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HARMONIC BOUNDARY VALUE PROBLEMS IN HALF DISC AND HALF RING

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: The Schwarz problem for the Cauchy-Riemann equation and the Dirichlet and Neumann problems for the Poisson equation are explicitly solved in a half disc and a half ring of the complex plane. The respective Poisson kernels and the Green and Neumann functions are given.

Keywords: Schwarz problem, Dirichlet problem, Neumann problem, Cauchy-Riemann equation, Poisson equation, half disc, half ring.

1. Introduction

Explicit solutions to some classical boundary value problems are well known for some particular domains as e.g. the unit disc [3]–[9], [11], [12], [14]–[27], half plane [10], [28], quarter plane [1], [2], [13] and circular ring [29]–[31]. The Poisson kernel for the upper half unit disc is given in [26] as an application of Cauchy's residue theorem. It is not likely that this kernel can be useful for solving the Schwarz problem for analytic functions. Here another Poisson kernel is developed on the basis of the reflection principle leading to the solution of the Schwarz problem for the inhomogeneous Cauchy-Riemann equation for the half unit disc

$$\mathbb{D}^+ := \{z : |z| < 1,\, 0 < \operatorname{Im} z\}.$$

With the help of the harmonic Green and Neumann functions, the Dirichlet and Neumann boundary value problems are also solved explicitly for the Poisson equation in the upper half disc. Particular attention is paid to the two corner points of the boundary for the Dirichlet problem. Using tools from [29], [31] these boundary value problems are treated in similar manners in the upper half circular ring

$$R^+ := \{z : r < |z| < 1,\, 0 < \operatorname{Im} z\},$$

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which of course is a simply connected domain. Therefore no multiple valued functions and no extra solvability conditions occur.

The fundamental tool for complex boundary value problems is the Cauchy-Pompeiu representation which just has to be properly modified.

Theorem I. *Any $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ for a regular complex domain $D \subset \mathbb{C}$ can be represented as*

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{D} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

or

$$w(z) = -\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int\limits_{D} w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}.$$

For this classical result see e.g. [3], [32]. As one knows the right-hand sides vanish if $z \notin \overline{D}$, the closure of D.

The purpose of these investigations is to provide explicit solutions and, if desirable, solvability conditions for some basic boundary value problems in two special domains. Just some model equations are considered. The results will help to treat more general equations. This will be done according to the treatment of the generalized Beltrami equation, which is repeatedly the subject of Bogdan Bojarski, see e.g. [23]–[25].

2. Upper half unit disc

With regard to the Schwarz problem, the Cauchy-Pompeiu formula is modified by combining it in a proper way with the Cauchy-Pompeiu formula evaluated at the reflections of the point z at the boundary ∂D, $1/\overline{z}$ and $\overline{z}$, and for symmetry also at the reflection $1/z$ of $1/\overline{z}$ at the real axis, or, what is the same, the reflection $1/z$ of $\overline{z}$ at the unit circle.

Theorem 1. *Any $w \in C^1(\mathbb{D}^+;\mathbb{C}) \cap C(\overline{\mathbb{D}^+};\mathbb{C})$ can be represented via*

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}^+} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{\mathbb{D}^+} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in \mathbb{D}^+, \tag{2.1}$$

and for $z \in \mathbb{D}^+$

$$\begin{aligned} w(z) = &\frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \operatorname{Re} w(\zeta)\left[\frac{\zeta + z}{\zeta - z} - \frac{\overline{\zeta} + z}{\overline{\zeta} - z}\right]\frac{d\zeta}{\zeta} + \frac{1}{\pi}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta} \\ &+ \frac{1}{\pi i}\int\limits_{-1}^{1} \operatorname{Re} w(t)\left[\frac{1}{t - z} - \frac{z}{1 - zt}\right]dt \\ &- \frac{1}{\pi i}\int\limits_{\mathbb{D}^+}\left\{w_{\overline{\zeta}}(\zeta)\left[\frac{1}{\zeta - z} - \frac{z}{1 - z\zeta}\right] - \overline{w_{\overline{\zeta}}(\zeta)}\left[\frac{1}{\overline{\zeta} - z} - \frac{z}{1 - z\overline{\zeta}}\right]\right\}d\xi d\eta, \end{aligned} \tag{2.2}$$

Proof. Besides formula (2.1) the Cauchy-Pompeiu formula for $z \in \mathbb{D}^+$ also ensures

$$0 = \frac{1}{2\pi i} \int_{\partial\mathbb{D}^+} w(\zeta) \frac{\overline{z} d\zeta}{1 - \overline{z}\zeta} - \frac{1}{\pi} \int_{\mathbb{D}^+} w_{\overline{\zeta}}(\zeta) \frac{\overline{z} d\xi d\eta}{1 - \overline{z}\zeta}, \tag{2.1$'$}$$

$$0 = \frac{1}{2\pi i} \int_{\partial\mathbb{D}^+} w(\zeta) \frac{d\zeta}{\zeta - \overline{z}} - \frac{1}{\pi} \int_{\mathbb{D}^+} w_{\overline{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - \overline{z}}, \tag{2.1$''$}$$

$$0 = \frac{1}{2\pi i} \int_{\partial\mathbb{D}^+} w(\zeta) \frac{z d\zeta}{1 - z\zeta} - \frac{1}{\pi} \int_{\mathbb{D}^+} w_{\overline{\zeta}}(\zeta) \frac{z d\xi d\eta}{1 - z\zeta}. \tag{2.1$'''$}$$

Adding (2.1) and the complex conjugate of (2.1$'$), i.e.

$$\begin{aligned} w(z) = \frac{1}{2\pi i} \int_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} & \left\{ \operatorname{Re} w(\zeta) \frac{\zeta + z}{\zeta - z} + i \operatorname{Im} w(\zeta) \right\} \frac{d\zeta}{\zeta} \\ & + \frac{1}{2\pi i} \int_{-1}^{1} \left\{ \operatorname{Re} w(t) \left[\frac{1}{t - z} - \frac{z}{1 - zt} \right] + i \operatorname{Im} w(t) \left[\frac{1}{t - z} + \frac{z}{1 - zt} \right] \right\} dt \\ & - \frac{1}{\pi} \int_{\mathbb{D}^+} \left\{ \frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} + \frac{z\overline{w_{\overline{\zeta}}(\zeta)}}{1 - z\overline{\zeta}} \right\} d\xi d\eta \end{aligned}$$

and then substracting the sum of of the complex conjugate of (2.1$''$) and (2.1$'''$), namely

$$\begin{aligned} 0 = \frac{1}{2\pi i} \int_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} & \left\{ \operatorname{Re} w(\zeta) \frac{\overline{\zeta} + z}{\overline{\zeta} - z} - i \operatorname{Im} w(\zeta) \right\} \frac{d\zeta}{\zeta} \\ & - \frac{1}{2\pi i} \int_{-1}^{1} \left\{ \operatorname{Re} w(t) \left[\frac{1}{t - z} - \frac{z}{1 - zt} \right] - i \operatorname{Im} w(t) \left[\frac{1}{t - z} + \frac{z}{1 - zt} \right] \right\} dt \\ & - \frac{1}{\pi} \int_{\mathbb{D}^+} \left\{ \frac{\overline{w_{\overline{\zeta}}(\zeta)}}{\overline{\zeta} - z} + \frac{z w_{\overline{\zeta}}(\zeta)}{1 - z\zeta} \right\} d\xi d\eta \end{aligned}$$

gives (2.2). ■

This formula provides the solution to the Schwarz problem for $\mathbb{D}^+$.

Theorem 2. *The Schwarz problem*

$$w_{\overline{z}} = f \;\; in \;\; \mathbb{D}^+, \quad f \in L_p(\mathbb{D}^+; \mathbb{C}), \;\; p > 2,$$

$$\operatorname{Re} w = \gamma \;\; in \;\; \partial\mathbb{D}^+, \quad \gamma \in C(\partial\mathbb{D}^+; \mathbb{C}), \quad \gamma(-1) = \gamma(1) = 0,$$

$$\frac{1}{\pi}\int_0^\pi \operatorname{Im} w(e^{i\varphi})d\varphi = c, \quad c \in \mathbb{R},$$

is uniquely solvable by

$$w(z) = \frac{1}{2\pi i}\int_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta)\left[\frac{\zeta+z}{\zeta-z} - \frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right]\frac{d\zeta}{\zeta} + ic + \frac{1}{\pi i}\int_{-1}^{1}\gamma(t)\left[\frac{1}{t-z} - \frac{z}{1-zt}\right]dt$$
$$-\frac{1}{\pi}\int_{\mathbb{D}^+}\left\{f(\zeta)\left[\frac{1}{\zeta-z} - \frac{z}{1-z\zeta}\right] - \overline{f(\zeta)}\left[\frac{1}{\overline{\zeta}-z} - \frac{z}{1-z\overline{\zeta}}\right]\right\}d\xi d\eta. \qquad (2.3)$$

Proof. The uniqueness part is an immediate consequence of the Cauchy-Pompeiu-Schwarz representation formula. Namely, if there is a solution the representation formula (2.2) shows that it necessarily has the form (2.3). That the related homogeneous problem is only trivially solvable is seen likewise from (2.2). It remains to verify that (2.3) in fact provides a solution. As the right-hand side of (2.3) up to the term

$$Tf(z) = -\frac{1}{\pi}\int_{\mathbb{D}^+} f(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

is an analytic function in $\mathbb{D}^+$ and as it is a weak solution to the Cauchy-Riemann equation $w_{\overline{z}} = f$ in $\mathbb{D}^+$ it remains to check the boundary behavior.

Let w_0 denote the area integral in (2.3). Then for $|z| = 1, 0 \leqslant \operatorname{Im} z$

$$w_0(z) = -\frac{1}{\pi}\int_{\mathbb{D}^+}\left\{f(\zeta)\left[\frac{1}{\zeta-z} - \frac{1}{\overline{z}-\zeta}\right] - \overline{f(\zeta)}\left[\frac{1}{\overline{\zeta}-z} - \frac{1}{\overline{z-\overline{\zeta}}}\right]\right\}d\xi d\eta$$

so that $\operatorname{Re} w_0(z) = 0$ there. Similarly, for $\operatorname{Im} z = 0$, i.e. for $z = \overline{z}$,

$$w_0(z) = -\frac{1}{\pi}\int_{\mathbb{D}^+}\left\{f(\zeta)\left[\frac{1}{\zeta-z} - \frac{z}{1-z\zeta}\right] - \overline{f(\zeta)}\left[\frac{1}{\overline{\overline{\zeta}-z}} - \frac{\overline{z}}{1-\overline{z\overline{\zeta}}}\right]\right\}d\xi d\eta,$$

thus also $\operatorname{Re} w_0(z) = 0$ there is seen. That

$$\int_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} w_0(z)\frac{dz}{z} = 0$$

holds follows from

$$\int_{\substack{|z|=1,\\ 0<\operatorname{Im} z}}\left[\frac{1}{\zeta-z} - \frac{z}{1-z\zeta}\right]\frac{dz}{z} = -\int_{\substack{|z|=1,\\ 0<\operatorname{Im} z}}\frac{1}{z-\zeta}\frac{dz}{z} + \int_{\substack{|z|=1,\\ 0<\operatorname{Im} z}}\frac{1}{\overline{z}-\zeta}\frac{d\overline{z}}{\overline{z}}$$
$$= -\int_{\substack{|z|=1,\\ 0<\operatorname{Im} z}}\frac{1}{z-\zeta}\frac{dz}{z} - \int_{\substack{|z|=1,\\ \operatorname{Im} z<0}}\frac{1}{z-\zeta}\frac{dz}{z} = -\int_{|z|=1}\frac{1}{z-\zeta}\frac{dz}{z} = 0 \quad \text{for } |\zeta| < 1.$$

