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GELFAND–SHILOV CLASSES OF MULTI-ANISOTROPIC TYPE

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: Aim of this paper is to introduce a generalization of Gelfand–Shilov classes by means of estimates based on a complete polyhedron (see for instance Gindikin–Volevich [10]). This class includes the standard Gelfand–Shilov functions and their anisotropic version.

Keywords: Gelfand–Shilov spaces, complete polyhedra.

1. Introduction

Let us begin by recalling the definition of Gevrey classes $G^s(\Omega)$, $1 < s < \infty$, Ω open subset of $\mathbb{R}^n$, and Gelfand–Shilov classes $S^s_r(\mathbb{R}^n)$, where we shall assume for simplicity $1 \leqslant s < \infty$, $1 \leqslant r < \infty$.

A function f belongs to $G^s(\Omega)$ if for every compact subset $K \subset\subset \Omega$ we have

$$\sup_{x\in K} |\partial_x^\alpha f(x)| \leqslant C^{|\alpha|+1}(\alpha!)^s, \quad \forall \alpha \in \mathbb{N}^n \tag{1}$$

for a suitable positive constant C independent on the multi-index α.

We then define $G^s_0(\Omega) = G^s(\Omega) \cap C^\infty_0(\Omega)$. A function f belongs to $G^s_0(\mathbb{R}^n)$ if and only if the Fourier transform $\hat{f}$ of f satisfies

$$|\hat{f}(\xi)| \leqslant C \exp(-\varepsilon|\xi|^{\frac{1}{s}}), \quad \forall \xi \in \mathbb{R}^n \tag{2}$$

for suitable constants $C < \infty$, $\varepsilon > 0$, see for example [12].

Willing to find a counterpart of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, we are then led to the classes of Gelfand–Shilov [9]. Namely, a function f belongs to the Gelfand–Shilov class $S^s_r(\mathbb{R}^n)$, for $r, s \geqslant 1$ if there exists a constant $C < \infty$ such that

$$\sup_{x\in \mathbb{R}^n} |x^\beta \partial_x^\alpha f(x)| \leqslant C^{|\alpha|+|\beta|+1}(\alpha!)^s(\beta!)^r, \quad \forall \alpha \in \mathbb{N}^n,\ \forall \beta \in \mathbb{N}^n. \tag{3}$$

According to [7], this definition is equivalent to the following one, reminiscent of the estimate (2).

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A function f belongs to the Gelfand–Shilov class $S_r^s(\mathbb{R}^n)$, for $r, s \geqslant 1$, if $f \in \mathcal{S}(\mathbb{R}^n)$ and there are constants $C < \infty$, $\varepsilon > 0$ such that f and its Fourier transform $\hat{f}$ satisfy the following two conditions

$$\begin{aligned} &(i) \quad |f(x)| \leqslant C \exp(-\varepsilon |x|^{\frac{1}{r}}), \quad \forall x \in \mathbb{R}^n, \\ &(ii) \quad |\hat{f}(\xi)| \leqslant C \exp(-\varepsilon |\xi|^{\frac{1}{s}}), \quad \forall \xi \in \mathbb{R}^n. \end{aligned} \tag{4}$$

The Gevrey classes $G^s(\Omega)$ have been generalized in different ways by several authors. Here we address in particular to the multi-anisotropic Gevrey classes, see Bouzar-Chaili [2, 3], Calvo [4], Calvo-Hakobyan [5], Gindikin-Volevich [10], Zanghirati [13, 14].

In short, we fix a complete polyhedron $\mathcal{P} \subset \mathbb{R}_+^n$. Let us denote by $\mathcal{V}(\mathcal{P})$ the set of the vertices, and let μ be the formal order of $\mathcal{P}$, see the next section 2 for details. We then define the weight function associated to $\mathcal{P}$

$$|\xi|_{\mathcal{P}} := \Bigg(\sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^v| \Bigg)^{\frac{1}{\mu}}, \quad \forall \xi \in \mathbb{R}^n. \tag{5}$$

We may introduce the multi-anisotropic class with compact support $G_0^{s,\mathcal{P}}(\mathbb{R}^n)$ of all the functions $f \in C_0^\infty(\mathbb{R}^n)$ satisfying for suitable $C < \infty$, $\varepsilon > 0$

$$|\hat{f}(\xi)| \leqslant C \exp(-\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}}), \quad \forall \xi \in \mathbb{R}^n. \tag{6}$$

We recapture (2) and the standard Gevrey classes $G_0^s(\mathbb{R}^n)$ when $\mathcal{P}$ is the polyhedron of vertices $\{0, me_j,\ j = 1, \ldots, n\}$ for some integer $m \geqslant 1$. Another relevant example is given by the anisotropic Gevrey classes, when $\mathcal{P}$ is the polyhedron of vertices $\{0, m_j e_j,\ j = 1, \ldots, n\}$ for some integers $m_j \geqslant 1$, see [13, 14]. The estimate (6) can be converted into estimates for $\partial_x^\alpha f(x)$, $\alpha \in \mathbb{N}^n$, and in this way we may also define $G^{s,\mathcal{P}}(\Omega)$ for an open subset $\Omega \subset \mathbb{R}^n$, see Calvo [4].

