BOOK REVIEWS

A course in probability theory, by Kai Lai Chung. Harcourt, New York, 1968. xiii+331 pp. $12.00.

Probability, by Leo Breiman. Addison-Wesley, Reading, Mass., 1968. ix+421 pp. $12.95.

In recent years books on probability theory have mushroomed. The two books under review have this in common: they are concerned with probability theory proper, rather than with its foundations (like the recent elegant but austere book by Neveu[1]), and they give a balanced view of the theory rather than dwelling at length on some specialized topics.

What is probability theory? Chung's spirited preface at first tells us what it is not: the "specious utterance" that "probability is just a chapter of measure theory" finds no more favor with him than the use of probability "as a front for certain types of analysis such as combinatorial, Fourier, functional and whatnot." Elsewhere[2] he says about the same utterance that it is "not so much false as it is fatuous, as it would be to say that number theory is just a chapter of algebra." Thus over the years Professor Chung continues his disagreement with Professor Doob who in his *Stochastic processes* writes: "Probability is simply a branch of measure theory, with its own special emphasis and field of application." On the positive side Professor Chung remarks in the preface of his book: "The random walks in Chapter 8 illustrate well the way probability theory transforms other parts of mathematics. It does so by introducing the trajectories of the process, thereby turning what was static into a dynamic structure."

As for Dr. Breiman, without defining probability he tells us about its right and left hand. "On the right is the rigorous foundational work using the tools of measure theory. The left hand 'thinks probabilistically,' reduces problems to gambling situations, coin-tossing, motions of a physical particle." Breiman acknowledges M. Loève as his teacher of the first aspect of probability (and D. Blackwell as the teacher of the second) and Professor Loève in *his* treatise on the subject (p. 170) defines a probabilistic property as one which can be described in terms of distribution functions. Professor Neveu (Preface, op. cit.) strongly disagrees: "Attempts have been made, it is true, to treat convergence problems of probability theory within the restricted framework of the study of distribution functions; but this

[1] J. Neveu, *Mathematical foundations of the calculus of probability*, Holden-Day, San Francisco, Calif., 1965.

[2] In the review of A. Rényi's textbook, Bull. Amer. Math. Soc. 70 (1964), 225.

procedure only gives a false simplification of the question and further conceals the intuitive basis of these problems." It is true that his definition of probability properties notwithstanding, Loève discusses at length topics which cannot be treated via distribution functions: martingales, stopping times, ergodic theory. (This is also true of Breiman, but to a lesser extent of Chung.) The reader confused by these conflicting opinions of luminaries may want to agree with Poincaré (Calcul des Probabilités, 1896): "L'on ne peut guère donner une definition satisfaisante de la Probabilité."

The description of Chung's book follows. Chapters 1–4 deal with preliminaries: distribution functions, measures, convergence concepts. The treatment is pleasantly detailed; in fact, the author prides himself in having abandoned the "decadent fashion of conciseness." In view of this expansiveness it is somewhat surprising that the Kolmogorov extension theorem has been stated but not proved. (When teaching a course in probability the reviewer omits this theorem, but his conscience is eased by the knowledge that the proof is present in the textbook.) Chapter 5 contains some "famous landmarks" of the theory: the three series theorem, the laws of large numbers, the Glivenko-Cantelli theorem, Wald's equation. Many results in this chapter could be established with no extra effort in the more general context of martingales or stationary processes. Chapter 6 develops at great length Paul Lévy's method of characteristic functions; author's mastery of this method is here in full view. A "peripheral" discussion of the logarithm of the characteristic function is included, and errors which have here befallen other books (in one instance pointed out by Professor Chung in a review) are proudly avoided. The heart of the book is in Chapters 7 and 8. Chapter 7 deals with the central limit problem under the independence assumption. The Lindeberg-Feller theorem is established and the becoming name "holospoudic" is given to the well-known property of random variables appearing in this theorem, loosely expressed by saying that each individual term is small in comparison with the sum. Professor Chung assures the reader that this is the only new term coined in the book; the use of the word *remote*, however, in the sense: *tail*-remote event for tail event, seems to be another innovation, justified perhaps by the difficulties appearing in the translation of the older term into certain European languages. (Neveu uses here the term "asymptotic".) A digression on the authors' language: Professor Chung's English is vigorous and imaginative, with some foreign influence perhaps reminiscent of Nabokov in its un-English richness and force of conviction. Dr. Breiman is also very lively, but very American, loose and informal almost to the point of incorrectness. Both authors write well, in a very different way.