In the same way for $|\zeta| = 1$, $0 < \operatorname{Im}\zeta$

$$\begin{aligned}\int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \left[\frac{\zeta+z}{\zeta-z} - \frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right]\frac{dz}{z} &= \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \left[\frac{\zeta+z}{\zeta-z} - \frac{\overline{z}+\zeta}{\overline{z}-\zeta}\right]\frac{dz}{z} \\ &= \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \frac{\zeta+z}{\zeta-z}\frac{dz}{z} - \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \frac{\zeta+\overline{z}}{\zeta-\overline{z}}\frac{d\overline{z}}{\overline{z}} \\ &= \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \frac{\zeta+z}{\zeta-z}\frac{dz}{z} + \int\limits_{\substack{|z|=1,\\ \operatorname{Im} z<0}} \frac{\zeta+z}{\zeta-z}\frac{dz}{z} \\ &= \int\limits_{|z|=1} \frac{\zeta+z}{\zeta-z}\frac{dz}{z} = -2\pi i\end{aligned}$$

is seen. Hence,

$$\frac{1}{\pi i}\int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} w(z)\frac{dz}{z} = ic - \frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta)\frac{d\zeta}{\zeta},$$

$$\frac{1}{\pi i}\int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \operatorname{Im} w(z)\frac{dz}{z} = c.$$

From

$$\begin{aligned}\operatorname{Re} w(z) = \frac{1}{2\pi i}&\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta)\left[\frac{\zeta}{\zeta-z} + \frac{\overline{\zeta}}{\overline{\overline{\zeta}-z}} - \frac{\overline{\zeta}}{\overline{\zeta}-z} - \frac{\zeta}{\zeta-\overline{z}}\right]\frac{d\zeta}{\zeta} \\ &+ \frac{1}{2\pi i}\int_{-1}^{1}\gamma(\zeta)\left[\frac{1}{t-z} - \frac{1}{t-\overline{z}} - \frac{z}{1-zt} + \frac{\overline{z}}{1-\overline{z}t}\right]dt + \operatorname{Re} w_0(z) \\ = \frac{1}{2\pi i}&\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta)\left[\frac{1-|z|^2}{|\zeta-z|^2} - \frac{1-|z|^2}{|\overline{\zeta}-z|^2}\right]\frac{d\zeta}{\zeta} \\ &+ \frac{1}{2\pi i}\int_{-1}^{1}\gamma(\zeta)\left[\frac{z-\overline{z}}{|t-z|^2} - \frac{z-\overline{z}}{|1-zt|^2}\right]dt + \operatorname{Re} w_0(z)\end{aligned}$$

it is seen that for $|\zeta_0| = 1$, $0 < \operatorname{Im}\zeta_0$

$$\lim_{z\to\zeta_0} \operatorname{Re} w(z) = \gamma(\zeta_0)$$

as $|\overline{\zeta} - \zeta_0|^2 \neq 0$, $1 - |\zeta_0|^2 = 0$, $|t - \zeta_0|^2 = |1 - \zeta_0 t|^2$, and for $\operatorname{Im}\zeta_0 = 0$, $|\zeta_0| < 1$

$$\lim_{z\to\zeta_0} \operatorname{Re} w(z) = \gamma(\zeta_0),$$

because $|\zeta - \zeta_0| = |\overline{\zeta} - \zeta_0|$, $|1 - \zeta_0 t|^2 \neq 0$, $\zeta_0 - \overline{\zeta_0} = 0$.

Here the corner points were excluded. To check the boundary behavior there, consider

$$\omega(z) = \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \left[\frac{\zeta + z}{\zeta - z} - \frac{\overline{\zeta} + z}{\overline{\zeta} - z} \right] \frac{d\zeta}{\zeta}$$

and

$$\widehat{\omega}(z) = \frac{1}{\pi i} \int\limits_{-1}^{1} \gamma(t) \left[\frac{1}{t - z} - \frac{z}{1 - zt} \right] dt.$$

From

$$\begin{aligned}
\omega(z) &= \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \frac{\overline{\zeta} + z}{\overline{\zeta} - z} \frac{d\overline{\zeta}}{\overline{\zeta}} \\
&= \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} - \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ \operatorname{Im}\zeta<0}} \gamma(\overline{\zeta}) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}
\end{aligned}$$

with

$$\Gamma(\zeta) = \begin{cases} \gamma(\zeta), & 0 \leqslant \operatorname{Im}\zeta, \\ -\gamma(\overline{\zeta}), & \operatorname{Im}\zeta < 0, \end{cases} \quad |\zeta| = 1$$

for $|\zeta_0| = 1$

$$\lim_{z\to\zeta_0,\, |z|=1} \operatorname{Re} \omega(z) = \Gamma(\zeta_0),$$

in particular

$$\lim_{z\to\pm 1,\, |z|<1} \operatorname{Re} \omega(z) = \gamma(\pm 1) = 0$$

is seen because of the continuity of Γ at ± 1.

Similarly with

$$\begin{aligned}
\int\limits_{-1}^{0} \gamma(t) \frac{z}{1 - zt} dt = -\int\limits_{-1}^{-\infty} \gamma\left(\frac{1}{t}\right) \frac{zt}{t - z} \frac{dt}{t^2} = \int\limits_{-\infty}^{-1} \gamma\left(\frac{1}{t}\right) \frac{z}{t - z} \frac{dt}{t} \\
= \int\limits_{-\infty}^{-1} \gamma\left(\frac{1}{t}\right) \left[\frac{1}{t - z} - \frac{1}{t} \right] dt,
\end{aligned}$$

$$\int_0^1 \gamma(t)\frac{z}{1-zt}dt = -\int_{+\infty}^{1} \gamma\left(\frac{1}{t}\right)\frac{zt}{t-z}\frac{dt}{t^2} = \int_1^{+\infty} \gamma\left(\frac{1}{t}\right)\frac{z}{t-z}\frac{dt}{t}$$

$$= \int_1^{+\infty} \gamma\left(\frac{1}{t}\right)\left[\frac{1}{t-z} - \frac{1}{t}\right]dt,$$

then

$$\widehat{\omega}(z) = -\frac{1}{\pi i}\int_{-\infty}^{-1} \gamma\left(\frac{1}{t}\right)\left[\frac{1}{t-z} - \frac{1}{t}\right]dt + \frac{1}{\pi i}\int_{-1}^{1}\gamma(t)\frac{dt}{t-z}$$

$$- \frac{1}{\pi i}\int_1^{+\infty} \gamma\left(\frac{1}{t}\right)\left[\frac{1}{t-z} - \frac{1}{t}\right]dt$$

follows so that

$$\mathrm{Re}\,\widehat{\omega}(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \widehat{\Gamma}(t)\frac{z-\overline{z}}{|t-z|^2}dt$$

with

$$\widehat{\Gamma}(\zeta) = \begin{cases} \gamma(t), & |t| \leqslant 1, \\ -\gamma\left(\frac{1}{t}\right), & 1 < |t|, \end{cases} \quad t \in \mathbb{R},$$

is seen. This implies for $t_0 \in \mathbb{R}$

$$\lim_{z \to t_0,\, 0<\mathrm{Im}\, z} \mathrm{Re}\,\widehat{\omega}(z) = \widehat{\Gamma}(t_0),$$

in particular for $t_0 = \pm 1$ because $\widehat{\Gamma}$ is continuous at these points as $\gamma(\pm 1) = 0$. ■

Iterating the Cauchy-Pompeiu formulas leads to second order representations, see e.g. [5]. Introducing the harmonic Green function in the respective formulas with regard to the Laplace operator gives a representation adjusted to the Dirichlet problem for the Poisson equation.

Theorem II. *Let $D \subset \mathbb{C}$ be a regular domain and $G_1 = 2G$ where G is the harmonic Green function for D, then any $w \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$ can be represented by*

$$w(z) = -\frac{1}{4\pi}\int_{\partial D} w(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta - \frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)G_1(z,\zeta)d\xi d\eta.$$

Here ν is the outward normal derivative on ∂D and s is the arc length parameter.

The harmonic Green function for the upper half unit disc $\mathbb{D}^+$ is

$$G_1(z,\zeta) = \log\left|\frac{(1-z\overline{\zeta})(\overline{\zeta}-z)}{(1-z\zeta)(\zeta-z)}\right|^2.$$

This can be directly verified. As one sees, the three reflection points, used for the Schwarz modification of the Cauchy-Pompeiu formula, appear here. In fact, the Poisson kernel, i.e. the real part of the Schwarz kernel coincides with the outward normal derivative of the Green function G_1. It is for $|z| = 1$, $0 < \operatorname{Im} z$

$$\partial_{\nu_z} G_1(z,\zeta) = (z\partial_z + \overline{z}\partial_{\overline{z}})G_1(z,\zeta) = 2\operatorname{Re}\left[\frac{\zeta+z}{\zeta-z} - \frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right] = 4\operatorname{Re}\left[\frac{\zeta}{\zeta-z} - \frac{\overline{\zeta}}{\overline{\zeta}-z}\right]$$

and for $z = \overline{z}$, $|z| < 1$

$$i\partial_{\nu_z} G_1(z,\zeta) = (\partial_z - \partial_{\overline{z}})G_1(z,\zeta) = 4i \operatorname{Im}\left[\frac{\zeta}{1-z\zeta} + \frac{1}{\zeta-z}\right] = 2\left[\frac{\zeta-\overline{\zeta}}{|1-z\zeta|^2} - \frac{\zeta-\overline{\zeta}}{|\zeta-z|^2}\right],$$

i.e.

$$\partial_{\nu_z} G_1(z,\zeta) = 4\operatorname{Im}\zeta\left[\frac{1}{|1-z\zeta|^2} - \frac{1}{|\zeta-z|^2}\right].$$

In explicit form the Green representation formula from Theorem 2 for $\mathbb{D}^+$ is

$$\begin{aligned} w(z) &= -\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} w(\zeta)\operatorname{Re}\left[\frac{z}{z-\zeta} - \frac{\overline{z}}{\overline{z}-\zeta}\right]\frac{d\zeta}{\zeta} \\ &\quad + \frac{1}{2\pi i}\int_{-1}^{1} w(t)\left[\frac{z-\overline{z}}{|t-z|^2} - \frac{z-\overline{z}}{|1-tz|^2}\right]dt - \frac{1}{\pi}\int\limits_{\mathbb{D}^+} w_{\zeta\overline{\zeta}}(\zeta)G_1(z,\zeta)d\xi d\eta \\ &= \frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} w(\zeta)\left[\frac{\zeta}{\zeta-z} + \frac{\overline{\zeta}}{\overline{\zeta-z}} - \frac{\zeta}{\zeta-\overline{z}} - \frac{\overline{\zeta}}{\overline{\zeta}-z}\right]\frac{d\zeta}{\zeta} \\ &\quad + \frac{1}{2\pi i}\int_{-1}^{1} w(t)\left[\frac{z-\overline{z}}{|t-z|^2} - \frac{z-\overline{z}}{|1-tz|^2}\right]dt - \frac{1}{\pi}\int\limits_{\mathbb{D}^+} w_{\zeta\overline{\zeta}}(\zeta)G_1(z,\zeta)d\xi d\eta. \end{aligned} \tag{2.4}$$

Theorem 3. *Any* $w \in C^2(\mathbb{D}^+;\mathbb{C}) \cap C^1(\overline{\mathbb{D}^+};\mathbb{C})$ *can be represented by formula* (2.4).

In fact (2.4) provides the solution to the Dirichlet problem. This will be verified for this particular domain $\mathbb{D}^+$ with a piecewise smooth boundary.

Theorem 4. *The Dirichlet problem*

$$w_{z\overline{z}} = f \quad in\ \mathbb{D}^+, \quad w = \gamma \quad on\ \partial\mathbb{D}^+,$$

for $f \in C(\mathbb{D}^+; \mathbb{C})$, $\gamma \in C(\partial\mathbb{D}^+; \mathbb{C})$, $\gamma(-1)) = \gamma(1) = 0$ *is uniquely solvable by*

$$w(z) = \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - \frac{\zeta}{\zeta - \overline{z}} - \frac{\overline{\zeta}}{\overline{\zeta} - z} \right] \frac{d\zeta}{\zeta}$$

$$+ \frac{1}{2\pi i} \int_{-1}^{1} \gamma(t) \left[\frac{z - \overline{z}}{|t - z|^2} - \frac{z - \overline{z}}{|1 - tz|^2} \right] dt - \frac{1}{\pi} \int\limits_{\mathbb{D}^+} f(\zeta) G_1(z, \zeta) d\xi d\eta.$$

Proof. By the properties of the Green function and the harmonicity of the boundary integrals w is seen to be a solution to the Poisson equation. It remains to verify the boundary conditions for the boundary integrals.

