'^k | k = 0, 1, \ldots, n\}$ is a basis for $H^*(X, Q)$ and since $e^{((n+1)/2)b}$ is a unit, $\{e^{(k+(n+1)/2)b} | k = 0, 1, \ldots, n\}$ is also a basis and by Poincaré duality $\{e^{(k+(n+1)/2)b} \cap [X]\}$ is a basis for $H_*(X, Q)$. Since the values of $\langle \hat{\mathscr{A}}(X), e^{(k+(n+1)/2)b} \cap [X]\rangle$ for $k = 0, 1, 2, \ldots, n$ determine $\hat{\mathscr{A}}(X)$, $\hat{\mathscr{A}}(X)$ is determined up to sign by the given data. However, $\hat{\mathscr{A}}(X) = 1 +$ terms of higher dimension and this fixes $\hat{\mathscr{A}}(X)$.

COROLLARY 2.12. *Suppose $\xi_i(t) = \xi(t)$ is independent of i. If $h: X \to CP^n$ is a homotopy equivalence, $h^*\hat{\mathscr{A}}(CP^n) = \hat{\mathscr{A}}(X)$.*

PROOF. Let $x \in H^2(CP^n)$ be the first Chern class of the Hopf bundle and $b = h^*(x)$. Set

$$D_k(t) = \sum_{i=0}^{n} t^{ka_i}\prod(1 - t^{a_j - a_i})^{-1}.$$

Then by Part I, (6.4) (iv) and (v) and the Remark following Part I, Proposition 2.8,

$$\langle e^{((n+1)/2+k)x}\hat{\mathscr{A}}(CP^n), [CP^n]\rangle = \mathrm{Id}_{S^1}^{CP^n}(\delta_{S^1}\mathscr{H}^k)(1) = D_k(1).$$

On the other hand, it follows from Theorem 2.10 that

$$\langle e^{((n+1)/2+k)b}\hat{\mathscr{A}}(X), [X]\rangle = \mathrm{Id}_{S^1}^{X}(\delta_{S^1}\eta^k)(1) = \xi(1)D_k(1).$$

Since $\xi_i(t) = t^{\gamma_i/2} \cdot \psi_i(t)$ for some integer γ_i, it follows from Lemma 2.3 that $\xi(1) = \pm 1$. Thus

$$\begin{aligned}\langle h^*\hat{\mathscr{A}}(CP^n), [X] \cap e^{((n+1)/2+k)b}\rangle \\ &= \pm\langle \hat{\mathscr{A}}(CP^n), [CP^n] \cap e^{((n+1)/2+k)b}\rangle \\ &= \pm\langle \hat{\mathscr{A}}(X)[X] \cap e^{((n+1)/2+k)b}\rangle.\end{aligned}$$

Since this holds for every integer k, $h^*\hat{\mathscr{A}}(CP^n) = \pm\hat{\mathscr{A}}(X)$. However, $\hat{\mathscr{A}}(CP^n) = 1 \pm$ terms of higher dimension and likewise for $\hat{\mathscr{A}}(X)$; so the plus sign must hold.

We end the section with an example due to G. Bredon which shows that the analogs of Lemma 1.9 and Corollary 1.14 are false without the assumption on the field k.

Let S^1 act on the complex plane C^2 via the representation $\rho = t^2 + t^3 \in R(S^1)$. Then S^4 is the one point compactification of C^2 and the S^1 action extends uniquely to a smooth action on S^4 with fixed point set 0 and ∞. The isotropy subgroups are 0, Z_2, Z_3 and S^1.

Let $\mathcal{O}$ be the S^1 orbit of the point $(1, 1) \in C^2$ and $\mathfrak{T}$ be an open equivariant tubular neighborhood of $\mathcal{O}$. Set $X = S^4 - \mathcal{O}$. Then X is diffeomorphic to $S^2 \times D^2 = CP^1 \times D^2$. Let $p_1 = 0$ and $p_2 = \infty \in X$. The integers associated to p_1 and p_2 via $\eta|_{p_i}(t) = t^{a_i}$ are a_1 and $a_1 + 6 = a_2$. Thus $a_2 - a_1 = 6$ but the fixed point set of Z_6 is the same as the fixed point set of S^1. Compare Corollary 1.14.

As an algebra

$$K_{S^1}^*(X) \cong \Lambda[x]/((x^2 + \phi_1\phi_2\phi_3 x))$$

where $\phi_d = \phi_d(t)$ is the dth cyclotomic polynomial. The algebra Γ in this case is

$$\Gamma = \Lambda[y]/((y^2 - (1 - t^6)y) = \Lambda[\bar{y}]/((\bar{y} - 1)(\bar{y} - t^6))$$

where $\bar{y} = y + t^6$.