In this paper we want to present a Gelfand–Shilov version of the multi-anisotropic Gevrey classes. Namely, taking (4) as a model and fixing two complete polyhedra $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}_+^n$, we define $S_{r,\mathcal{Q}}^{s,\mathcal{P}}(\mathbb{R}^n)$, $s \geqslant 1$, $r \geqslant 1$, as the subset of $\mathcal{S}(\mathbb{R}^n)$ of all the functions f satisfying

$$\begin{aligned} &(i) \quad |f(x)| \leqslant C \exp(-\varepsilon |x|_{\mathcal{Q}}^{\frac{1}{r}}), \quad \forall x \in \mathbb{R}^n, \\ &(ii) \quad |\hat{f}(\xi)| \leqslant C \exp(-\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}}), \quad \forall \xi \in \mathbb{R}^n \end{aligned} \tag{7}$$

for some positive constants $C < \infty$, $\varepsilon > 0$. Main result in the following will be to show the equivalence of (7) with suitable estimates of type (3), for $x^\alpha \partial_x^\beta f(x)$; let us address to the next Theorem 2.3 for a precise statement. We leave to future papers possible applications to partial differential equations in $\mathbb{R}^n$ with polynomial coefficients, cf. Boggiatto-Buzano-Rodino [1], and a generalization of the definition (7) to the case when $s < 1$ or $r < 1$, which presents difficult problems of nontriviality for the class $S_{r,\mathcal{Q}}^{s,\mathcal{P}}(\mathbb{R}^n)$.

2. Definition and main properties

To introduce our study of Gelfand–Shilov classes of multi-anisotropic type, we start by describing complete polyhedra and some related properties. For more properties and applications to the theory of partial differential equations of complete polyhedra, we can refer to [2, 3, 4, 5, 10, 13, 14].

Let $\mathcal{P}$ be a convex polyhedron in $\mathbb{R}^n$, then $\mathcal{P}$ can be obtained as convex hull of a finite set $\mathcal{V}(\mathcal{P}) \subset \mathbb{R}^n$ of convex-linearly-independent points, called the vertices of $\mathcal{P}$ and uniquely determined by $\mathcal{P}$. Moreover, if $\mathcal{P}$ has non-empty interior and the origin belongs to $\mathcal{P}$, there is a finite set $\mathcal{N}(\mathcal{P}) = \mathcal{N}_0(\mathcal{P}) \cup \mathcal{N}_1(\mathcal{P})$, with $|\nu| = 1$, $\forall \nu \in \mathcal{N}_0(\mathcal{P})$, such that

$$\mathcal{P} = \{z \in \mathbb{R}^n | \nu \cdot z \geqslant 0,\ \ \forall \nu \in \mathcal{N}_0(\mathcal{P}),\ \ \nu \cdot z \leqslant 1,\ \ \forall \nu \in \mathcal{N}_1(\mathcal{P})\},$$

$\mathcal{N}_1(\mathcal{P})$ is the set of the normal vectors to the faces of $\mathcal{P}$.

Definition 2.1. *A complete polyhedron is a convex polyhedron $\mathcal{P} \subset \mathbb{R}^n_+$ such that the following properties are satisfied*

1) $\mathcal{V}(\mathcal{P}) \subset \mathbb{Q}^n$ (i.e. all vertices have rational coordinates);
2) the origin $(0, 0, \ldots, 0)$ belongs to $\mathcal{P}$;
3) $\mathcal{N}_0(\mathcal{P}) = \{e_1, e_2, \ldots, e_n\}$, with $e_j = (0, \ldots, 0, 1_{j-th}, 0, \ldots, 0) \in \mathbb{R}^n$ for $j = 1, \ldots, n$;
4) every $\nu \in \mathcal{N}_1(\mathcal{P})$ has strictly positive components.

Remark 1. The condition 1, suggested by the applications to the partial differential equations, is actually superfluous in the following. The condition 4 implies that for every $x \in \mathcal{P}$ the set $Q(x) = \{y \in \mathbb{R}^n | 0 \leqslant y \leqslant x\}$ is included in $\mathcal{P}$ and if x belongs to a face of $\mathcal{P}$ and $y > x$, then $y \notin \mathcal{P}$ (where for $x, y \in \mathbb{R}^n$, $y \leqslant x$ means that $y_i \leqslant x_i$, $i = 1, \ldots, n$; and $y < x$ means $y \leqslant x$, $y \neq x$).