For $|\zeta_0| = 1$, $0 < \operatorname{Im}\zeta_0$

$$\begin{aligned} &\lim_{z \to \zeta_0, |z|<1} \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - \frac{\zeta}{\zeta - \overline{z}} - \frac{\overline{\zeta}}{\overline{\zeta} - z} \right] \frac{d\zeta}{\zeta} \\ &= \lim_{z \to \zeta_0, |z|<1} \left[\frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \right] \frac{d\zeta}{\zeta} \right. \\ &\qquad \left. - \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ \operatorname{Im}\zeta<0}} \gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \right] \frac{d\zeta}{\zeta} \right] \\ &= \lim_{z \to \zeta_0, |z|<1} \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \right] \frac{d\zeta}{\zeta} = \gamma(\zeta_0) \end{aligned} \tag{2.5}$$

with

$$\Gamma(\zeta) = \begin{cases} \gamma(\zeta), & 0 \leqslant \operatorname{Im}\zeta, \\ -\gamma(\overline{\zeta}), & \operatorname{Im}\zeta < 0, \end{cases} \quad |\zeta| = 1$$

and

$$\lim_{z \to \zeta_0, |z|<1} \frac{1}{2\pi i} \int_{-1}^{1} \gamma(t) \left[\frac{z - \overline{z}}{|t - z|^2} - \frac{z - \overline{z}}{|1 - tz|^2} \right] dt = 0.$$

For $t_0 = \overline{t_0}$, $|t_0| < 1$

$$\lim_{z \to t_0, 0<\operatorname{Im} z} \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - \frac{\zeta}{\zeta - \overline{z}} - \frac{\overline{\zeta}}{\overline{\zeta} - z} \right] \frac{d\zeta}{\zeta} = 0$$

and

$$\begin{aligned}\lim_{z\to t_0,\,0<\operatorname{Im} z}\frac{1}{2\pi i}\int_{-1}^{1}\gamma(t)\frac{z-\overline{z}}{|t-z|^2}dt&=\gamma(t_0)\\ \lim_{z\to t_0,\,0<\operatorname{Im} z}\frac{1}{2\pi i}\int_{-1}^{1}\gamma(t)\frac{z-\overline{z}}{|1-tz|^2}dt&=0.\end{aligned}\tag{2.6}$$

For the corner points ±1 formula (2.5) can be used. As $\gamma(\pm1)=0$, the function Γ is continuous in particular at the points ±1 and the limit in (2.5) is just $\Gamma(\pm1)=\gamma(\pm1)=0$ for $\zeta_0=\pm1$.

The second term in the boundary integral is transformed according to

$$\begin{aligned}\int_{-1}^{0}\gamma(t)\frac{dt}{|1-tz|^2}&=-\int_{-1}^{-\infty}\gamma\left(\frac{1}{t}\right)\frac{dt}{t^2|1-z/t|^2}=\int_{-\infty}^{-1}\gamma\left(\frac{1}{t}\right)\frac{dt}{|t-z|^2},\\ \int_{0}^{1}\gamma(t)\frac{dt}{|1-tz|^2}&=-\int_{+\infty}^{1}\gamma\left(\frac{1}{t}\right)\frac{dt}{t^2|1-z/t|^2}=\int_{1}^{+\infty}\gamma\left(\frac{1}{t}\right)\frac{dt}{|t-z|^2}.\end{aligned}$$

Then for $t_0=\pm1$

$$\begin{aligned}&\lim_{z\to\pm1,\,0<\operatorname{Im} z}\frac{1}{2\pi i}\int_{-1}^{1}\gamma(t)\left[\frac{z-\overline{z}}{|t-z|^2}-\frac{z-\overline{z}}{|1-tz|^2}\right]dt\\ &\qquad=\lim_{z\to\pm1,\,0<\operatorname{Im} z}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\widehat{\Gamma}(t)\frac{z-\overline{z}}{|t-z|^2}dt=0=\gamma(\pm1)\end{aligned}$$

with

$$\widehat{\Gamma}(\zeta)=\begin{cases}\gamma(t), & |t|\leqslant1,\\ -\gamma\left(\frac{1}{t}\right), & 1<|t|,\end{cases}\quad t\in\mathbb{R}. \qquad\blacksquare$$

Another representation formula of second order besides the one in Theorem 2 is available, where instead of the Green function the harmonic Neumann function is used.

Theorem III. *Let $D\subset\mathbb{C}$ be a regular domain and $N_1=2N$ where N is the harmonic Neumann function for D, then any $w\in C^2(D;\mathbb{C})\cap C^1(\overline{D};\mathbb{C})$ can be represented as*

$$w(z)=-\frac{1}{4\pi}\int_{\partial D}\{w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)-\partial_{\nu_\zeta}w(\zeta)N_1(z,\zeta)\}ds_\zeta-\frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta.$$

As can be verified, the harmonic Neumann function N_1 for the upper half unit disc $\mathbb{D}^+$ is

$$N_1(z,\zeta) = 2\log|z\zeta|^2 - \log|(\zeta - z)(\overline{\zeta} - z)(1 - z\overline{\zeta})(1 - z\zeta)|^2.$$

For $|z| = 1$, $|\zeta| < 1$

$$\partial_{\nu_z} N_1(z,\zeta) = (z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta) = 4 + \frac{z}{\zeta - z} + \frac{\overline{z}}{\overline{\zeta - z}} + \frac{z}{\overline{\zeta} - z} + \frac{\overline{z}}{\zeta - \overline{z}}$$
$$+ \frac{z\zeta}{1 - z\zeta} + \frac{\overline{z\zeta}}{1 - \overline{z\zeta}} + \frac{z\overline{\zeta}}{1 - z\overline{\zeta}} + \frac{\overline{z}\zeta}{1 - \overline{z}\zeta} = 0$$

and for $|z| < 1$, $|\zeta| = 1$

$$N_1(z,\zeta) = 2\Big[\log|z|^2 - \log|(1 - z\overline{\zeta})(1 - z\zeta)|^2\Big],$$

$$(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta) = 2\Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 + \frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\zeta}{\zeta - \overline{z}} - 1\Big],$$

for $z = \overline{z}$, $0 < \operatorname{Re}\zeta$

$$\partial_{\nu_z} N_1(z,\zeta) = -i(\partial_z - \partial_{\overline{z}})N_1(z,\zeta) = -i\Big[\frac{2}{z} - \frac{2}{\overline{z}} + \frac{1}{\zeta - z} - \frac{1}{\overline{\zeta - z}} + \frac{1}{\overline{\zeta} - z}$$
$$- \frac{1}{\zeta - \overline{z}} + \frac{\zeta}{1 - z\zeta} - \frac{\overline{\zeta}}{1 - \overline{z\zeta}} + \frac{\overline{\zeta}}{1 - z\overline{\zeta}} - \frac{\zeta}{1 - \overline{z}\zeta}\Big] = 0,$$

and for $0 < \operatorname{Im} z$, $\operatorname{Im}\zeta = 0$

$$N_1(z,\zeta) = 2\Big[\log|z\zeta|^2 - \log|(\zeta - z)(1 - z\zeta)|^2\Big],$$

$$-i(\partial_z - \partial_{\overline{z}})N_1(z,\zeta) = -2i(z - \overline{z})\Big[\frac{1}{|\zeta - z|^2} + \frac{\zeta^2}{|1 - z\zeta|^2} - \frac{1}{|z|^2}\Big].$$

In explicit form the Neumann representation formula from Theorem 2 for $\mathbb{D}^+$ is

$$w(z) = \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \partial_\nu w(\zeta) \left[\log|z|^2 - \log|(1 - z\overline{\zeta})(1 - z\zeta)|^2\right] \frac{d\zeta}{\zeta}$$
$$+ \frac{1}{2\pi} \int_{-1}^{1} \partial_\nu w(t) \left[\log|tz|^2 - \log|(t - z)(1 - zt)|^2\right] dt \tag{2.7}$$
$$- \frac{1}{\pi} \int\limits_{\mathbb{D}^+} w_{\zeta\overline{\zeta}}(\zeta) \left[2\log|z\zeta|^2 - \log|(\zeta - z)(\overline{\zeta} - z)(1 - z\overline{\zeta})(1 - z\zeta)|^2\right] d\xi d\eta.$$

Theorem 5. *Any* $w \in C^2(\mathbb{D}^+;\mathbb{C}) \cap C^1(\overline{\mathbb{D}^+};\mathbb{C})$ *can be represented according to formula* (2.7).

Formula (2.7) provides the solution to the Neumann boundary value problem for the Poisson equation.

Theorem 6. *The Neumann problem*

$$w_{z\overline{z}} = f \quad in\ \mathbb{D}^+, \qquad \partial_\nu w = \gamma \quad on\ \partial\mathbb{D}^+,$$

for $f \in C(\mathbb{D}^+;\mathbb{C})$, $\gamma \in C(\partial\mathbb{D}^+;\mathbb{C})$, *is uniquely solvable. The solution is given by*

$$\begin{aligned} w(z) = \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} & \gamma(\zeta)\left[\log|z|^2 - \log|(1-z\overline{\zeta})(1-z\zeta)|^2\right]\frac{d\zeta}{\zeta} \\ & + \frac{1}{2\pi}\int\limits_{-1}^{1}\gamma(t)\left[\log|tz|^2 - \log|(t-z)(1-zt)|^2\right]dt \\ & - \frac{1}{\pi}\int\limits_{\mathbb{D}^+} f(\zeta)\left[2\log|z\zeta|^2 - \log|(\zeta-z)(\overline{\zeta}-z)(1-z\overline{\zeta})(1-z\zeta)|^2\right]d\xi d\eta. \end{aligned} \tag{2.8}$$

Proof. As the area integral is a solution to the Poisson equation satisfying the homogeneous boundary condition, only the boundary integrals representing a harmonic function have to be shown to satisfy the boundary condition.

1. For $z \in \mathbb{D}^+$

$$\begin{aligned} (z\partial_z + \overline{z}\partial_{\overline{z}})w(z) = \frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} & \gamma(\zeta)\left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 + \frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\zeta}{\zeta - \overline{z}} - 1\right]\frac{d\zeta}{\zeta} \\ & + \frac{1}{2\pi}\int\limits_{-1}^{1}\gamma(t)\left[2 + \frac{z}{t-z} + \frac{\overline{z}}{t-\overline{z}} + \frac{zt}{1-zt} + \frac{\overline{z}t}{1-\overline{z}t}\right]dt \\ & - \frac{1}{\pi}\int\limits_{\mathbb{D}^+} f(\zeta)(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta)d\xi d\eta, \end{aligned}$$

so that for $|\zeta_0| = 1$, $0 < \operatorname{Im}\zeta_0$

$$\partial_\nu w(\zeta_0) = \lim_{z\to\zeta_0}(z\partial_z + \overline{z}\partial_{\overline{z}})w(z) = \gamma(\zeta_0),$$

because for $0 < \operatorname{Im}\zeta$

$$\frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\zeta}{\zeta - \overline{z}} - 1 = \frac{1 - |z|^2}{|\overline{\zeta} - z|^2},$$

and for $|t| < 1$, $t = \overline{t}$,

$$2+\frac{z}{t-z}+\frac{\overline{z}}{t-\overline{z}}+\frac{zt}{1-zt}+\frac{\overline{z}t}{1-\overline{z}t} = (1-|z|^2)\left[\frac{t}{(t-z)(1-\overline{z}t)}+\frac{t}{(t-\overline{z})(1-zt)}\right].$$