Let k be the cyclotomic field of primitive sixth roots of unity. The map of algebras A from Γ to $K_{S^1}^*(X)$ is defined by $A(y) = \phi_6(t)x$, so $A \otimes_\Lambda 1_k$: $\Gamma \otimes_\Lambda k \to K_{S^1}^*(X) \otimes_\Lambda k$ is not an isomorphism. Compare Lemma 1.9.

3. **Speculation: Bilinear forms on $K_G^*(X)$.** When X is a closed oriented manifold of dimension $2n$, the cup product pairing on $\hat{H}^n(X, Z)$

$= H^n(X, Z)/\text{Torsion}$ is nondegenerate. This means that the homomorphism $\Phi: \hat{H}^n(X, Z) \to \mathrm{Hom}_Z(\hat{H}^n(X, Z), Z)$ defined by $\Phi(x)[y] = \langle x \cup y, [X]\rangle$ for $x, y \in \hat{H}^n(X, Z)$ and $[X] \in H_{2n}(X, Z)$ the orientation class is an isomorphism. This is a consequence of

(a) Poincaré duality $\hat{H}^n(X, Z) \cong \hat{H}_n(X, Z)$.

(b) The universal coefficient theorem: Cap product defines an isomorphism; $\hat{H}^n(X, Z) \to \mathrm{Hom}_Z(\hat{H}_n(X), Z)$.

The statement of the universal coefficient theorem may be turned around to

(b′) $\hat{H}_n(X, Z) \to \mathrm{Hom}_Z(\hat{H}^n(X, Z), Z)$ is an isomorphism.

The concept of the nondegenerate bilinear form constructed in this way has been a powerful tool in the development of the topology of manifolds. The purpose of this section is to set up an analogous situation for $K_G^*(X)$, ask some questions and discuss examples.

Suppose that G is a compact connected Lie group acting smoothly on a closed G oriented manifold X. Then $K_*^G(X) = K_G^*(TX)$ and Poincaré duality holds; i.e., we have an isomorphism $\psi_G: K_G^*(X) \to K_G^*(TX)$. In addition we have an $R(G)$ homomorphism

$$\hat{D}: K_G^*(TX) \to \mathrm{Hom}_{R(G)}(K_G^*(X), R(G))$$

defined by

$$\hat{D}(x)[y] = \mathrm{Id}_G^X(x \cdot y), \qquad x \in K_G^*(TX), \qquad y \in K_G^*(X). \tag{3.1}$$

The composition $\hat{D}\psi_G$ gives rise to a bilinear form $\langle\ \rangle$ on $\hat{K}_G(X) = K_G^*(X)/\mathfrak{T}$ where $\mathfrak{T}$ is the $R(G)$ torsion subgroup of $K_G^*(X)$. Precisely

$$\langle z, y\rangle = \mathrm{Id}_G^X(\psi_G(z) \cdot y) = \hat{D}(\psi_G(z))[y]$$

for $z, y \in K_G^*(X)$.

Question 3.2. When is the bilinear form $\langle\ \rangle$ on $\hat{K}_G(X)$ nondegenerate? This means that the map $\hat{K}_G(X) \to \mathrm{Hom}_{R(G)}(\hat{K}_G(X), R(G))$ defined by $\hat{D}\psi_G$ is an isomorphism of $R(G)$ modules.

Question 3.3. When the preceding question has an affirmative answer, can one relate the algebraic invariants of the bilinear form $\langle\ \rangle$ to the representations of G on the fibers normal to the fixed point sets?

One hopes that the bilinear form $\langle\ \rangle$ is nondegenerate when $\hat{K}_G(X)$ is free over $R(G)$. We discuss some interesting cases when this is true.

Case 3.4. $X = U(Z)$ is the group of isometries of a complex G module Z and G acts on X by

$$g \cdot x = \rho(g)x\rho(g)^{-1}, \qquad g \in G,\ x \in U(Z),$$

where $\rho: G \to U(Z)$ is the representation given by the structure of Z as a complex G module. In Part I, §4, Example 3, we constructed an orientation class α_G. See Part I, Proposition 4.11.

PROPOSITION 3.5. *The bilinear form* $\langle\ \rangle$ *on* $\hat{K}_G(X)$ *defined by*

$$\langle x, y\rangle = \mathrm{Id}_G^X(\alpha_G x \cdot y)$$

is nondegenerate.