Let us now summarize some notations related to a complete polyhedron $\mathcal{P}$:

$$k(s, \mathcal{P}) = \inf\{t > 0 : t^{-1}s \in \mathcal{P}\} = \max_{\nu \in \mathcal{N}_1(\mathcal{P})} \nu \cdot s, \quad \forall s \in \mathbb{R}^n_+;$$

$$\mu_j(\mathcal{P}) = \max_{\nu \in \mathcal{N}_1(\mathcal{P})} \nu_j^{-1};$$

$$\mu = \mu(\mathcal{P}) = \max_{j=1,\ldots,n} \mu_j \quad \text{the formal order of } \mathcal{P};$$

$$\mu^{(0)} = \mu^{(0)}(\mathcal{P}) = \min_{\gamma \in \mathcal{V}(\mathcal{P}) \setminus \{0\}} |\gamma| \quad \text{the minimum order of } \mathcal{P};$$

$$\mu^{(1)} = \mu^{(1)}(\mathcal{P}) = \max_{\gamma \in \mathcal{V}(\mathcal{P})} |\gamma| \quad \text{the maximum order of } \mathcal{P}.$$

Finally, we define the weight function associated to $\mathcal{P}$:

$$|\xi|_{\mathcal{P}} := \left(\sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^v| \right)^{\frac{1}{\mu}}, \quad \forall \xi \in \mathbb{R}^n. \tag{8}$$

It is a weight function according to the definition of Liess-Rodino [11].

The definition of the previous quantities is clarified by the following result (for the proof we refer to [4]).

Proposition 2.1. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$ with vertices $s^l = (s_1^l, \ldots, s_n^l)$, for $l = 1, \ldots, N(\mathcal{P})$. Then*

1) *for every $j = 1, 2, \ldots, n$, there is a vertex s^{l_j} of $\mathcal{P}$ such that $s^{l_j} = s_j^{l_j} e_j$, $s_j^{l_j} = \max_{s \in \mathcal{P}} s_j =: m_j(\mathcal{P})$;*
2) *the boundary of $\mathcal{P}$ has at least one vertex lying outside the coordinate axes if and only if $\mu_j > m_j$, $\forall j = 1, \ldots n$, that is equivalent to ask that the formal order $\mu(\mathcal{P})$ is greater than the maximum order $\mu^{(1)}(\mathcal{P})$;*
3) *if s belongs to $\mathcal{P}$, then $|\xi^s| \leqslant \sum_{l=1}^{N(\mathcal{P})} |\xi^{s^l}|$, $\forall \xi \in \mathbb{R}^n$, where $\xi^s = \prod_{j=1}^n \xi_j^{s_j}$ and $N(\mathcal{P})$ is the number of vertices of $\mathcal{P}$, including the origin;*
4) *$\frac{\alpha}{k(\alpha,\mathcal{P})}$, for any $\alpha \in \mathbb{N}^n$, belongs to the boundary of $\mathcal{P}$, and therefore $\alpha = k(\alpha, \mathcal{P}) \sum_{i=1}^m \lambda^i s^{l^i}$, $\lambda^i \geqslant 0$, $i = 1, \ldots, m$, $\sum_{i=1}^m \lambda^i = 1$, where $s^{l^1}, \ldots, s^{l^m}$ are the vertices of the face of $\mathcal{P}$ where $\frac{\alpha}{k(\alpha,\mathcal{P})}$ lies;*
5) *For all $x \in \mathbb{R}^n$, saying $N(\mathcal{P})$ the number of vertices of $\mathcal{P}$, the following inequality is satisfied $N(\mathcal{P})^{j-1} \sum_{v \in \mathcal{V}(\mathcal{P})} |x^{vj}| \leqslant |x|_{\mathcal{P}}^j \leqslant 2^{N(\mathcal{P})(j-1)} \sum_{v \in \mathcal{V}(\mathcal{P})} |x^{vj}|$. for any $j = 1, 2, \ldots$.*

Proposition 2.2. *For any complete polyhedron $\mathcal{P}$ and any $s \in \mathbb{R}_+^n$, $k(s, \mathcal{P})$ is bounded as follows:*

$$\frac{|s|}{\mu^{(1)}} \leqslant k(s, \mathcal{P}) \leqslant \frac{|s|}{\mu^{(0)}}.$$

The associated weight function $|\xi|_{\mathcal{P}}$ satisfies for some constants $0 < C_1, C_2 < \infty$ and all $\xi \in \mathbb{R}^n$:

$$C_1 \langle \xi \rangle^{\frac{\mu^{(0)}}{\mu}} \leqslant |\xi|_{\mathcal{P}} \leqslant C_2 \langle \xi \rangle^{\frac{\mu^{(1)}}{\mu}}.$$

Considering a polynomial with complex coefficients, we can regard it as the symbol of a differential operator and associate a polyhedron to it. Namely, if $P(D) = \sum_{|\alpha| \leqslant m} c_\alpha D^\alpha$, $c_\alpha \in \mathbb{C}$, is a differential operator in $\mathbb{R}^n$ with complex coefficients and $P(\xi) = \sum_{|\alpha| \leqslant m} c_\alpha \xi^\alpha, \xi \in \mathbb{R}^n$, its symbol, we define the Newton polyhedron or characteristic polyhedron associated to $P(D)$ (or $P(\xi)$) as the convex hull of the set $\{0\} \bigcup \{\alpha \in \mathbb{Z}_+^n : c_\alpha \neq 0\}$.

The Newton polyhedron of an hypoelliptic operator is complete (cf. Friberg [8]), but the converse is not true in general (cf. Bouzar-Chaili [2, 3], Calvo-Hakobyan [5] and Zanghirati [13, 14]). To clarify our treatment, we give now some examples of complete polyhedra (for more details cf. [4]).