Chapter 8 of Chung's book deals with *random walk*; i.e. with sums $S_n = X_1 + \cdots + X_n$ where the X_n's are independent and identically distributed. After the proof of the Hewitt-Savage zero one-law and the strong Markov property of the random walk, the Fourier-analytic methods are used to probe the subject in depth. The theorem that the one-dimensional random walk is recurrent if (and only if) $E(X_i) = 0$ is in fact proved twice; one proof due to the author and Donald Ornstein is measure-theoretic; a second, earlier, proof due to the author and W. H. J. Fuchs is Fourier-analytic. (Measure-theoretic proofs of several random walk results can be found in a recent paper of Donald Ornstein, Trans. Amer. Math. Soc. **138** (1969), 1–60.) The chapter continues with some quite recent results of F. Spitzer and G. Baxter and culminates in Sparre Andersen's arc sine law.

The last chapter gives the elements of martingales and Markov processes. Both notions generalize independence, and after such an extensive study of random walk, the reader may have the impression that the main ideas of Chapter 9 were already fully present in the older theory of independence. This does not do full justice to martingales (very little of Markov processes is done). Actually, some martingale ideas (stopping times) appear in the treatment of earlier chapters. There is further the point that the martingale theory has its origins not only in the probabilistic theory of addition of random variables (Paul Lévy) but also in the measure-theoretic theory of derivatives of set-functions (R. de Possel, Andersen and Jessen), and not much of this motivation is visible here. But this is a book on probability, and probability theory is sometimes defined as a study of independence and its generalizations. We submit that this definition is among the best, provided that the reader knows, intuitively or otherwise, what is independence.

Breiman's book, which despite its slim appearance is longer than Chung's book and also more concise, covers nearly all topics appearing in the first book, and substantially more. A notable exception is that here the Lindeberg-Feller theorem is unaccountably omitted. It would seem that a necessary and sufficient condition for the central limit theorem to hold should have found its way into a book of such a scope, but a reader interested in holospoudicity will be disappointed. There is, however, ample compensation. A chapter on stationary processes includes a proof of Birkhoff's ergodic theorem and a nice application of this theorem due to Kesten, Spitzer and Whitman: if R_n is the number of distinct values taken on by a random walk on integers in the first n steps, then R_n/n converges almost surely to the probability that the random walk will never return to the origin. The chapter on discrete parameter Markov chains is remarkable for its elegant and compact presentation of limit theorems, including Orey's theorem, the most important recent result in the theory now seemingly essentially completed. This theorem reads: If $p^{(n)}(i,j)$ is the n-step transition probability matrix of an irreducible (=indecomposable) recurrent aperiodic Markov chain, then for any i, k

$$\lim_{n\to\infty} \sum_j \left| p^{(n)}(i,j) - p^{(n)}(k,j) \right| = 0.$$

The main limit theorem of Kolmogorov: $p^{(n)}(i,j)$ converges as $n \to \infty$, easily follows. There is a chapter on Brownian motion containing a recent result of Dworetski, Erdös and Kakutani that almost every Brownian path is nowhere differentiable. There is a substantial chapter on invariance theorems, including Donsker's invariance principle, the Doob-Donsker proof of the Kolmogorov-Smirnov theorem and finally a general invariance principle of Skorokhod, the proof of which is only sketched. The author has nearly all the material necessary to establish the recent important result of V. Strassen on the law of iterated logarithm, but he stops short of it. The last chapters are concerned with continuous parameter stochastic processes: martingales, Poisson processes, Markov processes, diffusion. They seem to present a good introduction into a field in which a great deal of research is now being done, by some of the leading probabilists. A parenthetical remark: Breiman's book, unlike Chung's, has an excellent table of contents from which the reader may for himself find out what the book contains.

In conclusion, both works under review can be highly recommended as first-year graduate textbooks; even though in this role we would give a slight edge to Chung's book, a very carefully and lucidly written account by a master of the subject, and a product, the author tells us, of many years of teaching. Breiman's book may be preferred by an instructor who wishes to rapidly introduce a very good class into modern ideas more general than independence; furthermore, a specialist will want to read this book for the rich account it gives of recent developments in fields other than his own. Each book constitutes a very valuable addition to the literature.

LOUIS SUCHESTON

An introduction to harmonic analysis, by Yitzhak Katznelson. Wiley, New York, 1968. 266 pp. $12.95.

This is a text, on the modern theory of Fourier series and integrals, designed to be used by students who know the basics of real and