PROOF. Suppose the complex dimension of Z is n. Then

$$K_G^*(X) = R(G) \otimes_Z \Lambda(\theta_1, \ldots, \theta_n)$$

is the exterior algebra over $R(G)$ generated by n basic generators $\theta_1, \theta_2, \ldots, \theta_n$ **[10]**. Here $\Lambda(\theta_1, \ldots, \theta_n)$ is the Z exterior algebra generated by $\theta_1, \theta_2, \ldots, \theta_n$. Moreover, the homomorphism $\rho^*: K_{U(Z)}^*(X) \to K_G^*(X)$ induced by the homomorphism $\rho: G \to U(Z)$ is given by $\rho^* = \tilde{\rho} \otimes 1$ where $\tilde{\rho}: RU(Z) \to R(G)$ is induced by the homomorphism ρ. This means that a Z basis $\{e_i\}$ for $\Lambda(\theta_1, \ldots, \theta_n)$ gives an $R(G)$ basis for $K_G^*(X)$ and $K_{U(Z)}^*(X)$. Note that $\langle\ \rangle$ is nondegenerate on $K_G^*(X)$ if and only if the determinant of the matrix $\langle e_i, e_j\rangle$ is a unit of $R(G)$. Let us denote this determinant by $\det_{R(G)}\langle e_i, e_j\rangle$. By the above remarks and the compatibility axiom (Part I, (1.3))

$$\begin{aligned}\det_{R(G)}\langle e_i, e_j\rangle &= \mathrm{Id}_G^X(\alpha_G e_i \cdot e_j)\\ &= \mathrm{Id}_G^X(\rho^*\alpha_{U(Z)} e_i \cdot e_j) = \tilde{\rho}\mathrm{Id}_{U(Z)}^X(\alpha_{U(Z)} e_i \cdot e_j)\\ &= \tilde{\rho}\det_{R(U(Z))}\langle e_i, e_j\rangle.\end{aligned}$$

Thus it suffices to prove the proposition when $G = U(Z)$. For convenience we set $U(Z) = U$. As properties of the maximal torus of U play a key role in our proof we discuss them before proceeding.

Let $j: \mathfrak{T} \to U$ be the inclusion of the maximal torus $\mathfrak{T}$ viewed as a homomorphism of groups and $i: \mathfrak{T} \to U = X$ the same map viewed as a continuous map of topological spaces. Let $\mathfrak{T}$ operate on itself and X via inner automorphisms i.e., $x \in X$, $t \in \mathfrak{T}$,

$$t \circ x = txt^{-1}.$$

Then i is equivariant with respect to the action of $\mathfrak{T}$ and $X^{\mathfrak{T}} = \mathfrak{T}$ i.e., the fixed point set of the action of $\mathfrak{T}$ on X is $i\mathfrak{T}$ which we briefly write as $\mathfrak{T}$.

Since $\mathfrak{T}$ acts trivially on itself, $T\mathfrak{T} = \mathfrak{T} \times W$ where $W = R^n$ with trivial action of $\mathfrak{T}$. Either W or $W \times R^1$ is a complex $\mathfrak{T}$ module $\hat{W} \cong C^{[(n+1)/2]}$ with trivial action and the class

$$\begin{aligned}\alpha_{\mathfrak{T}} = 1 \otimes \Delta_{\mathfrak{T}} &\in K_{\mathfrak{T}}^0(\mathfrak{T}) \otimes_{R(\mathfrak{T})} K_{\mathfrak{T}}^0(\hat{W})\\ &= K_{\mathfrak{T}}^0(\mathfrak{T} \times \hat{W})\end{aligned}$$

is an orientation for $\mathfrak{T}$. See Part I, §4, Example 3. Moreover, if $\hat{V}$ is the complex U module used in constructing Δ_U we have

$$\hat{V}|_{\mathfrak{T}} = \hat{W} \oplus O \tag{3.6}$$

where O is the tangent space of $U/\mathfrak{T}$ at $[\mathfrak{T}] \in U/\mathfrak{T}$. The representation of $\mathfrak{T}$ on O is given by the adjoint action. This gives a representation $\omega:\mathfrak{T} \to U(d)$ where $2d =$ real dim O and the composition of ω with the map $\psi_0: U(d) \to \mathrm{spin}^c(2n)$ (Part I, (2.3)) gives a homomorphism $\widetilde{Ad} = \psi_0\omega: \mathfrak{T} \to \mathrm{spin}^c(2n)$. Let $\Omega = Ad^*(\Delta_+ - \Delta_-) \in R(\mathfrak{T})$.

It follows from (3.6) that

$$s^*j^*\Delta_U = \Delta_{\mathfrak{T}} \cdot \Omega \in K^0_{\mathfrak{T}}(\hat{W}) = K^{[(n+1)/2]}_{\mathfrak{T}}(T\mathfrak{T}). \tag{3.7}$$

Here $s: \hat{W} \to \hat{W} \oplus O$ is the zero section of this bundle over $\hat{W}$.