1) If $P(D)$ is an elliptic operator of order m, then its Newton polyhedron is the complete polyhedron of vertices $\{0, me_j,\ j = 1, \ldots, n\}$. The set $\mathcal{N}_1(\mathcal{P})$ is reduced to the point $\nu = m^{-1} \sum_{j=1}^m e_j$, and $m_j(\mathcal{P}) = \mu_j(\mathcal{P}) = \mu^{(0)}(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = \mu(\mathcal{P}) = m$, for all $j = 1, 2, \ldots, n$; the weight function $|\xi|_{\mathcal{P}}$ associated to $\mathcal{P}$ is equivalent to $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. It is the standard case.

2) If $P(D)$ is a quasi-elliptic operator of order m (cf. for instance [10], [12], [14]), its characteristic polyhedron $\mathcal{P}$ is complete and has vertices $\{0, m_j e_j,\ j = 1, \dots, n\}$, where $m_j = m_j(\mathcal{P})$ are fixed integers. The set $\mathcal{N}_1(\mathcal{P})$ is reduced to a point $\nu = \sum_{j=1}^{n} m_j^{-1} e_j$; then $\mu_j(\mathcal{P}) = m_j$, for all $j = 1, \dots, n$, $\mu^{(0)}(\mathcal{P}) = \min_{j=1,\dots,n} m_j$, $\mu(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = \max_{j=1,\dots,n} m_j = m$. The weight function associated to $\mathcal{P}$ is $|\xi|_{\mathcal{P}} = (1 + |\xi_1|^{m_1} + \dots + |\xi_n|^{m_n})^{\frac{1}{m}}$. It is the anisotropic case.
3) If $\mathcal{P} \subset \mathbb{R}^2$ is the polyhedron of vertices $\mathcal{V}(\mathcal{P}) = \{(0,0), (0,3), (1,2), (2,0)\}$, then $\mathcal{P}$ is complete and $\mathcal{N}_1(\mathcal{P}) = \left\{\nu_1 = \left(\frac{1}{3}, \frac{1}{3}\right),\ \nu_2 = \left(\frac{1}{2}, \frac{1}{4}\right)\right\}$. We have $m_1(\mathcal{P}) = \mu^{(0)}(\mathcal{P}) = 2$, $m_2(\mathcal{P}) = m(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = 3$, $\mu(\mathcal{P}) = 4$. We observe that in this case the formal order $\mu(\mathcal{P})$ is bigger than the maximal order $m(\mathcal{P})$, as $\mathcal{P}$ has a vertex lying outside the coordinate axes (cf. Proposition 2.1). The weight function associated to $\mathcal{P}$ is $|\xi|_{\mathcal{P}} = (1 + |\xi_1|^2 + |\xi_2|^3 + |\xi_1 \xi_2^2|)^{\frac{1}{4}}$.

Basing on the definition of complete polyhedra, we now introduce the multi-anisotropic version of the standard Gelfand–Shilov classes (cf. [9]).

Definition 2.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two complete polyhedra in $\mathbb{R}^n$. We say that a function f belongs to the Gelfand Shilov class $S_{\mathcal{Q},r}^{\mathcal{P},s}(\mathbb{R}^n)$, for $r, s \geqslant 1$ if there are constants $C < \infty$, $\varepsilon > 0$ such that f and its Fourier transform $\hat{f}$ satisfy the following two conditions*

$$\begin{aligned} &(i) \quad |f(x)| \leqslant C \exp\Big(-\varepsilon |x|_{\mathcal{Q}}^{\frac{1}{r}}\Big), \quad \forall x \in \mathbb{R}^n, \\ &(ii) \quad |\hat{f}(\xi)| \leqslant C \exp\Big(-\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}}\Big), \quad \forall \xi \in \mathbb{R}^n \end{aligned} \tag{9}$$

From now on, $r, s \geqslant 1$ and $\mathcal{P}$, $\mathcal{Q}$ will be complete polyhedra according to Definition 2.1. Moreover, to simplify the computations we may suppose that the formal order of $\mathcal{P}$ and $\mathcal{Q}$ is equal to 1; it is not restrictive in the definition of Gelfand–Shilov classes of multi-anisotropic type, since it is easy to check that similar polyhedra $\mathcal{P}$ and $\mathcal{P}'$, and $\mathcal{Q}$ and $\mathcal{Q}'$ define the same class.

By means of the inversion formula for the Fourier transform, it is possible to prove the following result.

Theorem 2.1. *A function $f \in \mathcal{S}(\mathbb{R}^n)$ belongs to $S_{\mathcal{P},s}^{\mathcal{Q},r}(\mathbb{R}^n)$ if and only if its Fourier transform $\hat{f}$ belongs to $S_{\mathcal{Q},r}^{\mathcal{P},s}(\mathbb{R}^n)$.*

In particular, the spaces $S_{\mathcal{Q},r}^{\mathcal{Q},r}(\mathbb{R}^n)$ are invariant under the action of the Fourier transform.

By using the properties of complete polyhedra and the associated weight function, we can prove an equivalent definition of Gelfand–Shilov classes of multi-anisotropic type.