2. For $z \in \mathbb{D}^+$

$$\begin{aligned}-i(\partial_z - \partial_{\overline{z}})w(z) = &-\frac{1}{4\pi}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta)\left[\frac{2}{z}-\frac{2}{\overline{z}}+\frac{1}{\zeta-z}-\frac{1}{\overline{\overline{\zeta-z}}}+\frac{1}{\overline{\zeta}-z}\right.\\ &\left.-\frac{1}{\zeta-\overline{z}}+\frac{\zeta}{1-z\zeta}-\frac{\overline{\zeta}}{1-\overline{\overline{z\zeta}}}+\frac{\overline{\zeta}}{1-z\overline{\zeta}}-\frac{\zeta}{1-\overline{z}\zeta}\right]\frac{d\zeta}{\zeta}\\ &+\frac{z-\overline{z}}{2\pi i}\int_{-1}^{1}\gamma(t)\left[\frac{1}{|t-z|^2}+\frac{t^2}{|1-zt|^2}-\frac{1}{|z|^2}\right]dt\\ &+\frac{i}{\pi}\int\limits_{\mathbb{D}^+} f(\zeta)(\partial_z-\partial_{\overline{z}})N_1(z,\zeta)d\xi d\eta.\end{aligned}$$

Hence, for $z \in \mathbb{D}^+$, $|t_0| < 1$, $t_0 = \overline{t_0}$,

$$\partial_\nu w(t_0) = \lim_{z\to t_0}[-i(\partial_z - \partial_{\overline{z}})]w(z) = \gamma(t_0),$$

as for $0 < \operatorname{Im}\zeta$

$$\begin{aligned}&\frac{2}{z}-\frac{2}{\overline{z}}+\frac{1}{\zeta-z}-\frac{1}{\overline{\overline{\zeta-z}}}+\frac{1}{\overline{\zeta}-z}-\frac{1}{\zeta-\overline{z}}+\frac{\zeta}{1-z\zeta}-\frac{\overline{\zeta}}{1-\overline{\overline{z\zeta}}}+\frac{\overline{\zeta}}{1-z\overline{\zeta}}-\frac{\zeta}{1-\overline{z}\zeta}\\ &=(z-\overline{z})\times\left[\frac{1}{(\zeta-z)(\zeta-\overline{z})}+\frac{1}{(\overline{\zeta}-z)(\overline{\overline{\zeta-z}})}\right.\\ &\quad\left.+\frac{\zeta^2}{(1-z\zeta)(1-\overline{z}\zeta)}+\frac{\overline{\zeta}^2}{(1-\overline{\overline{z\zeta}})(1-z\overline{\zeta})}-\frac{2}{|z|^2}\right]\end{aligned}$$

and for $|t_0| < 1$, $t_0 = \overline{t_0}$,

$$\lim_{z\to t_0}\left[\frac{t^2}{|1-zt|^2}-\frac{1}{|z|^2}\right] = \frac{t^2}{|1-t_0t|^2}-\frac{1}{t_0^2}. \qquad \blacksquare$$

Remark. In $\mathbb{D}^+$ the Neumann problem is unconditionally solvable.

3. Upper half circular ring

In order to modify the Cauchy-Pompeiu formula according to the Schwarz boundary condition at first the Cauchy kernel is replaced by a proper one for the ring

$R = \{z : 0 < r < |z| < 1\}$, see [30], [31]. Then this resulting formula is complemented by the respective expressions evaluated at the reflected points $1/\overline{z}$, $\overline{z}$ and $1/z$ of z and then these four expressions are properly composed.

Let R^+ denote the upper half part of the ring R.

Theorem 7. *Any $w \in C^1(R^+;\mathbb{C}) \cap C(\overline{R^+};\mathbb{C})$ can be represented as*

$$\begin{aligned} w(z) = \frac{1}{2\pi i} \int\limits_{\partial R^+} w(\zeta) \left\{ \frac{\zeta}{\zeta - z} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{z}{\zeta - r^{2n} z} \right] \right\} \frac{d\zeta}{\zeta} \\ - \frac{1}{\pi} \int\limits_{R^+} w_{\overline{\zeta}}(\zeta) \left\{ \frac{1}{\zeta - z} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{1}{r^{2n}\zeta - z} + \frac{z}{\zeta(\zeta - r^{2n} z)} \right] \right\} d\xi d\eta \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} w(z) = & \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1, \\ 0<\operatorname{Im}\zeta}} \operatorname{Re} w(\zeta) \left\{ \frac{\zeta + z}{\zeta - z} - \frac{\overline{\zeta} + z}{\overline{\zeta} - z} \right. \\ & \left. + 2 \sum_{n=1}^{\infty} r^{2n} \left[\frac{\zeta}{r^{2n}\zeta - z} - \frac{z}{r^{2n} z - \zeta} - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - z} + \frac{z}{r^{2n} z - \overline{\zeta}} \right] \right\} \frac{d\zeta}{\zeta} \\ & - \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=r, \\ 0<\operatorname{Im}\zeta}} \operatorname{Re} w(\zeta) \left\{ \frac{\zeta + z}{\zeta - z} - \frac{\overline{\zeta} + z}{\overline{\zeta} - z} \right. \\ & \left. + 2 \sum_{n=1}^{\infty} r^{2n} \left[\frac{\zeta}{r^{2n}\zeta - z} - \frac{z}{r^{2n} z - \zeta} - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - z} + \frac{z}{r^{2n} z - \overline{\zeta}} \right] \right\} \frac{d\zeta}{\zeta} \\ & + \frac{1}{\pi i} \int\limits_{[-1,r]\cup[r,1]} \operatorname{Re} w(t) \left\{ \frac{1}{t - z} - \frac{z}{1 - zt} \right. \\ & \left. + \sum_{n=1}^{\infty} r^{2n} \left[\frac{1}{r^{2n} t - z} - \frac{z}{t(r^{2n} z - t)} - \frac{1}{t(r^{2n} - zt)} + \frac{z}{r^{2n} zt - 1} \right] \right\} dt \\ & + \frac{1}{\pi} \int\limits_{\substack{|\zeta|=1, \\ 0<\operatorname{Im}\zeta}} \operatorname{Im} w(\zeta) \frac{d\zeta}{\zeta} - \frac{1}{\pi} \int\limits_{R^+} \left\{ w_{\overline{\zeta}}(\zeta) \left[\frac{1}{\zeta - z} - \frac{z}{1 - z\zeta} \right. \right. \\ & \left. + \sum_{n=1}^{\infty} r^{2n} \left(\frac{1}{r^{2n}\zeta - z} - \frac{z}{\zeta(r^{2n} z - \zeta)} - \frac{1}{\zeta(z\zeta - r^{2n})} + \frac{z}{r^{2n} z\zeta - 1} \right) \right] \\ & - \overline{w_{\overline{\zeta}}(\zeta)} \left[\frac{1}{\overline{\zeta} - z} - \frac{z}{1 - z\overline{\zeta}} \right. \\ & \left. \left. + \sum_{n=1}^{\infty} r^{2n} \left(\frac{1}{r^{2n}\overline{\zeta} - z} - \frac{z}{\overline{\zeta}(r^{2n} z - \overline{\zeta})} - \frac{1}{\overline{\zeta}(r^{2n} - z\overline{\zeta})} + \frac{z}{r^{2n} z\overline{\zeta} - 1} \right) \right] \right\} d\xi d\eta. \end{aligned} \tag{3.2}$$

Proof. Both parts of the Cauchy-Pompeiu formula give for $z \in R^+$

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial R^+} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{R^+} w_{\overline{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z}$$

and

$$\begin{aligned} 0 = \frac{1}{2\pi i} \int\limits_{\partial R^+} w(\zeta) \sum_{n=1}^{\infty} r^{2n} \left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{z}{\zeta - r^{2n} z} \right] \frac{d\zeta}{\zeta} \\ - \frac{1}{\pi} \int\limits_{R^+} w_{\overline{\zeta}}(\zeta) \sum_{n=1}^{\infty} r^{2n} \left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{z}{\zeta - r^{2n} z} \right] d\xi d\eta. \end{aligned}$$

Thus

$$\begin{aligned} w(z) = \frac{1}{2\pi i} \int\limits_{\partial R^+} w(\zeta) \left\{ \frac{\zeta}{\zeta - z} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{z}{\zeta - r^{2n} z} \right] \right\} \frac{d\zeta}{\zeta} \\ - \frac{1}{\pi} \int\limits_{R^+} w_{\overline{\zeta}}(\zeta) \left\{ \frac{1}{\zeta - z} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{1}{r^{2n}\zeta - z} + \frac{z}{\zeta(\zeta - r^{2n} z)} \right] \right\} d\xi d\eta. \end{aligned} \tag{3.3}$$

Evaluating the right-hand side for the reflections $1/\overline{z}$, $\overline{z}$ and $1/z$ of $z \in R^+$ shows

$$\begin{aligned} 0 = \frac{1}{2\pi i} \int\limits_{\partial R^+} w(\zeta) \left\{ \frac{\overline{z}\zeta}{\overline{z}\zeta - 1} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{\overline{z}\zeta}{r^{2n}\overline{z}\zeta - 1} + \frac{1}{\overline{z}\zeta - r^{2n}} \right] \right\} \frac{d\zeta}{\zeta} \\ - \frac{1}{\pi} \int\limits_{R^+} w_{\overline{\zeta}}(\zeta) \left\{ \frac{\overline{z}}{\overline{z}\zeta - 1} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{\overline{z}}{r^{2n}\overline{z}\zeta - 1} + \frac{1}{\zeta(\overline{z}\zeta - r^{2n})} \right] \right\} d\xi d\eta, \end{aligned} \tag{3.3′}$$

$$\begin{aligned} 0 = \frac{1}{2\pi i} \int\limits_{\partial R^+} w(\zeta) \left\{ \frac{\zeta}{\zeta - \overline{z}} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{\zeta}{r^{2n}\zeta - \overline{z}} + \frac{\overline{z}}{\zeta - r^{2n}\overline{z}} \right] \right\} \frac{d\zeta}{\zeta} \\ - \frac{1}{\pi} \int\limits_{R^+} w_{\overline{\zeta}}(\zeta) \left\{ \frac{1}{\zeta - \overline{z}} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{1}{r^{2n}\zeta - \overline{z}} + \frac{\overline{z}}{\zeta(\zeta - r^{2n}\overline{z})} \right] \right\} d\xi d\eta, \end{aligned} \tag{3.3″}$$

and

$$\begin{aligned} 0 = \frac{1}{2\pi i} \int\limits_{\partial R^+} w(\zeta) \left\{ \frac{z\zeta}{z\zeta - 1} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{z\zeta}{r^{2n}z\zeta - 1} + \frac{1}{z\zeta - r^{2n}} \right] \right\} \frac{d\zeta}{\zeta} \\ - \frac{1}{\pi} \int\limits_{R^+} w_{\overline{\zeta}}(\zeta) \left\{ \frac{z}{z\zeta - 1} + \sum_{n=1}^{\infty} r^{2n} \left[\frac{z}{r^{2n}z\zeta - 1} + \frac{1}{\zeta(z\zeta - r^{2n})} \right] \right\} d\xi d\eta. \end{aligned} \tag{3.3‴}$$

Subtracting the complex conjugate of (3.3′) from (3.3) and the complex conjugate of (3.3″) from (3.3‴) and adding both results leads to the representation (3.2). ■

Formula (3.2) serves to solve the Schwarz problem for R^+.