Let $Ti: T\mathfrak{T} \to TU$ be the inclusion which is equivariant with respect to the $\mathfrak{T}$ action on each. Then from the factorization

$$K^*_{\mathfrak{T}}(TU) = K^*_{\mathfrak{T}}(U \times \hat{V}) \xrightarrow{i^*} K^*_{\mathfrak{T}}(\mathfrak{T} \times (\hat{W} \oplus O)) \xrightarrow{s^*} K^*_{\mathfrak{T}}(\mathfrak{T} \times \hat{W}) = K^*_{\mathfrak{T}}(T\mathfrak{T})$$

of Ti^* and from (3.7) we see that

$$Ti^*j^*\alpha_U = \Omega \cdot \alpha_{\mathfrak{T}}. \tag{3.8}$$

Let U act trivially on $\mathfrak{T}$ and by left translation on $U/\mathfrak{T}$. Then we have an equivariant map

$$\pi: U/\mathfrak{T} \times \mathfrak{T} \to U$$

defined by

$$\pi(u\mathfrak{T}, t) = utu^{-1}$$

for $t \in \mathfrak{T}$ and $u \in U$. Note that $\mathfrak{T}$ is included in $U/\mathfrak{T} \times \mathfrak{T}$ via $f(t) = [\mathfrak{T}] \times t$. This is equivariant with respect to the $\mathfrak{T}$ action. We then have a commutative diagram of $\mathfrak{T}$ spaces and π is U equivariant

$$\begin{tikzcd}
\mathfrak{T} \arrow[r, "f"] \arrow[dr, "i"'] & U/\mathfrak{T} \times \mathfrak{T} \arrow[d, "\pi"] \\
& U.
\end{tikzcd}$$

Atiyah [**0**] has shown that the coordinates of $\mathfrak{T}$ define elements $\beta_i \in K^1(\mathfrak{T})$ such that

(i) $\pi^*(\theta_1 \cdot \theta_2 \cdots \theta_n) = \xi'\Omega \otimes \beta_1 \cdot \beta_2 \cdots \beta_n \in R(\mathfrak{T}) \otimes_Z K^*(\mathfrak{T}) = K^*_U(U/\mathfrak{T} \times \mathfrak{T})$ where ξ' is a unit of $R(\mathfrak{T})$.

Let $\phi: \mathfrak{T} \to 1$ be the trivial homomorphism. Then because $\mathfrak{T}$ acts on itself trivially,

(ii) $\mathrm{Id}^{\mathfrak{T}}_{\mathfrak{T}}(\alpha_{\mathfrak{T}}\beta_1 \cdots \beta_n) = \mathrm{Id}^{\mathfrak{T}}_{\mathfrak{T}}(\phi^*\alpha_1\beta_1 \cdots \beta_n) = \phi\mathrm{Id}^{\mathfrak{T}}_1(\alpha_1\beta_1 \cdots \beta_n) = 1.$

By the localization theorem (Part I, 1.6),

$$\begin{aligned}&\mathrm{Id}^X_{\mathfrak{T}}(j^*\alpha_U\theta_1\cdot\theta_2\cdots\theta_n)\\&=\mathrm{Id}^{\mathfrak{T}}_{\mathfrak{T}}(Ti^*j^*\alpha_U\theta_1\cdots\theta_n/\lambda_{-1}(0\otimes C))\\&=\mathrm{Id}^{\mathfrak{T}}_{\mathfrak{T}}(\Omega\alpha_{\mathfrak{T}}\cdot\xi'\Omega\beta_1\cdots\beta_n/\lambda_{-1}(0\otimes C))\end{aligned}$$

because

$$\begin{aligned}Ti^*j^*\alpha_U\theta_1\cdots\theta_n&=(Ti^*j^*\alpha_U)\cdot(i^*\theta_1\cdots\theta_n)\\&=\Omega\alpha_{\mathfrak{T}}\cdot f^*\pi^*\theta_1\cdots\theta_n=\xi'\Omega^2\cdot\alpha_{\mathfrak{T}}\cdot\beta_1\cdots\beta_n\end{aligned}$$

by (3.8) and (i). Let $-$ be the involution of $R(\mathfrak{T})$ defined by sending a complex $\mathfrak{T}$ module W to $\mathrm{Hom}_C(W, C)$. Then a basic property of the element Ω is that $\Omega\bar{\Omega} = \lambda_{-1}(0 \otimes C)$ and $\bar{\Omega} = \xi \cdot \Omega$ where ξ is a unit of $R(\mathfrak{T})$ [**8**]. Putting all this information together gives

(iii) $\mathrm{Id}^X_{\mathfrak{T}}(j^*\alpha_U\theta_1\cdots\theta_n) = \xi'\xi^{-1}$ a unit of $R(\mathfrak{T})$.

Note that $\tilde{j}$ is an inclusion of $R(U)$ in $R(\mathfrak{T})$ as the subring of invariants of the Weyl group. This implies that if $x \in R(U)$ and $\tilde{j}(x)$ is a unit of $R(\mathfrak{T})$, then x is a unit of $R(U)$. From this and

(iv) $\tilde{j}'\,\mathrm{Id}^X_U(\alpha_U\theta_1\cdots\theta_n) = \xi'\xi^{-1}$ by (iii,)

we see that

(v) $\mathrm{Id}^X_U(\alpha_U\theta_1\cdots\theta_n)$ is a unit of $R(U)$.