Theorem 2.2. *A function $f \in \mathcal{S}(\mathbb{R}^n)$ belongs to $S_{\mathcal{Q},r}^{\mathcal{P},s}(\mathbb{R}^n)$, for $r, s \geqslant 1$ if $f \in \mathcal{S}(\mathbb{R}^n)$ and there is a constant $C < \infty$ such that f and its Fourier transform $\hat{f}$*

satisfy

$$\begin{aligned} &(i) \quad \sup_{x\in\mathbb{R}^n} |x^\alpha f(x)| \leqslant C^{|\alpha|+1}(\mu k(\alpha,\mathcal{Q}))^{r\mu k(\alpha,\mathcal{Q})}, \quad \forall \alpha \in \mathbb{N}^n, \\ &(ii) \quad \sup_{\xi\in\mathbb{R}^n} |\xi^\alpha \hat{f}(\xi)| \leqslant C^{|\alpha|+1}(\mu k(\alpha,\mathcal{P}))^{s\mu k(\alpha,\mathcal{P})}, \quad \forall \alpha \in \mathbb{N}^n. \end{aligned} \tag{10}$$

Proof. We prove that if f belongs to $\mathcal{S}(\mathbb{R}^n)$, then condition $i)$ of (9) is equivalent to $i)$ of (10). The equivalence of $ii)$ of (9) with $ii)$ of (10) can be analogously checked.

Therefore, let us suppose that $f \in \mathcal{S}(\mathbb{R}^n)$ satisfies, with some $C < \infty$ and $\varepsilon > 0$

$$|f(x)| \leqslant C \exp\Big(-\varepsilon|x|_{\mathcal{Q}}^{\frac{1}{r}}\Big), \quad \forall x \in \mathbb{R}^n, \tag{11}$$

that can be rewritten (for a new ε) in the form

$$|f(x)|^{\frac{1}{r}} \leqslant C \exp\Big(-\varepsilon|x|_{\mathcal{Q}}^{\frac{1}{r}}\Big), \quad \forall x \in \mathbb{R}^n,$$

and, taking the Taylor expansion of the exponential, we get

$$\sum_{j=0}^{\infty} \frac{\varepsilon^j |x|_{\mathcal{Q}}^{\frac{j}{r}}}{j!} |f(x)|^{\frac{1}{r}} < \infty, \quad \forall x \in \mathbb{R}^n.$$

The convergence of the series of functions implies that its terms are uniformly bounded:

$$\frac{\varepsilon^j |x|_{\mathcal{Q}}^{\frac{j}{r}}}{j!} |f(x)|^{\frac{1}{r}}, \quad j = 0, 1, \ldots.$$

It follows that for a positive constant C

$$|x|_{\mathcal{Q}}^{\frac{j}{r}} |f(x)|^{\frac{1}{r}} \leqslant C^{j+1} j!, \quad j = 0, 1, \ldots.$$

Therefore, for any $\alpha \in \mathbb{N}^n$, recalling Proposition 2.1, 5, we get for $k(\alpha, \mathcal{Q}) \leqslant j$:

$$|x^\alpha f(x)| \leqslant |x|_{\mathcal{Q}}^{k(\alpha,\mathcal{Q})} |f(x)| \leqslant (C^{j+1} j!)^r.$$

Taking $j = \min\{k \in \mathbb{N} : \ k(\alpha, \mathcal{Q}) \leqslant k\}$, and recalling that $j! \leqslant j^j$, we get that $|x^\alpha f(x)| \leqslant C^{j+1} j^{rj}$, $\forall \alpha \in \mathbb{N}^n$ for a new constant $C < \infty$, that is equivalent to

$$|x^\alpha f(x)| \leqslant C^{|\alpha|+1} k(\alpha, \mathcal{Q})^{rk(\alpha,\mathcal{Q})}, \quad \forall \alpha \in \mathbb{N}^n.$$

Conversely, let us suppose that $i)$ of (10) is satisfied. Taking $\alpha = vj$ and recalling that $|v| \leqslant 1$, $\forall v \in \mathcal{V}(\mathcal{P})$ (since the formal order $\mu \leqslant 1$), then for any $v \in \mathcal{V}(\mathcal{P})$ and any $j \in \mathbb{N}$, we get

$$\sum_{v\in\mathcal{V}(\mathcal{P})} |x^{vj}||f(x)| \leqslant C^{j+1}(j!)^r. \tag{12}$$

Thanks to 5) of Proposition 2.1, taking the $\frac{1}{r}$-th power of (12), for $\varepsilon = \frac{1}{2C^r}$ we get

$$|x|_{\mathcal{Q}}^{\frac{j}{r}}|f(x)|^{\frac{1}{r}}\varepsilon^j\frac{1}{j!} \leqslant \frac{C}{2^j}.$$

Summing for $j = 0, 1, \ldots$ we get

$$\sum_{j=0}^{\infty} |x|_{\mathcal{Q}}^{\frac{j}{r}}|f(x)|^{\frac{1}{r}}\frac{\varepsilon^j}{j!} \leqslant C.$$