Theorem 8. *The Schwarz problem*

$$w_{\overline{z}} = f \ \ in \ \ R^+, \quad f \in L_p(R^+;\mathbb{C}), \ \ 2 < p,$$

$$\operatorname{Re} w = \gamma \ \ on \ \ \partial R^+, \ \ \gamma \in C(\partial R^+;\mathbb{C}), \ \ \gamma(-1) = \gamma(-r) = \gamma(r) = \gamma(1) = 0,$$

$$\frac{1}{\pi}\int_0^{\pi} \operatorname{Im} w(e^{i\varphi})d\varphi = c, \quad c \in \mathbb{R},$$

is uniquely solvable. The solution is given as

$$\begin{aligned} w(z) = \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \Bigg\{ \frac{\zeta+z}{\zeta-z} - \frac{\overline{\zeta}+z}{\overline{\zeta}-z} \\ +2\sum_{n=1}^{\infty} r^{2n}\left[\frac{\zeta}{r^{2n}\zeta - z} - \frac{z}{r^{2n}z-\zeta} - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta}-z} + \frac{z}{r^{2n}z-\overline{\zeta}}\right]\Bigg\}\frac{d\zeta}{\zeta} \\ + \frac{1}{\pi i}\int\limits_{\partial R^+\cup\mathbb{R}} \gamma(t)\Bigg\{\frac{1}{t-z} - \frac{z}{1-zt} \\ + \sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{r^{2n}t-z} - \frac{z}{t(r^{2n}z-t)} - \frac{1}{t(r^{2n}-zt)} + \frac{z}{r^{2n}zt-1}\right]\Bigg\} dt + ic \\ - \frac{1}{\pi}\int\limits_{R^+}\Bigg\{ f(\zeta)\Bigg[\frac{1}{\zeta-z} - \frac{z}{1-z\zeta} \\ + \sum_{n=1}^{\infty} r^{2n}\left(\frac{1}{r^{2n}\zeta-z} - \frac{z}{\zeta(r^{2n}z-\zeta)} - \frac{1}{\zeta(r^{2n}-z\zeta)} + \frac{z}{r^{2n}z\zeta-1}\right)\Bigg] \\ - \overline{f(\zeta)}\Bigg[\frac{1}{\overline{\zeta}-z} - \frac{z}{1-z\overline{\zeta}} \\ + \sum_{n=1}^{\infty} r^{2n}\left(\frac{1}{r^{2n}\overline{\zeta}-z} - \frac{z}{\overline{\zeta}(r^{2n}z-\overline{\zeta})} - \frac{1}{\overline{\zeta}(r^{2n}-z\overline{\zeta})} + \frac{z}{r^{2n}z\overline{\zeta}-1}\right)\Bigg]\Bigg\} d\xi d\eta. \end{aligned} \tag{3.4}$$

Proof. The area integral in (3.4) is obviously an analytic function up to the term $Tf(z)$, where $Tf(z)$ is defined at the beginning of the proof of Theorem 2. As the boundary integrals are also analytic functions the equation (3.4) provides a weak solution to the inhomogeneous Cauchy-Riemann equation. Moreover, the area integral is purely imaginary on the boundary ∂R^+. In order to check the side condition

$$\frac{1}{\pi i} \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \operatorname{Im} w(z) \frac{dz}{z} = c$$

the relations

$$\frac{1}{2\pi i} \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \left[\frac{\zeta+z}{\zeta-z} - \frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right] \frac{dz}{z} = \frac{1}{2\pi i} \int\limits_{|z|=1} \frac{\zeta+z}{\zeta-z}\frac{dz}{z} = 0 \quad \text{for } |\zeta|=1,$$

$$\frac{1}{2\pi i} \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \left[\frac{\zeta}{r^{2n}\zeta-z} - \frac{z}{r^{2n}z-\zeta} - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta}-z} + \frac{z}{r^{2n}z-\overline{\zeta}}\right] \frac{dz}{z}$$

$$= \frac{1}{2\pi i} \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \left[\frac{1}{r^{2n}-z\overline{\zeta}} - \frac{1}{r^{2n}-\overline{z}\zeta} - \frac{1}{r^{2n}-z\zeta} + \frac{1}{r^{2n}-\overline{z\zeta}}\right] \frac{dz}{z} \quad \text{for } |\zeta|=1,$$

$$\frac{1}{2\pi i} \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \left[\frac{1}{\zeta-z} - \frac{z}{1-z\zeta}\right] \frac{dz}{z} = -\frac{1}{2\pi i} \int\limits_{|z|=1} \frac{1}{z-\zeta}\frac{dz}{z} = 0 \quad \text{for } |\zeta|<1,$$

and

$$\frac{1}{2\pi i} \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \left[\frac{1}{r^{2n}\zeta-z} - \frac{z}{\zeta(r^{2n}z-\zeta)} - \frac{1}{\zeta(r^{2n}-z\zeta)} + \frac{z}{r^{2n}z\zeta-1}\right] \frac{dz}{z}$$

$$= \frac{1}{2\pi i} \int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z}} \left[\frac{1}{r^{2n}\zeta-z} - \frac{1}{\zeta(r^{2n}-\overline{z}\zeta)} - \frac{1}{\zeta(r^{2n}-z\zeta)} + \frac{1}{r^{2n}\zeta-\overline{z}}\right] \frac{dz}{z}$$

$$= \frac{1}{2\pi i} \int\limits_{|z|=1} \left[\frac{1}{r^{2n}\zeta-z} - \frac{1}{\zeta(r^{2n}-z\zeta)}\right] \frac{dz}{z} = 0 \quad \text{for } r<|\zeta|<1$$

are to be used. Hence, just the boundary integrals in (3.4) have to be shown to satisfy the boundary relations. Let w_0 denote the sum of these two boundary integrals.

Then

$$
\begin{aligned}
\operatorname{Re} w_0(z) = \frac{1}{2\pi i} \int\limits_{\substack{\partial R^+ \\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \Bigg\{ & \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - \frac{\overline{\zeta}}{\overline{\zeta} - z} - \frac{\zeta}{\zeta - \overline{z}} \\
& + \sum_{n=1}^{\infty} r^{2n} \Bigg[\frac{\zeta}{r^{2n}\zeta - z} + \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - \overline{z}} - \frac{z}{r^{2n} z - \zeta} - \frac{\overline{z}}{r^{2n}\overline{z} - \overline{\zeta}} \\
& - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - z} - \frac{\zeta}{r^{2n}\zeta - \overline{z}} + \frac{z}{r^{2n} z - \overline{\zeta}} + \frac{\overline{z}}{r^{2n}\overline{z} - \zeta} \Bigg] \Bigg\} \frac{d\zeta}{\zeta} \\
& + \frac{1}{2\pi i} \int\limits_{\partial R^+ \cap \mathbb{R}} \gamma(t) \Bigg\{ \frac{1}{t - z} - \frac{1}{t - \overline{z}} - \frac{z}{1 - zt} + \frac{\overline{z}}{1 - \overline{z}t} \\
& + \sum_{n=1}^{\infty} r^{2n} \Bigg[\frac{1}{r^{2n}t - z} - \frac{1}{r^{2n}t - \overline{z}} - \frac{z}{t(r^{2n}z - t)} + \frac{\overline{z}}{t(r^{2n}\overline{z} - t)} \\
& - \frac{1}{t(r^{2n} - zt)} + \frac{1}{t(r^{2n} - \overline{z}t)} + \frac{z}{r^{2n}zt - 1} - \frac{\overline{z}}{r^{2n}\overline{z}t - 1} \Bigg] \Bigg\} dt
\end{aligned}
$$

Next

$$
\begin{aligned}
\operatorname{Re} w_0(z) = \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1 \\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \Bigg\{ & \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - \frac{\overline{\zeta}}{\overline{\zeta} - z} - \frac{\zeta}{\zeta - \overline{z}} \\
& + \sum_{n=1}^{\infty} r^{2n} \Bigg[\frac{1}{r^{2n} - z\overline{\zeta}} + \frac{1}{r^{2n} - \overline{z}\zeta} - \frac{|z|^2}{r^{2n}|z|^2 - \overline{z}\zeta} - \frac{|z|^2}{r^{2n}|z|^2 - z\overline{\zeta}} \\
& - \frac{1}{r^{2n} - z\zeta} - \frac{1}{r^{2n} - \overline{z\zeta}} + \frac{|z|^2}{r^{2n}|z|^2 - z\zeta} + \frac{|z|^2}{r^{2n}|z|^2 - \overline{z\zeta}} \Bigg] \Bigg\} \frac{d\zeta}{\zeta} \\
& - \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=r \\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \Bigg\{ \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - \frac{\overline{\zeta}}{\overline{\zeta} - z} - \frac{\zeta}{\zeta - \overline{z}} \\
& + \sum_{n=1}^{\infty} r^{2n} \Bigg[\frac{r^2}{r^{2(n+1)} - z\overline{\zeta}} + \frac{r^2}{r^{2(n+1)} - \overline{z}\zeta} - \frac{|z|^2}{r^{2n}|z|^2 - \overline{z}\zeta} - \frac{|z|^2}{r^{2n}|z|^2 - z\overline{\zeta}} \\
& - \frac{r^2}{r^{2(n+1)} - z\zeta} - \frac{r^2}{r^{2(n+1)} - \overline{z\zeta}} + \frac{|z|^2}{r^{2n}|z|^2 - z\zeta} + \frac{|z|^2}{r^{2n}|z|^2 - \overline{z\zeta}} \Bigg] \Bigg\} \frac{d\zeta}{\zeta} \\
& + \frac{z - \overline{z}}{2\pi i} \int\limits_{\partial R^+ \cap \mathbb{R}} \gamma(t) \Bigg\{ \frac{1}{|t - z|^2} - \frac{1}{|1 - zt|^2} \\
& + \sum_{n=1}^{\infty} r^{2n} \Bigg[\frac{1}{|r^{2n}t - z|^2} + \frac{1}{|r^{2n}z - t|^2} - \frac{1}{|r^{2n} - zt|^2} - \frac{1}{|r^{2n}zt - 1|^2} \Bigg] \Bigg\} dt.
\end{aligned}
$$

Thus for $|\zeta_0| = 1$, $0 < \operatorname{Im}\zeta_0$, $z \in R^+$,

$$\begin{aligned}
\lim_{z\to\zeta_0} \operatorname{Re} w_0(z) &= \lim_{z\to\zeta_0} \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - \frac{\overline{\zeta}}{\overline{\zeta} - z} - \frac{\zeta}{\zeta - \overline{z}}\right] \frac{d\zeta}{\zeta} \\
&\quad - \lim_{z\to\zeta_0} \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=r\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \left\{\sum_{n=0}^{\infty} r^{2n} \left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - \overline{z}}\right.\right. \\
&\quad \left. - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - z} - \frac{\zeta}{r^{2n}\zeta - \overline{z}}\right] - \sum_{n=1}^{\infty} r^{2n} \left[\frac{|z|^2\zeta}{r^{2n}|z|^2\zeta - r^2 z}\right. \\
&\quad \left.\left. + \frac{|z|^2\overline{\zeta}}{r^{2n}|z|^2\overline{\zeta} - r^2\overline{z}} - \frac{|z|^2\overline{\zeta}}{r^{2n}|z|^2\overline{\zeta} - r^2 z} - \frac{|z|^2\zeta}{r^{2n}|z|^2\zeta - r^2\overline{z}}\right]\right\} \frac{d\zeta}{\zeta} \\
&= \lim_{z\to\zeta_0} \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\right] \frac{d\zeta}{\zeta} = \gamma(\zeta_0)
\end{aligned}$$

with

$$\Gamma(\zeta) = \begin{cases} \gamma(\zeta), & 0 \leqslant \operatorname{Im}\zeta, \\ -\gamma(\overline{\zeta}), & \operatorname{Im}\zeta < 0, \end{cases} \quad |\zeta| = 1.$$