It follows from (v) and the algebra structure of the exterior algebra $\Lambda(\theta_1, \ldots, \theta_n)$ that the map $\hat{D}\psi_U : K^*_U(X) \to \mathrm{Hom}_{R(U)}(K^*_U(X), R(U))$ is an isomorphism and this completes the proof of Proposition 3.5. Compare [**0**].

PROPOSITION 3.9. *Let S^1 act on CP^n as described in Part* I, (6.4) (*Standard example*). *Then the bilinear form $\langle\ \rangle$ is nondegenerate.*

PROOF. Atiyah has shown [**1**]

$$K^*_{S^1}(CP^n) = R(S^1)[\mathscr{H}]\Big/\left(\prod_{i=0}^{n}(\mathscr{H} - t^{a_i})\right)$$

as an $R(S^1)$ algebra. The restriction homomorphism

$$i^* : K^*_{S^1}(CP^n) \to \prod_{i=0}^{n+1} K^*_{S^1}(p_i) = \prod_{i=0}^{n+1} R(S^1)$$

is given by

$$i^*\mathscr{H}^k = (t^{ka_0}, t^{ka_1}, \ldots, t^{ka_n}). \tag{3.10}$$

Thus $1, \mathscr{H}, \ldots, \mathscr{H}^n$ is a basis for $K^*_{S^1}(CP^n)$ over $R(S^1)$ and i^* is a monomorphism and induces an isomorphism

$$K^*_{S^1}(CP^n) \otimes_{R(S^1)} F(S^1) \to \prod_{i=0}^{n} F(S^1)$$

where $F(S^1)$ is the field of fractions of $R(S^1)$. Let

$$e_i = \prod_{j \neq i}(t^{a_i} - t^{a_j})^{-1}(\mathscr{H} - t^{a_j}).$$

Then $e_0, e_1, \dots, e_n$ is a base for $K^*_{S^1}(CP^n) \otimes_{R(S^1)} F(S^1)$ and

$$\mathscr{H}^k = \sum t^{ka_i} e_i \tag{3.11}$$

because of (3.10) and the fact that i^* induces an isomorphism over $F(s')$.

Observe that $e_i^2 = e_i$ because $i^*e_i^2 = i^*e_i \cdot i^*e_i = i^*e_i$. Thus in terms of this new basis $\langle\ \rangle$ gives a diagonal matrix with

$$\langle e_i, e_i\rangle = \prod_{j\neq i} (1 - t^{a_j - a_i})^{-1}. \tag{3.12}$$

This follows from Part I, Corollary 5.6, by setting $u_j(t) = e_i|_{p_j}(t)$. Then $u_i(t) = 1$ and $u_j(t) = 0$ for $j \neq i$. Furthermore, $\lambda_j = 0$ because the action of S^1 on the principle spin$^c(2n)$ bundle P is defined using the lifting ψ_0 of $U(n)$ to spin$^c(2n)$ (Part I, 6.3).

Thus if $\Delta_i = \prod_{j\neq i}(t^{a_i} - t^{a_j})$, then

$$\det((\langle e_i, e_j\rangle)) = t^{n\Sigma a_j} \cdot \prod_{i=0}^{n} \Delta_i^{-1}.$$

Let $e_i = \sum_j S_{ij}\mathscr{H}^j$. Then

$$\det((\langle \mathscr{H}^i, \mathscr{H}^j\rangle)) = \det S^{-2} \det((\langle e_i, e_j\rangle)).$$

By (3.11), S is the inverse of the Vandermone matrix V so S^{-1} is V and

$$\det S^{-1} = \det V = \prod_{j<k} (t^{a_j} - t^{a_k}).$$

But

$$(\det V)^2 = \prod_k \prod_{j\neq k} (t^{a_j} - t^{a_k}) = \prod_{k=0}^{n} \Delta_k.$$

Hence $\det((\langle \mathscr{H}^i, \mathscr{H}^j\rangle)) = t^{n\Sigma a_j}$, a unit of $R(S^1)$. Since $\mathscr{H}^i$, $i = 0, 1, 2, \dots, n$, is a free $R(S^1)$ base for $K^*_{S^1}(CP^n)$, the proof is complete.

Finally, I mention the interesting analysis of Vasquez [**18**]. He studies the case in which $H \subset G$ are compact connected Lie groups with H of maximal rank in G and such that the homogenous space $X = G/H$ has a spin$^c(m)$ structure, $m = \dim X$. Left translation by G makes X a G space and Vasquez shows that the bilinear form $\langle\ \rangle$ is nondegenerate on $K^*_G(X)$.

These examples can be multiplied by taking cartesian products. Anyway, Question 3.2 has an affirmative answer in enough cases to make it interesting and hopefully useful.

4. **An exotic action of S^1 on CP^3.** We offer an example of an action of S^1 on CP^3 which is definitely distinct from the linear actions of Part I, (6.4). This example is distinguished from the linear actions by the representation of S^1 on the tangent space at the four isolated fixed points.