Therefore $|f(x)|^{\frac{1}{r}}\exp(\varepsilon|x|_{\mathcal{Q}}^{\frac{1}{r}}) \leqslant C$ for a constant $C < \infty$, and, taking the r-th power, we obtain $|f(x)| \leqslant C\exp(-\varepsilon|x|_{\mathcal{Q}}^{\frac{1}{r}})$, that gives the desired equivalence. ■

Lemma 2.1. *Conditions* (10) *are equivalent to the following ones:*

$$\begin{aligned}
&(i) \quad \sup_{x\in\mathbb{R}^n} |x^\alpha f(x)| \leqslant C^{j+1}j^{rj}, \forall\alpha\in\mathbb{N}^n, \quad \textit{where } j = \min\{i\in\mathbb{N}: \ k(\alpha,\mathcal{Q}) \leqslant i\}\\
&(ii) \quad \sup_{\xi\in\mathbb{R}^n} |\xi^\alpha \hat{f}(\xi)| \leqslant C^{j+1}j^{sj}, \forall\alpha\in\mathbb{N}^n, \quad \textit{where } j = \min\{i\in\mathbb{N}: \ k(\alpha,\mathcal{P}) \leqslant i\}.
\end{aligned} \tag{13}$$

In analogy with Chung-Chung-Kim [7], we prove for Gelfand–Shilov classes of multi-anisotropic type the following result.

Theorem 2.3. *For any $f \in \mathcal{S}(\mathbb{R}^n)$, the following conditions are equivalent:*

i) f belongs to $S_{\mathcal{Q},r}^{\mathcal{P},s}(\mathbb{R}^n)$;

ii) There exists a constant $C < \infty$ such that:

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n} |x^\alpha f(x)| \leqslant C^{|\alpha|+1}(\mu k(\alpha,\mathcal{Q}))^{r\mu k(\alpha,\mathcal{Q})}, \quad \forall\alpha\in\mathbb{N}^n,\\
&\sup_{\xi\in\mathbb{R}^n} |\xi^\alpha \hat{f}(\xi)| \leqslant C^{|\alpha|+1}(\mu k(\alpha,\mathcal{P}))^{s\mu k(\alpha,\mathcal{P})}, \quad \forall\alpha\in\mathbb{N}^n;
\end{aligned} \tag{14}$$

iii) There exists a constant $C < \infty$ such that:

$$\begin{aligned}
&\|x^\alpha f(x)\|_{L^2} \leqslant C^{|\alpha|+1}(\mu k(\alpha,\mathcal{Q}))^{r\mu k(\alpha,\mathcal{Q})}, \quad \forall\alpha\in\mathbb{N}^n,\\
&\|\xi^\alpha \hat{f}(\xi)\|_{L^2} \leqslant C^{|\alpha|+1}(\mu k(\alpha,\mathcal{P}))^{s\mu k(\alpha,\mathcal{P})}, \quad \forall\alpha\in\mathbb{N}^n;
\end{aligned} \tag{15}$$

iv) There exists a constant $C < \infty$ such that:

$$\begin{aligned}
&\|x^\alpha f(x)\|_{L^2} \leqslant C^{|\alpha|+1}(\mu k(\alpha,\mathcal{Q}))^{r\mu k(\alpha,\mathcal{Q})}, \quad \forall\alpha\in\mathbb{N}^n,\\
&\|\partial^\alpha f(x)\|_{L^2} \leqslant C^{|\alpha|+1}(\mu k(\alpha,\mathcal{P}))^{s\mu k(\alpha,\mathcal{P})}, \quad \forall\alpha\in\mathbb{N}^n;
\end{aligned} \tag{16}$$

v) There exists a constant $C < \infty$ such that:

$$\|x^\alpha\partial^\beta f(x)\|_{L^2} \leqslant C^{|\alpha|+|\beta|+1}(\mu k(\alpha,\mathcal{Q}))^{r\mu k(\alpha,\mathcal{Q})}(\mu k(\beta,\mathcal{P}))^{s\mu k(\beta,\mathcal{P})}, \quad \textit{for all } \alpha,\beta\in\mathbb{N}^n; \tag{17}$$

vi) There exists a constant $C < \infty$ such that:

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| \leqslant C^{|\alpha|+|\beta|+1}(\mu k(\alpha,\mathcal{Q}))^{r\mu k(\alpha,\mathcal{Q})}(\mu k(\beta,\mathcal{P}))^{s\mu k(\beta,\mathcal{P})}, \quad \text{for all } \alpha,\beta \in \mathbb{N}^n. \tag{18}$$

Proof. Theorem 2.2 proves the equivalence of $i)$ and $ii)$. We now prove that $ii) \Rightarrow iii) \Rightarrow iv) \Rightarrow v) \Rightarrow vi) \Rightarrow ii)$.