For $|\zeta_0| = r$, $0 < \operatorname{Im}\zeta_0$, $z \in R^+$,

$$\begin{aligned}
&\lim_{z\to\zeta_0} \operatorname{Re} w_0(z) \\
&= \lim_{z\to\zeta_0} \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \left\{\sum_{n=1}^{\infty} r^{2n} \left[\frac{z}{r^{2n}z - |z|^2\zeta} + \frac{\overline{z}}{r^{2n}\overline{z} - |z|^2\overline{\zeta}} - \frac{\overline{z}}{r^{2n}\overline{z} - |z|^2\zeta}\right.\right. \\
&\quad \left.\left. - \frac{z}{r^{2n}z - |z|^2\overline{\zeta}}\right] - \sum_{n=0}^{\infty} r^{2n} \left[\frac{z}{r^{2n}z - \zeta} + \frac{\overline{z}}{r^{2n}\overline{z} - \overline{\zeta}} - \frac{\overline{z}}{r^{2n}\overline{z} - \zeta} - \frac{z}{r^{2n}z - \overline{\zeta}}\right]\right\} \frac{d\zeta}{\zeta} \\
&\quad - \lim_{z\to\zeta_0} \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=r\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \left\{\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - \frac{\overline{\zeta}}{\overline{\zeta} - z} - \frac{\zeta}{\zeta - \overline{z}}\right. \\
&\quad + \sum_{n=1}^{\infty} r^{2n} \left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - \overline{z}} - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - z} - \frac{\zeta}{r^{2n}\zeta - \overline{z}}\right] - \sum_{n=1}^{\infty} r^{2n} \\
&\quad \times \left.\left[\frac{|z|^2\zeta}{r^{2n}|z|^2\zeta - r^2 z} + \frac{|z|^2\overline{\zeta}}{r^{2n}|z|^2\overline{\zeta} - r^2\overline{z}} - \frac{|z|^2\overline{\zeta}}{r^{2n}|z|^2\overline{\zeta} - r^2 z} - \frac{|z|^2\zeta}{r^{2n}|z|^2\zeta - r^2\overline{z}}\right]\right\} \frac{d\zeta}{\zeta} \\
&= -\lim_{z\to\zeta_0} \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \Gamma(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\right] \frac{d\zeta}{\zeta} = \gamma(\zeta_0)
\end{aligned}$$

with

$$\Gamma(\zeta)=\begin{cases}\gamma(\zeta), & 0\leqslant \operatorname{Im}\zeta,\\ -\gamma(\overline{\zeta}), & \operatorname{Im}\zeta<0,\end{cases}\quad |\zeta|=r.$$

Finally, for $t_0\in(-1,-r)\cup(r,1)$, $z\in R^+$

$$\begin{aligned}\lim_{z\to t_0}\operatorname{Re}w_0(z)&=\lim_{z\to t_0}\frac{1}{2\pi i}\int\limits_{\partial R^+\cap\mathbb{R}}\gamma(t)\left\{\frac{z-\overline{z}}{|t-z|^2}-\frac{z-\overline{z}}{|1-tz|^2}+\sum_{n=1}^{\infty}r^{2n}\right.\\ &\quad\times\left.\left[\frac{z-\overline{z}}{|r^{2n}t-z|^2}+\frac{z-\overline{z}}{|r^{2n}z-t|^2}-\frac{z-\overline{z}}{|r^{2n}-zt|^2}-\frac{z-\overline{z}}{|1-r^{2n}zt|}\right]\right\}dt\\ &=\lim_{z\to t_0}\frac{1}{2\pi i}\left[\int\limits_{-\frac{1}{r}}^{-r}+\int\limits_{r}^{\frac{1}{r}}\right]\widehat{\Gamma}(t)\frac{z-\overline{z}}{|t-z|^2}dt=\gamma(t_0),\end{aligned}$$

with

$$\widehat{\Gamma}(t)=\begin{cases}\gamma(t), & r\leqslant |t|\leqslant 1,\\ -\gamma\left(\dfrac{1}{t}\right), & 1\leqslant |t|\leqslant \dfrac{1}{r}\end{cases}\quad t\in\mathbb{R}.$$

In order to check the boundary behavior in the corner points $\pm r$ and ± 1 decompose

$$\operatorname{Re}w_0=w_1+w_2+\widehat{w}$$

according to the boundary integrals along $|\zeta|=1$, $0<\operatorname{Im}\zeta$, and $|\zeta|=r$, $0<\operatorname{Im}\zeta$ and $r<|t|<1$, $t\in\mathbb{R}$. From

$$\begin{aligned}w_1(z)&=\frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}}\gamma(\zeta)\left\{\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\right.\\ &\quad\left.+\sum_{n=1}^{\infty}r^{2n}\left[\frac{\zeta}{r^{2n}\zeta-z}+\frac{\overline{\zeta}}{r^{2n}\overline{\zeta}-\overline{z}}-\frac{z}{r^{2n}z-\zeta}-\frac{\overline{z}}{r^{2n}\overline{z}-\overline{\zeta}}\right]\right\}\frac{d\zeta}{\zeta}\\ &\quad-\frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}}\gamma(\zeta)\left\{\frac{\overline{\zeta}}{\overline{\zeta}-z}+\frac{\zeta}{\zeta-\overline{z}}-1\right.\\ &\quad\left.+\sum_{n=1}^{\infty}r^{2n}\left[\frac{\overline{\zeta}}{r^{2n}\overline{\zeta}-z}+\frac{\zeta}{r^{2n}\zeta-\overline{z}}-\frac{z}{r^{2n}z-\overline{\zeta}}-\frac{\overline{z}}{r^{2n}\overline{z}-\zeta}\right]\right\}\frac{d\zeta}{\zeta}\\ &=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Gamma(\zeta)\left\{\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\right.\\ &\quad\left.+\sum_{n=1}^{\infty}r^{2n}\left[\frac{\zeta}{r^{2n}\zeta-z}+\frac{\overline{\zeta}}{r^{2n}\overline{\zeta}-\overline{z}}-\frac{z}{r^{2n}z-\zeta}-\frac{\overline{z}}{r^{2n}\overline{z}-\overline{\zeta}}\right]\right\}\frac{d\zeta}{\zeta},\end{aligned}$$

$$
\begin{aligned}
w_2(z) = &-\frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta}}\gamma(\zeta)\left\{\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\right.\\
&\left.+\sum_{n=1}^{\infty}r^{2n}\left[\frac{\zeta}{r^{2n}\zeta-z}+\frac{\overline{\zeta}}{r^{2n}\overline{\zeta}-\overline{z}}-\frac{z}{r^{2n}z-\zeta}-\frac{\overline{z}}{r^{2n}\overline{z}-\overline{\zeta}}\right]\right\}\frac{d\zeta}{\zeta}\\
&+\frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta}}\gamma(\zeta)\left\{\frac{\overline{\zeta}}{\overline{\zeta}-z}+\frac{\zeta}{\zeta-\overline{z}}-1\right.\\
&\left.+\sum_{n=1}^{\infty}r^{2n}\left[\frac{\overline{\zeta}}{r^{2n}\overline{\zeta}-z}+\frac{\zeta}{r^{2n}\zeta-\overline{z}}-\frac{z}{r^{2n}z-\overline{\zeta}}-\frac{\overline{z}}{r^{2n}\overline{z}-\zeta}\right]\right\}\frac{d\zeta}{\zeta}\\
=&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\Gamma(\zeta)\left\{\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\right.\\
&\left.+\sum_{n=1}^{\infty}r^{2n}\left[\frac{\zeta}{r^{2n}\zeta-z}+\frac{\overline{\zeta}}{r^{2n}\overline{\zeta}-\overline{z}}-\frac{z}{r^{2n}z-\zeta}-\frac{\overline{z}}{r^{2n}\overline{z}-\overline{\zeta}}\right]\right\}\frac{d\zeta}{\zeta},
\end{aligned}
$$

and as well

$$
\begin{aligned}
\widehat{w}(z) = &\frac{z-\overline{z}}{2\pi i}\left[\int_{-1}^{-r}+\int_{r}^{1}\right]\gamma(t)\left\{\frac{1}{|t-z|^2}+\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{|r^{2n}t-z|^2}+\frac{1}{|r^{2n}z-t|^2}\right]\right\}dt\\
&-\frac{z-\overline{z}}{2\pi i}\left[\int_{-1}^{-r}+\int_{r}^{1}\right]\gamma(t)\left\{\frac{1}{|1-tz|^2}+\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{|r^{2n}-tz|^2}+\frac{1}{|r^{2n}zt-1|^2}\right]\right\}dt\\
=&\frac{1}{2\pi i}\left[\int_{-\frac{1}{r}}^{-r}+\int_{r}^{\frac{1}{r}}\right]\widehat{\Gamma}(t)\left\{\frac{z-\overline{z}}{|t-z|^2}+\sum_{n=1}^{\infty}r^{2n}\left[\frac{z-\overline{z}}{|r^{2n}t-z|^2}+\frac{z-\overline{z}}{|r^{2n}z-t|^2}\right]\right\}dt
\end{aligned}
$$

as

$$
\widehat{w}(z) = \frac{1}{2\pi i}\left[\int_{-1}^{-r^2}+\int_{r^2}^{1}\right]\widetilde{\Gamma}(t)\left\{\frac{z-\overline{z}}{|t-z|^2}+\sum_{n=1}^{\infty}r^{2n}\left[\frac{z-\overline{z}}{|r^{2n}t-z|^2}+\frac{z-\overline{z}}{|r^{2n}z-t|^2}\right]\right\}dt
$$

with

$$
\widetilde{\Gamma}(t) = \begin{cases}\gamma(t), & r\leqslant|t|\leqslant 1,\\ -\gamma\left(\dfrac{r^2}{t}\right), & r^2\leqslant|t|\leqslant r,\end{cases}\qquad t\in\mathbb{R}.
$$

By the properties of the Poisson kernels for discs and half planes, see e.g. [5], [28], and the continuity of Γ, $\widehat{\Gamma}$, $\widetilde{\Gamma}$ in the respective corner points

$$\lim_{z\to\pm1} w_1(z) = \Gamma(\pm1) = 0 = \gamma(\pm1),$$
$$\lim_{|z|\to r} w_1(z) = 0, \quad \lim_{z-\overline{z}\to 0} w_1(z) = 0,$$
$$\lim_{z\to\pm r} w_2(z) = \Gamma(\pm r) = 0 = \gamma(\pm r),$$
$$\lim_{|z|\to 1} w_2(z) = 0, \quad \lim_{z-\overline{z}\to 0} w_2(z) = 0,$$
$$\lim_{z\to\pm1} \widehat{w}(z) = \widehat{\Gamma}(\pm1) = 0 = \gamma(\pm1)$$
$$\lim_{z\to\pm r} \widehat{w}(z) = \widetilde{\Gamma}(\pm r) = 0 = \gamma(\pm r),$$
$$\lim_{|z|\to 1} \widehat{w}(z) = 0, \quad \lim_{|z|\to r} \widehat{w}(z) = 0.$$

Thus for $\zeta_0 \in \{-1, -r, r, 1\}$

$$\lim_{z\to\zeta_0} \operatorname{Re} w_0(z) = 0 = \gamma(\zeta_0). \qquad \blacksquare$$

For the Dirichlet problem for the Poisson equation the harmonic Green function is used. On the basis of [29], [31] and using the reflections on the boundary parts as in the case of the half disc this Green function for the half ring R^+ is

$$G_1(z,\zeta) = \log\left|\frac{1-z\overline{\zeta}}{\zeta - z}\,\frac{\overline{\zeta}-z}{1-z\zeta}\prod_{n=1}^{\infty}\frac{z-r^{2n}\overline{\zeta}}{z-r^{2n}\zeta}\,\frac{z\overline{\zeta}-r^{2n}}{z\zeta-r^{2n}}\,\frac{\overline{\zeta}-r^{2n}z}{\zeta-r^{2n}z}\,\frac{1-r^{2n}z\overline{\zeta}}{1-r^{2n}z\zeta}\right|^2. \tag{3.5}$$

To obtain the Poisson kernel the outward normal derivatives are calculated. On $|z| = 1$

$$\partial_{\nu_z} G_1(z,\zeta) = 4\operatorname{Re}\left\{\frac{\zeta}{\zeta - z} - \frac{\overline{\zeta}}{\overline{\zeta}-z} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{\zeta}{r^{2n}\zeta - z} - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta}-z} + \frac{1}{r^{2n}-z\zeta} - \frac{1}{r^{2n}-z\overline{\zeta}}\right]\right\},$$

on $|z| = r$

$$\partial_{\nu_z} G_1(z,\zeta) = -4\operatorname{Re}\left\{\frac{\zeta}{\zeta - z} - \frac{\overline{\zeta}}{\overline{\zeta}-z} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{\zeta}{r^{2n}\zeta - z} - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta}-z} + \frac{z}{r^{2n}z-\overline{\zeta}} - \frac{z}{r^{2n}z-\zeta}\right]\right\},$$

on $z = \overline{z}$

$$\partial_{\nu_z} G_1(z,\zeta) = -4\operatorname{Im}\zeta\left\{\frac{1}{|\zeta - z|^2} - \frac{1}{|1-z\zeta|^2} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{|r^{2n}\zeta - z|^2} + \frac{1}{|\zeta - r^{2n}z|^2} - \frac{1}{|r^{2n}-z\zeta|^2} - \frac{1}{|1-r^{2n}z\zeta|^2}\right]\right\}$$

Thus Theorem 2 in the case of R^+ is reformulated.