Representations of S^1 in $U(2)$ are defined by the

$$\omega(t) = \begin{pmatrix} t & 0 \\ 0 & t^6 \end{pmatrix}, \qquad \psi(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^7 \end{pmatrix},$$

$$\beta(t) = \begin{pmatrix} t^2 & 0 \\ 0 & t^3 \end{pmatrix}, \qquad t \in S^1.$$

The sphere S^3 is identified with $SU(2)$ via the map

$$f(\vec{z}) = \begin{pmatrix} z_0 & z_1 \\ -\bar{z}_1 & \bar{z}_0 \end{pmatrix}, \qquad \vec{z} = (z_0, z_1), \quad z_i \in C.$$

We let $\vec{z} \cdot u$ for $\vec{z} \in S^3$ and $u \in U(2)$ denote the standard action of $U(2)$ on S^3.

Lemma 4.1. *In $U(2)$ we have*

$$\psi(t) f(\vec{z} \cdot \omega(t)) = f(\vec{z})\, \omega(t).$$

Lemma 4.2. *There is a degree one map $\Phi: S^3 \to S^3$ satisfying*

$$\Phi(\vec{z} \cdot \beta(t)) = \Phi(\vec{z})\, \omega(t).$$

Proof. Let

$$\Phi(z_0, z_1) = \frac{(\bar{z}_0 z_1, z_0^3 + z_1^2)}{\|(\bar{z}_0 z_1, z_0^3 + z_1^2)\|}$$

for $(z_0, z_1) \in S^3$. Here $\bar{z}_0$ denotes the complex conjugate of z_0.

Define a diffeomorphism $g: U(2) \times S^3 \to U(2) \times S^3$ by

$$g(u, \vec{z}) = (u f(\Phi(\vec{z})), \vec{z}).$$

Let X_i, $i = 0, 1$, be the S^1 manifold whose underlying space is $U(2) \times D^4$ with S^1 action defined by

$$(u, \vec{z})t = (u\psi(t), \vec{z}\beta(t)), \qquad i = 0,$$

$$(u, \vec{z})t = (u\omega(t), \vec{z}\beta(t)), \qquad i = 1,$$

$u \in U(2)$, $|\vec{z}| \leqq 1$.

Lemma 4.3. *$g: \partial X_0 \to \partial X_1$ is an equivariant diffeomorphism.*

Proof. Equivariance follows from Lemmas 4.1 and 4.2. Since g is 1-1, it is a diffeomorphism.

Let $H = U(1) \times U(1) \subset U(2)$. Then H acts on the left of $U(2)$ by left multiplication and on X_i by $\alpha(u, \vec{z}) = (\alpha u, \vec{z})$, $\alpha \in U(1) \times U(1)$, $u \in U(2)$ and $\vec{z} \in D^4$. The left action of H on X_i commutes with the right action of S^1 on X_i so the orbit space $\bar{X}_i = X_i/H$ inherits an S^1 action. Moreover g commutes with the action of H on ∂X_0 and ∂X_1 and induces a diffeomorphism

$\bar{g}:\partial\bar{X}_0 \to \partial\bar{X}_1$ which is equivariant with respect to the S^1 action on each.

In a similar manner the left action of H on $U(2)$ commutes with the two right actions of S^1 defined by

$$u \circ t = u\psi(t), \qquad u \circ t = u\omega(t),$$

for $u \in U(2)$, $t \in S^1$. The orbit space $U(2)/H$ is $CP^1 = S^2$ and inherits two S^1 actions. We let Z_i, for $i = 0, 1$, denote the resulting S^1 manifolds.

Observe that as an S^1 manifold $\bar{X}_i = Z_i \times D(M)$ where $D(M)$ is the unit disk in the complex 2 dimensional S^1 module M with S^1 acting via the representation $t \to \beta(t) \in U(2)$. This means that $\bar{X}_i$ is the total space of a disk bundle of a complex S^1 bundle over Z_i. Thus we have a Thom isomorphism

$$\lambda_i : K^*_{S^1}(Z_i) \to K^*_{S^1}(\bar{X}_i, \partial\bar{X}_i).$$

LEMMA 4.4. *Let $X = \bar{X}_0 \cup_{\bar{g}} \bar{X}_1$ denote the manifold obtained by identifying $x \in \partial\bar{X}_0$ with $\bar{g}(x) \in \partial\bar{X}_1$. Then X is homotopy equivalent to CP^3.*

PROOF. Since $\Phi : S^3 \to S^3$ is homotopic to the identity (Lemma 4.2), $\bar{g}$ is homotopic to the map $H : S^2 \times S^3 \to S^2 \times S^3$ defined by $H(u, \vec{z}) = (uf(\vec{z}), z)$ and $CP^3 = S^2 \times D^4 \cup_H S^2 \times D^4$.