Let us assume $ii)$ and prove $iii)$. Fixing an integer $M > \frac{n}{4}$, so that $\|(1+|x|^2)^{-M}\|_{L^2} < \infty$, we have

$$\|x^\alpha f(x)\|_{L^2} \leqslant C \sup_{x\in\mathbb{R}^n}(1+|x|^2)^M |x^\alpha f(x)|, \quad \forall \alpha \in \mathbb{N}^n.$$

We then write $(1+|x|^2)^M = \sum_{|\gamma|\leqslant M} c_\gamma x^{2\gamma}$, with c_γ positive integers that can be estimated in terms of M. Therefore

$$\|x^\alpha f(x)\|_{L^2} \leqslant C \sum_{|\gamma|\leqslant M} \sup_{x\in\mathbb{R}^n} |x^{\alpha+2\gamma} f(x)|, \quad \forall \alpha \in \mathbb{N}^n.$$

Applying the first estimate of $ii)$, we obtain

$$\|x^\alpha f(x)\|_{L^2} \leqslant C \sum_{|\gamma|\leqslant M} C^{|\alpha+2\gamma|+1} k(\alpha+2\gamma,\mathcal{Q})^{rk((\alpha+2\gamma),\mathcal{Q})}, \quad \forall \alpha \in \mathbb{N}^n.$$

By taking $\nu \in \mathcal{N}_1(\mathcal{Q})$ such that $k(\alpha+2\gamma,\mathcal{Q}) = (\alpha+2\gamma)\cdot\nu$, we obtain

$$k(\alpha+2\gamma,\mathcal{Q}) = \alpha\cdot\nu + 2\gamma\cdot\nu \leqslant \max_{\nu\in\mathcal{N}_1(\mathcal{Q})} \alpha\cdot\nu + \max_{\nu\in\mathcal{N}_1(\mathcal{Q})} 2\gamma\cdot\nu = k(\alpha,\mathcal{Q}) + k(2\gamma,\mathcal{Q}), \tag{19}$$

having the equality only in the case that $\alpha = k\gamma$, $k \in \mathbb{R}$. Therefore, since the γ's are fixed and in finite number:

$$\begin{aligned}\|x^\alpha f(x)\|_{L^2} &\leqslant \sum_{|\gamma|\leqslant M} C^{|2\gamma|+1} C^{|\alpha|}(k(\alpha,\mathcal{Q}) + k(2\gamma,\mathcal{Q}))^{r(k(\alpha,\mathcal{Q})+k(2\gamma,\mathcal{Q}))} \\ &\leqslant C^{|\alpha|+1}(k(\alpha,\mathcal{Q}))^{rk(\alpha,\mathcal{Q})}.\end{aligned}$$

Arguing similarly for the ξ variables, we obtain the second inequality.

By Plancharel's formula, $iii)$ is equivalent to $iv)$.

Let us now prove that $iv) \Rightarrow v)$. Integrating by parts and using Leibniz formula, we have

$$\begin{aligned}\|x^\alpha \partial^\beta f(x)\|^2_{L^2} &= (\partial^\beta f(x), x^{2\alpha}\partial^\beta f(x))_{L^2} = (f, \partial^\beta(x^{2\alpha}\partial^\beta f(x))_{L^2} \\ &\leqslant \sum_{\gamma\leqslant\beta,\ \gamma\leqslant 2\alpha} \binom{\beta}{\gamma}\binom{2\alpha}{\gamma}\gamma! \left|(x^{2\alpha-\gamma}f, \partial^{2\beta-\gamma}f)_{L^2}\right|\end{aligned}$$

Since $\binom{\beta}{\gamma}\binom{2\alpha}{\gamma} \leqslant 2^{|\beta|+2|\alpha|}$, applying the Cauchy-Schwartz inequality we obtain

$$\|x^\alpha \partial^\beta f(x)\|^2_{L^2} \leqslant 2^{|\beta|+2|\alpha|} \sum_{\gamma\leqslant\beta,\ \gamma\leqslant 2\alpha} \gamma! \|x^{2\alpha-\gamma}f\|_{L^2} \|\partial^{2\beta-\gamma}f\|_{L^2}. \tag{20}$$

Using now the assumptions $iv)$, where we can assume for simplicity that $C > 1$, we have for $\gamma \leqslant 2\beta$, $\gamma \leqslant 2\alpha$

$$\begin{aligned}
&\gamma!\|x^{2\alpha-\gamma}f\|_{L^2}\|\partial^{2\beta-\gamma}f\|_{L^2}\\
&\qquad \leqslant C^{2|\alpha|+2|\beta|+2}\gamma!(k((2\alpha-\gamma),\mathcal{Q}))^{rk((2\alpha-\gamma),\mathcal{Q})}(k((2\beta-\gamma),\mathcal{P}))^{sk((2\beta-\gamma),\mathcal{P})}\\
&\qquad \leqslant C^{2|\alpha|+2|\beta|+2}(k((2\alpha),\mathcal{Q}))^{rk((2\alpha),\mathcal{Q})}(k((2\beta),\mathcal{P}))^{sk((2\beta),\mathcal{P})}.
\end{aligned}$$