Theorem 9. *Any $w \in C^2(R^+;\mathbb{C}) \cap C(\overline{R^+};\mathbb{C})$ is representable as*

$$
\begin{aligned}
w(z) = \frac{1}{2\pi i} \int\limits_{\substack{\partial R^+,\\ 0<\operatorname{Im}\zeta}} & w(\zeta) \Bigg\{ \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\overline{\zeta} - z}} - \frac{\overline{\zeta}}{\overline{\zeta} - z} - \frac{\zeta}{\zeta - \overline{z}} \\
&+ \sum_{n=1}^{\infty} r^{2n} \Bigg[\frac{\zeta}{r^{2n}\zeta - z} - \frac{\zeta}{r^{2n}\zeta - \overline{z}} + \frac{\overline{z}}{r^{2n}\overline{z} - \zeta} - \frac{z}{r^{2n}z - \zeta} \\
&+ \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - \overline{z}} - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - z} + \frac{z}{r^{2n}z - \overline{\zeta}} - \frac{\overline{z}}{r^{2n}\overline{z} - \overline{\zeta}} \Bigg] \Bigg\} \frac{d\zeta}{\zeta} \\
&+ \frac{z - \overline{z}}{2\pi i} \int\limits_{\partial R^+ \cap \mathbb{R}} w(t) \Bigg\{ \frac{1}{|t - z|^2} - \frac{1}{|1 - zt|^2} \\
&+ \sum_{n=1}^{\infty} r^{2n} \Bigg[\frac{1}{|r^{2n}z - t|^2} + \frac{1}{|z - r^{2n}t|^2} - \frac{1}{|r^{2n} - zt|^2} - \frac{1}{|1 - r^{2n}zt|^2} \Bigg] \Bigg\} dt \\
&- \frac{1}{\pi} \int\limits_{R^+} w_{\zeta\overline{\zeta}}(\zeta) G_1(z,\zeta) d\xi d\eta.
\end{aligned}
\tag{3.6}
$$

Theorem 10. *The Dirichlet problem*

$$w_{z\overline{z}} = f \ \text{ in } \ R^+, \quad w = \gamma \ \text{ on } \ \partial R^+,$$

for $f \in L_2(R^+;\mathbb{C}) \cap C(R^+;\mathbb{C})$, $\gamma \in C(\partial R^+;\mathbb{C})$, $\gamma(-1) = \gamma(-r) = \gamma(r) = \gamma(1) = 0$, is uniquely solvable. The solution is

$$
\begin{aligned}
w(z) = \frac{1}{2\pi i} \int\limits_{\substack{\partial R^+,\\ 0<\operatorname{Im}\zeta}} & \gamma(\zeta) \Bigg\{ \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\overline{\zeta} - z}} - \frac{\overline{\zeta}}{\overline{\zeta} - z} - \frac{\zeta}{\zeta - \overline{z}} \\
&+ \sum_{n=1}^{\infty} r^{2n} \Bigg[\frac{\zeta}{r^{2n}\zeta - z} - \frac{\zeta}{r^{2n}\zeta - \overline{z}} + \frac{\overline{z}}{r^{2n}\overline{z} - \zeta} - \frac{z}{r^{2n}z - \zeta} \\
&+ \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - \overline{z}} - \frac{\overline{\zeta}}{r^{2n}\overline{\zeta} - z} + \frac{z}{r^{2n}z - \overline{\zeta}} - \frac{\overline{z}}{r^{2n}\overline{z} - \overline{\zeta}} \Bigg] \Bigg\} \frac{d\zeta}{\zeta} \\
&+ \frac{z - \overline{z}}{2\pi i} \int\limits_{\partial R^+ \cap \mathbb{R}} \gamma(t) \Bigg\{ \frac{1}{|t - z|^2} - \frac{1}{|1 - zt|^2} \\
&+ \sum_{n=1}^{\infty} r^{2n} \Bigg[\frac{1}{|r^{2n}z - t|^2} + \frac{1}{|z - r^{2n}t|^2} - \frac{1}{|r^{2n} - zt|^2} - \frac{1}{|1 - r^{2n}zt|^2} \Bigg] \Bigg\} dt \\
&- \frac{1}{\pi} \int\limits_{R^+} f(\zeta) G_1(z,\zeta) d\xi d\eta.
\end{aligned}
\tag{3.7}
$$

Proof. Let w_1, w_2 and $\widehat{w}$ denote the three different parts of the boundary integral along the respective paths on $|\zeta| = 1$, $|\zeta| = r$ and $\mathbb{R}$. Then as before for the Schwarz problem

$$w_1(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Gamma(\zeta) \Big\{ \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 - \sum_{n=1}^{\infty} r^{2n} \Big[\frac{1 - |z|^2}{(r^{2n}\zeta - z)(r^{2n}\overline{z} - \overline{\zeta})} + \frac{1 - |z|^2}{(r^{2n}\overline{\zeta} - \overline{z})(r^{2n}z - \zeta)} \Big] \Big\} \frac{d\zeta}{\zeta},$$

with

$$\Gamma(\zeta) = \begin{cases} \gamma(\zeta), & 0 \leqslant \operatorname{Im} \zeta, \\ -\gamma(\overline{\zeta}), & \operatorname{Im} \zeta < 0, \end{cases} \quad |\zeta| = 1,$$

$$w_2(z) = -\frac{1}{2\pi i} \int\limits_{|\zeta|=r} \Gamma(\zeta) \Big\{ \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 - \sum_{n=1}^{\infty} r^{2n} \Big[\frac{r^2 - |z|^2}{(r^{2n}\zeta - z)(r^{2n}\overline{z} - \overline{\zeta})} + \frac{r^2 - |z|^2}{(r^{2n}\overline{\zeta} - \overline{z})(r^{2n}z - \zeta)} \Big] \Big\} \frac{d\zeta}{\zeta},$$

with

$$\Gamma(\zeta) = \begin{cases} \gamma(\zeta), & 0 \leqslant \operatorname{Im} \zeta, \\ -\gamma(\overline{\zeta}), & \operatorname{Im} \zeta < 0, \end{cases} \quad |\zeta| = r,$$

$$\widehat{w}(z) = \frac{z - \overline{z}}{2\pi i} \Big[\int\limits_{-\frac{1}{r}}^{-r} + \int\limits_{r}^{\frac{1}{r}} \Big] \widehat{\Gamma}(t) \Big\{ \frac{1}{|t - z|^2} + \sum_{n=1}^{\infty} r^{2n} \Big[\frac{1}{|r^{2n}t - z|^2} + \frac{1}{|r^{2n}z - t|^2} \Big] \Big\} dt,$$

with

$$\widehat{\Gamma}(t) = \begin{cases} \gamma(t), & r \leqslant |t| \leqslant 1, \\ -\gamma\Big(\dfrac{1}{t}\Big), & 1 \leqslant |t| \leqslant \dfrac{1}{r}, \end{cases} \quad t \in \mathbb{R},$$

and

$$\widehat{w}(z) = \frac{z - \overline{z}}{2\pi i} \Big[\int\limits_{-1}^{-r^2} + \int\limits_{r^2}^{1} \Big] \widetilde{\Gamma}(t) \Big\{ \frac{1}{|t - z|^2} + \sum_{n=1}^{\infty} r^{2n} \Big[\frac{1}{|r^{2n}t - z|^2} + \frac{1}{|r^{2n}z - t|^2} \Big] \Big\} dt$$

with

$$\widetilde{\Gamma}(t) = \begin{cases} \gamma(t), & r \leqslant |t| \leqslant 1, \\ -\gamma\Big(\dfrac{r^2}{t}\Big), & r^2 \leqslant |t| \leqslant r, \end{cases} \quad t \in \mathbb{R}.$$

Thus for any point of continuity ζ_0 of γ on ∂R^+

$$\lim_{z \to \zeta_0} [w_1(z) + w_2(z) + \widehat{w}(z)] = \gamma(\zeta_0)$$

holds. This is even true at the corner points of ∂R^+ in analogy of the proof of the Schwarz problem, due to the vanishing of the boundary function there. ■

The harmonic Neumann function for R^+ is

$$\begin{aligned} N_1(z,\zeta) &= 2\log\frac{|z\zeta|^2}{r^2} - \log|(\zeta-z)(\overline{\zeta}-z)(1-z\overline{\zeta})(1-z\zeta)|^2 \\ &\quad + \sum_{n=1}^{\infty}\Big[4\log|z\zeta|^2 - \log\Big|(z-r^{2n}\zeta)(z-r^{2n}\overline{\zeta})(z\zeta-r^{2n})(z\overline{\zeta}-r^{2n}) \\ &\quad \times(\zeta-r^{2n}z)(\overline{\zeta}-r^{2n}z)(1-r^{2n}z\zeta)(1-r^{2n}z\overline{\zeta})\Big|^2\Big]. \end{aligned} \tag{3.8}$$

Its behavior for $0<|z|<1$ is

$$\begin{aligned} N_1(z,\zeta) = 2\Big\{&\log\frac{|z|^2}{r^2} - \log|(1-z\zeta)(1-z\overline{\zeta})|^2 \\ &-\sum_{n=1}^{\infty}\log\Big|\Big(1-\frac{r^{2n}\zeta}{z}\Big)\Big(1-\frac{r^{2n}\overline{\zeta}}{z}\Big)(1-r^{2n}z\zeta)(1-r^{2n}z\overline{\zeta})\Big|^2\Big\} \end{aligned}$$

on $|\zeta|=1$ and

$$\begin{aligned} N_1(z,\zeta) = 2\Big\{&\log|z\zeta|^2 - \log|(1-z\zeta)(1-z\overline{\zeta})|^2 \\ &-\sum_{n=1}^{\infty}\log\Big|\Big(1-\frac{r^{2n}}{z\zeta}\Big)\Big(1-\frac{r^{2n}}{z\overline{\zeta}}\Big)(1-r^{2n}z\zeta)(1-r^{2n}z\overline{\zeta})\Big|^2\Big\} \quad \text{on } |\zeta|=r. \end{aligned}$$

For $0<\operatorname{Im} z$ on $\operatorname{Im}\zeta=0$

$$\begin{aligned} (z\partial_z+\overline{z}\partial_{\overline{z}})N_1(z,\zeta) = 2\Big\{&\frac{\zeta}{\zeta-z}+\frac{\zeta}{\zeta-\overline{z}}+\frac{z\zeta}{1-z\zeta}+\frac{\overline{z}\zeta}{1-\overline{z}\zeta} \\ &-\sum_{n=1}^{\infty}r^{2n}\Big[\frac{\zeta}{z-r^{2n}\zeta}+\frac{\zeta}{\overline{z}-r^{2n}\zeta}+\frac{1}{z\zeta-r^{2n}}+\frac{1}{\overline{z}\zeta-r^{2n}} \\ &-\frac{z}{\zeta-r^{2n}z}-\frac{\overline{z}}{r^{2n}\zeta-\overline{z}}-\frac{z\zeta}{1-r^{2n}z\zeta}-\frac{\overline{z}\zeta}{1-r^{2n}\overline{z}\zeta}\Big]\Big\}. \end{aligned}$$

Thus for $\operatorname{Im}\zeta=0$

$$\partial_{\nu_z}N_1(z,\zeta) = \begin{cases} 0, & \text{if } |z|=1, \\ 4, & \text{if } |z|=r. \end{cases}$$