The manifold X inherits a unique S^1 action compatible with the given action on $\bar{X}_i$, for $i = 0, 1$. The fixed point set of this action consists of four isolated points p_0, p_1, p_2, p_3 labeled so that $p_0, p_1 \in \bar{X}_0$ and $p_2, p_3 \in \bar{X}_1$. The representations of S^1 on $TX|_{p_i}$ are given by

$$TX|_{p_0}(t) = t^7 \oplus t^2 \oplus t^3,$$
$$TX|_{p_1}(t) = t^{-7} \oplus t^2 \oplus t^3,$$
$$TX|_{p_2}(t) = t^5 \oplus t^2 \oplus t^3,$$
$$TX|_{p_3}(t) = t^{-5} \oplus t^2 \oplus t^3.$$

Actually we have listed complex representations which define the real representations we seek.

Here is an explicit description of the S^1 bundle η over X. The total space of η, $E(\eta)$ is given as

$$E(\eta) = C^1 \times_H X_0 \cup_G C^1 \times_H X_1$$

where $[c, x_0]$ in $C^1 \times_H \partial X_0$ is identified with $G[c, x_0] = [c, g(x_0)]$ in $C^1 \times_H \partial X_1$. The action of $E(\eta)$ is defined by the condition

$$[c, x_i]t = [c, x_i t], \qquad x_i \in X_i, \; i = 0, 1.$$

$c \in C^1$, $t \in S^1$. The projection π of $E(\eta)$ on X is described by $\pi[c, x_i] = p_i(x_i)$, $i = 0, 1$, and $p_i : X_i \to \bar{X}_i$ is the orbit map. We find

$$\eta|_{p_0}(t) = t^0, \quad \eta|_{p_1}(t) = t^7, \quad \eta|_{p_2}(t) = t^1, \quad \eta|_{p_3}(t) = t^6.$$

COROLLARY 4.5. *This action of S^1 on X is not equivalent to any linear action of S^1 on CP^3. It is distinguished from such a linear action by the representations of S^1 on TX at the isolated fixed points p_i.*

PROOF. By Part I, (6.4), the representations of S^1 on $TCP|_{p_i}$ are completely determined by the equivariant Hopf bundle $\mathscr{H}$ as follows: $\mathscr{H}|_{p_i}(t) = t^{a_i}$ and $TX|_{p_i}(t)$ is the real representation defined by the complex representation $\chi_i = \sum_{j \neq i} t^{a_j - a_i}$.

Suppose $F: X \to CP^3$ is a diffeomorphism equivariant with respect to some linear action on CP^3. By composing F with the map which conjugates the coordinates in CP^3 if necessary, we may suppose that F is orientation preserving. Then by Stewart's theorem

$F^*\mathscr{H} = t^k \cdot \eta$ as S^1 line bundles for some integer k. Then $a_0 = k$, $a_1 = 7 + k$, $a_2 = 1 + k$, $a_3 = 6 + k$. But the real representations on TX_{p_i} are not of the form

$$\rho_i(e^{i\theta}) = \operatorname{diag}\begin{pmatrix} \cos(a_k - a_i)\theta & \sin(a_k - a_i)\theta \\ -\sin(a_k - a_i)\theta & \cos(a_k - a_i)\theta \end{pmatrix}, \qquad k \neq 1,$$

as dictated by the linear case.

COROLLARY 4.6. *X is diffeomorphic to CP^3.*

PROOF.

$$\xi_i(t) = \frac{(t^{-1/2} - t^{1/2})(t^{-6/2} - t^{6/2})}{(t^{-2/2} - t^{2/2})(t^{-3/2} - t^{3/2})} = \xi(t)$$

is independent of i. Choose an orientation preserving homotopy equivalence $h: X \to CP^3$. By Corollary 2.12, $h^*\hat{\mathscr{A}}(CP^3) = \hat{\mathscr{A}}(X)$ and $\hat{\mathscr{A}}(X) = 1 - p_1 h^*(x^2)/24$ where x is the first Chern class of the Hopf bundle and p_1 is an integer such that $p_1 h^*(x^2)$ is the first Pontrjagin class of X, $P_1(X)$. Montgomery-Yang have shown [19] that the manifolds homotopy equivalent to CP^3 are in 1-1 correspondence with the integers. The correspondence is characterized by $i \to W_i$ where $P_1(W_i) = (24i + 4)h_i^*(x^2)$ and $h_i: W_i \to CP^3$ is a homotopy equivalence.

Since $h^*\hat{\mathscr{A}}(CP^3) = \hat{\mathscr{A}}(X)$ we have $P_1(X) = 4h^*(x^2)$ so by the Montgomery-Yang theorem X is diffeomorphic to CP^3.