In fact, by Proposition 2.1,

$$\begin{aligned}
\gamma! &\leqslant \mu^{(1)}(\mathcal{Q})k(\alpha,\mathcal{Q}) \leqslant k(\alpha,\mathcal{Q}),\\
\gamma! &\leqslant \mu^{(1)}(\mathcal{P})k(\beta,\mathcal{P}) \leqslant k(\beta,\mathcal{P}),
\end{aligned}$$

since we are supposing that $\mu \leqslant 1$ and $\mu^{(1)} \leqslant \mu$ for both $\mathcal{P}$ and $\mathcal{Q}$. Therefore,

$$\gamma! \leqslant k(\alpha,\mathcal{Q})^{rk(\alpha,\mathcal{Q})}k(\beta,\mathcal{P})^{sk(\beta,\mathcal{P})},$$

since we are supposing $r, s \geqslant 1$. Observe that the number of the terms in the sum (20) does not exceed $2^{2|\alpha|+|\beta|+2n}$ and that $k((2\alpha),\mathcal{Q}) = 2k(\alpha,\mathcal{Q})$, $k((2\beta),\mathcal{P}) = 2k(\beta,\mathcal{P})$ an for (19) we conclude that for a new constant $C < \infty$ it is satisfied

$$\|x^\alpha\partial^\beta f(x)\|^2_{L^2} \leqslant C^{2|\alpha|+2|\beta|+2}k(\alpha,\mathcal{Q})^{2rk(\alpha,\mathcal{Q})}k(\beta,\mathcal{P})^{2sk(\beta,\mathcal{P})}.$$

To prove that $v)$ implies $vi)$ we use the Sobolev embedding theorem. Namely, fixing an integer $d > \frac{n}{2}$, we get for a $C < \infty$

$$\begin{aligned}
|x^\alpha\partial^\beta f(x)| &\leqslant C\|x^\alpha\partial^\beta f(x)\|_{H^d}\\
&= C\sum_{|\gamma|\leqslant d}\|\partial^\gamma(x^\alpha\partial^\beta f(x))\|_{L^2}, \quad \forall x\in\mathbb{R}^n,\ \forall\alpha,\beta\in\mathbb{N}^n.
\end{aligned} \tag{21}$$

By Leibniz rule, we estimate the right hand side by

$$\sum_{|\gamma|\leqslant d}\ \sum_{\delta\leqslant\gamma,\ \delta\leqslant\alpha}\binom{\gamma}{\delta}\binom{\alpha}{\delta}\delta!\|x^{\alpha-\delta}\partial^{\beta+\gamma-\delta}f(x)\|_{L^2}. \tag{22}$$

Note that $\binom{\gamma}{\delta}\binom{\alpha}{\delta}\delta! \leqslant C2^{|\alpha|}$, with C independent of α. Moreover, the number of the terms in the sum (22) can be estimated by an integer independent of α. On the other hand, using assumption $v)$ and (19), we get for a new constant $C' < \infty$

$$\begin{aligned}
&\|x^{\alpha-\delta}\partial^{\beta+\gamma-\delta}f(x)\|_{L^2}\\
&\qquad \leqslant C^{|\alpha|+|\beta|-2|\delta|+|\gamma|+1}k((\alpha-\delta),\mathcal{Q})^{rk((\alpha-\delta),\mathcal{Q})}k((\beta+\gamma-\delta),\mathcal{P})^{sk((\beta+\gamma-\delta),\mathcal{P})}\\
&\qquad \leqslant C^{|\alpha|+|\beta|+1}k(\alpha,\mathcal{Q})^{rk(\alpha,\mathcal{Q})}k((\beta+\gamma),\mathcal{P})^{sk((\beta+\gamma),\mathcal{P})}\\
&\qquad \leqslant C'^{|\alpha|+|\beta|+1}k(\alpha,\mathcal{Q})^{rk(\alpha,\mathcal{Q})}k(\beta,\mathcal{P})^{sk(\beta,\mathcal{P})}.
\end{aligned} \tag{23}$$

Combining (21), (22) and (23), we get

$$|x^\alpha\partial^\beta f(x)| \leqslant C'^{|\alpha|+|\beta|+1}k(\alpha,\mathcal{Q})^{rk(\alpha,\mathcal{Q})}k(\beta,\mathcal{P})^{sk(\beta,\mathcal{P})}, \quad \forall x\in\mathbb{R}^n,\ \forall\alpha,\beta\in\mathbb{N}^n.$$

Finally, we prove that $vi)$ implies $ii)$. We first observe that $vi)$ for $\beta = 0$ gives the first estimate of $ii)$.

By performing the Fourier transform we get

$$|\xi^\beta \hat{f}(\xi)| = |(\partial^\beta f)\hat{}(\xi)| \leqslant \|\partial^\beta f\|_{L^1}, \quad \forall \xi \in \mathbb{R}^n. \tag{24}$$

Fixing an integer $M > \frac{n}{2}$, so that $\|(1+|x|^2)^{-M}\|_{L^1} < \infty$, we obtain

$$\|\partial^\beta f(x)\|_{L^1} \leqslant C \sup_{x \in \mathbb{R}^n} (1+|x|^2)^M |\partial^\beta f(x)|. \tag{25}$$

Condition $vi)$ with $\alpha = 0$ gives (since in this case M is fixed)

$$(1+|x|^2)^M |\partial^\beta f(x)| \leqslant C^{|\beta|+1} k(\beta, \mathcal{P})^{sk(\beta,\mathcal{P})}, \quad \forall x \in \mathbb{R}^n, \ \forall \beta \in \mathbb{N}^n. \tag{26}$$

Combining (24), (25) and (26) we get $ii)$. ■

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