5. **The bilinear form $\langle\ \rangle$ on $K^*_{S^1}(X)$, $X = CP^3$.** Let $i: Z_0 \to X$ be the inclusion and denote by i_* the composition of the Thom isomorphism λ_0 and the natural map $K^*_{S^1}(\bar{X}_0, \partial\bar{X}_0) \to K^*_{S^1}(X)$. Then

$$i^* i_*(x) = \lambda_{-1}(\nu) \cdot x, \quad \text{for} \quad x \in K^*_{S^1}(Z_0),$$

and v the S^1 normal bundle of Z_0 in X. From the exact sequence of the pair $(X, \bar{X}_1)$ we obtain this short exact sequence

$$\text{(i)} \qquad 0 \to K^*_{S^1}(Z_0) \xrightarrow{i_*} K^*_{S^1}(X) \xrightarrow{j^*} K^*_{S^1}(Z_1) \to 0$$

where $j: Z_1 \to X$ is the inclusion.

Let $\eta_i \in K^*_{S^1}(Z_i)$ be the equivariant Hopf bundle for Z_i and $\eta \in K^*_{S^1}(X)$ the equivariant Hopf bundle for X. Since 1, η_i gives an $R(S^1)$ base for $K^*_{S^1}(Z_i)$, it follows readily from (i) that $i_*(1)$, $i_*(\eta_0)$, 1 and η gives an $R(S^1)$ base for $K^*_{S^1}(X)$. Let

$$e_i = \prod_{j \neq i} (\eta - t^{a_j})(t^{a_i} - t^{a_j})^{-1} \in K^*_{S^1}(X) \otimes_{R(S^1)} F(S^1) = K^*_{S^1}(X)_0.$$

LEMMA 5.1. *In* $K^*_{S^1}(X)_0$ *we have*

$$\begin{aligned} i_*(1) &= \lambda_{-1}(v)(e_0 + e_1), \\ i_*(\eta_0) &= \lambda_{-1}(v)(e_0 + t^7 e_1), \\ 1 &= e_0 + e_1 + e_2 + e_3, \\ \eta &= e_0 + t^7 e_1 + t^1 e_2 + t^6 e_3. \end{aligned}$$

REMARK. *Note* $\lambda_{-1}(v) = (1 - t^2)(1 - t^3) \cdot 1$, $1 \in K^*_{S^1}(X)$; *so we regard* $\lambda_{-1}(v)$ *as* $(1 - t^2)(1 - t^3) \in R(S^1)$ *as well as an element of* $K^*_{S^1}(X)$.

PROOF. $K^*_{S^1}(X)$ is a free $F(S^1)$ module so the restriction

$$r^*: K^*_{S^1}(X)_0 \to K^*_{S^1}(X^{S^1})_0 = \prod_{i=0}^{3} K^*_{S^1}(p_i)_0$$

is an isomorphism; hence, in order to establish the equation of the lemma it is sufficient to show that they hold in $K^*_{S^1}(X^{S^1})_0$. E.g., since $j^* i_* = 0$, $i_*(1) = \alpha_0 e_0 + \alpha_1 e_1$ and $\alpha_i = i_*(1)|_{p_i}$ for $i = 0, 1$. But $i_*(1)|_{p_i} = i^* i_*(1)|_{p_i} = \lambda_{-1}(v)|_{p_i} = \lambda_{-1}(v)$. Thus $\alpha_0 = \alpha_1 = \lambda_{-1}(v)$.

THEOREM 5.2. *The bilinear form* $\langle\ \rangle$ *on* $K^*_{S^1}(X)$ *is nondegenerate.*

PROOF. With respect to the basis e_i, the matrix of the bilinear form $\langle\ \rangle$ is diagonal. In fact,

$$\begin{aligned} \langle e_0, e_0 \rangle &= t^{-\lambda_0}/(1 - t^7)\lambda_{-1}(v), \\ \langle e_1, e_1 \rangle &= t^{-\lambda_1}/(1 - t^{-7})\lambda_{-1}(v), \\ \langle e_2, e_2 \rangle &= t^{-\lambda_2}/(1 - t^5)\lambda_{-1}(v), \\ \langle e_3, e_3 \rangle &= t^{-\lambda_3}/(1 - t^{-5})\lambda_{-1}(v), \end{aligned}$$

where λ_i are integers. See Part I, Corollary 5.6.

So in terms of this basis the determinant of $\langle\ \rangle$ is

$$D = u/(1 - t^7)^2(1 - t^5)^2 \cdot \lambda_{-1}(v)^4$$

where u is a unit of $Z[t, t^{-1}]$.

Let S be the matrix which expresses the basis $i_*(1), i_*(\eta_0), 1, \eta$ in terms of the basis $\{e_i\}$. From Lemma 5.1 we see that the determinant of S, written $|S|$, is $\lambda_{-1}(v)^2 \cdot (1 - t^7)(1 - t^5) \cdot t$. Thus the determinant of $\langle\ \rangle$ with respect to the "integral basis" $i_*(1), i_*(\eta), 1, \eta$ is $|S|^2 \cdot D$ and this is a unit of $Z[t, t^{-1}]$.

I regard this example optimistically as strong evidence that the bilinear form $\langle\ \rangle$ is nondegenerate under rather general circumstances.

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