Also for $r<|z|<1$ on $\operatorname{Im}\zeta=0$

$$\begin{aligned} -i(\partial_z-\partial_{\overline{z}})N_1(z,\zeta) &= 2i(z-\overline{z})\Big\{\frac{1}{|z|^2}-\frac{1}{|\zeta-z|^2}+\frac{z+\overline{z}}{|1-z\zeta|^2} \\ &+\sum_{n=1}^{\infty}r^{2n}\Big[\frac{(r^{2n}\zeta-z-\overline{z})\zeta}{|z|^2|r^{2n}\zeta-z|^2}+\frac{r^{2n}(z+\overline{z})}{|1-r^{2n}z\zeta|^2}-\frac{r^{2n}+z+\overline{z}}{|z|^2|z\zeta-r^{2n}|^2}+\frac{r^{2n}}{|\zeta-r^{2n}z|^2}\Big]\Big\}, \end{aligned}$$

so that for $\operatorname{Im}\zeta = 0$

$$\partial_{\nu_z} N_1(z,\zeta) = -2i\Big[\frac{z-\overline{z}}{|\zeta - z|^2} + \frac{z-\overline{z}}{|z|^2} - \frac{(z-\overline{z})(z+\overline{z})}{|1-z\zeta|^2}\Big], \quad \text{if } z = \overline{z}.$$

On $|\zeta| = 1$, $0 < \operatorname{Im}\zeta$ for $r < |z| < 1$

$$\begin{aligned}(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta) = 2\Big\{ & \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 + \frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\zeta}{\zeta - \overline{z}} - 1 \\ & - \sum_{n=1}^{\infty} r^{2n}\Big[\frac{\overline{\zeta}}{z - r^{2n}\overline{\zeta}} + \frac{\zeta}{\overline{z} - r^{2n}\zeta} + \frac{\zeta}{z - r^{2n}\zeta} + \frac{\overline{\zeta}}{\overline{z} - r^{2n}\overline{\zeta}} \\ & - \frac{z}{\zeta - r^{2n}z} - \frac{\overline{z}}{\overline{\zeta} - r^{2n}\overline{z}} - \frac{z}{\overline{\zeta} - r^{2n}z} - \frac{\overline{z}}{\zeta - r^{2n}\overline{z}}\Big]\Big\},\end{aligned}$$

so that

$$\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} 2\Big[\dfrac{\zeta}{\zeta - z} + \dfrac{\overline{\zeta}}{\overline{\zeta - z}} - 1 + \dfrac{\overline{\zeta}}{\overline{\zeta} - z} + \dfrac{\zeta}{\zeta - \overline{z}} - 1\Big], & \text{if } |z| = 1, \\ 4, & \text{if } |z| = r. \end{cases}$$

Moreover, for $|\zeta| = 1$, $0 < \operatorname{Im}\zeta$

$$\begin{aligned}-i(\partial_z - \partial_{\overline{z}})N_1(z,\zeta) = 2i\Big\{ & \frac{z-\overline{z}}{|z|^2} + \frac{z-\overline{z}}{(\zeta - z)(\zeta - \overline{z})} - \frac{z-\overline{z}}{(\overline{\zeta - z})(\overline{\zeta} - z)} \\ & - \sum_{n=1}^{\infty} r^{2n}\Big[\frac{\overline{\zeta}}{z(z - r^{2n}\overline{\zeta})} - \frac{\zeta}{\overline{z}(\overline{z} - r^{2n}\zeta)} + \frac{\zeta}{z(z - r^{2n}\zeta)} \\ & - \frac{\overline{\zeta}}{\overline{z}(\overline{z} - r^{2n}\overline{\zeta})} - \frac{\overline{\zeta}}{1 - r^{2n}z\overline{\zeta}} + \frac{\zeta}{1 - r^{2n}\overline{z}\zeta} - \frac{\zeta}{1 - r^{2n}z\zeta} \\ & + \frac{\overline{\zeta}}{1 - r^{2n}\overline{z\zeta}}\Big]\Big\},\end{aligned}$$

so that

$$\partial_{\nu_z} N_1(z,\zeta) = 0 \ \text{ if } z = \overline{z}.$$

For $r < |z| < 1$ on $|\zeta| = r$, $0 < \operatorname{Im}\zeta$

$$\begin{aligned}(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta) = 2\Big\{ & \frac{z\overline{\zeta}}{1 - z\overline{\zeta}} + \frac{z\zeta}{1 - z\zeta} + \frac{\overline{z}\zeta}{1 - \overline{z}\zeta} + \frac{\overline{z\zeta}}{1 - \overline{z\zeta}} \\ & - \sum_{n=1}^{\infty} r^{2n}\Big[\frac{1}{z\zeta - r^{2n}} + \frac{1}{z\overline{\zeta} - r^{2n}} + \frac{1}{\overline{z\zeta} - r^{2n}} + \frac{1}{\overline{z}\zeta - r^{2n}} \\ & - \frac{r^2 z}{\overline{\zeta} - r^{2(n+1)}z} - \frac{r^2 z}{\zeta - r^{2(n+1)}z} - \frac{r^2\overline{z}}{\zeta - r^{2(n+1)}\overline{z}} \\ & - \frac{r^2\overline{z}}{\overline{\zeta} - r^{2(n+1)}\overline{z}}\Big]\Big\}.\end{aligned}$$

Hence,

$$\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} 0, & \text{if } |z| = 1, \\ 2\Big[\dfrac{\zeta}{\zeta - z} + \dfrac{\overline{\zeta}}{\overline{\overline{\zeta} - z}} + \dfrac{\overline{\zeta}}{\overline{\zeta} - z} + \dfrac{\zeta}{\zeta - \overline{z}}\Big], & \text{if } |z| = r. \end{cases}$$

Finally, for $|\zeta| = r$, $0 < \operatorname{Im}\zeta$

$$\begin{aligned} -i(\partial_z - \partial_{\overline{z}})N_1(z,\zeta) = 2i\Big\{ &-\frac{\zeta}{1 - z\zeta} - \frac{\overline{\zeta}}{1 - z\overline{\zeta}} + \frac{\overline{\zeta}}{1 - \overline{z\zeta}} + \frac{\zeta}{1 - \overline{z}\zeta} \\ &+ \sum_{n=1}^{\infty} r^{2n}\Big[\frac{1}{z(z\zeta - r^{2n})} + \frac{1}{z(z\overline{\zeta} - r^{2n})} - \frac{1}{\overline{z}(\overline{z\zeta} - r^{2n})} \\ &- \frac{1}{\overline{z}(\overline{z}\zeta - r^{2n})} - \frac{\overline{\zeta}}{1 - r^{2n}z\overline{\zeta}} + \frac{\zeta}{1 - r^{2n}\overline{z}\zeta} \\ &- \frac{\zeta}{1 - r^{2n}z\zeta} + \frac{\overline{\zeta}}{1 - r^{2n}\overline{z\zeta}}\Big]\Big\}, \end{aligned}$$

so that

$$\partial_{\nu_z} N_1(z,\zeta) = 0 \quad \text{if } z = \overline{z}.$$

For $\zeta \in R^+$

$$\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} 0, & |z| = 1,\ 0 < \operatorname{Im} z, \\ 4, & |z| = r,\ 0 < \operatorname{Im} z, \\ 0, & z = \overline{z},\ r < |z| < 1, \end{cases}$$

holds.

Finally, N_1 satisfies the normalization condition

$$\begin{aligned} \frac{1}{\pi i}\int\limits_{\substack{|z|=r,\\ 0<\operatorname{Im} z}} N_1(z,\zeta)\frac{dz}{z} = &-\frac{1}{\pi i}\int\limits_{\substack{|z|=r,\\ 0<\operatorname{Im} z}} \log\Big|\Big(1 - \frac{z}{\zeta}\Big)\Big(1 - \frac{z}{\overline{\zeta}}\Big)(1 - z\overline{\zeta})(1 - z\zeta)\Big|^2 \frac{dz}{z} \\ &- \sum_{n=1}^{\infty}\frac{1}{\pi i}\int\limits_{\substack{|z|=r,\\ 0<\operatorname{Im} z}} \log\Big|\Big(1 - \frac{r^{2n}z}{\zeta}\Big)\Big(1 - \frac{r^{2n}z}{\overline{\zeta}}\Big)(1 - r^{2n}z\overline{\zeta})(1 - r^{2n}z\zeta) \\ &\times \Big(1 - \frac{r^{2n}\zeta}{z}\Big)\Big(1 - \frac{r^{2n}\overline{\zeta}}{z}\Big)\Big(1 - \frac{r^{2n}}{z\zeta}\Big)\Big(1 - \frac{r^{2n}}{z\overline{\zeta}}\Big)\Big|^2 \frac{dz}{z} \\ = &-\frac{1}{\pi i}\int\limits_{|z|=r} \log\Big|\Big(1 - \frac{z}{\zeta}\Big)(1 - z\zeta)\Big|^2 \frac{dz}{z} \\ &- \sum_{n=1}^{\infty}\frac{1}{\pi i}\int\limits_{|z|=r} \log\Big|\Big(1 - \frac{r^{2n}\zeta}{z}\Big)\Big(1 - \frac{r^{2n}}{z\zeta}\Big)\Big(1 - \frac{r^{2n}z}{\zeta}\Big)(1 - r^{2n}z\zeta)\Big|^2 \frac{dz}{z} = 0 \end{aligned}$$

by the Gauss mean value theorem for harmonic functions and because

$$\int_0^\pi \log(4\sin^2\varphi)d\varphi = 0.$$

With this Neumann function Theorem 2 provides the Neumann representation for R^+.

Theorem 11. *Any $w \in C^2(R^+;\mathbb{C}) \cap C^1(\overline{R^+};\mathbb{C})$ is representable in the form*

$$\begin{aligned} w(z) = & \frac{1}{\pi}\int_0^\pi w(re^{i\varphi})d\varphi + \frac{1}{2\pi i}\int_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \partial_{\nu_\zeta} w(\zeta)\Big\{\log\frac{|z|^2}{r^2} - \log|(1-z\zeta)(1-z\overline{\zeta})|^2 \\ & - \sum_{n=1}^\infty \log\Big|\Big(1-\frac{r^{2n}\zeta}{z}\Big)\Big(1-\frac{r^{2n}\overline{\zeta}}{z}\Big)(1-r^{2n}z\zeta)(1-r^{2n}z\overline{\zeta})\Big|^2\Big\}\frac{d\zeta}{\zeta} \\ & + \frac{1}{2\pi i}\int_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta}} \partial_{\nu_\zeta} w(\zeta)\big\{\log|(1-z\zeta)(1-z\overline{\zeta})|^2 - \log|z\zeta|^2 \\ & + \sum_{n=1}^\infty \log\Big|\Big(1-\frac{r^{2n}}{z\zeta}\Big)\Big(1-\frac{r^{2n}}{z\overline{\zeta}}\Big)(1-r^{2n}z\zeta)(1-r^{2n}z\overline{\zeta})\Big|^2\Big\}\frac{d\zeta}{\zeta} \\ & + \frac{1}{2\pi}\int_{\partial R^+\cap\mathbb{R}} \partial_{\nu_t} w(t)\Big\{\log\frac{|z|^2}{r^2} - \log\Big|\Big(1-\frac{z}{t}\Big)(1-zt)\Big|^2 \\ & - \sum_{n=1}^\infty \log\Big|\Big(1-\frac{r^{2n}t}{z}\Big)\Big(1-\frac{r^{2n}}{zt}\Big)\Big(1-\frac{r^{2n}z}{t}\Big)(1-r^{2n}zt)\Big|^2\Big\}dt \\ & - \frac{1}{\pi}\int_{R^+} w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta. \end{aligned} \tag{3.9}$$

Theorem 12. *The Neumann problem*

$$w_{z\overline{z}} = f \ \text{ in } R^+, \quad \partial_\nu w = \gamma \ \text{ on } \partial R^+, \quad \frac{1}{\pi}\int_0^\pi w(re^{i\varphi})d\varphi = c,$$

with $f \in L_2(R^+;\mathbb{C}) \cap C(R^+;\mathbb{C})$, $\gamma \in C(\partial R^+;\mathbb{C})$, $c \in \mathbb{C}$, $\gamma(-1) = \gamma(-r) = \gamma(r) = \gamma(1) = 0$ is solvable if and only if

$$\frac{1}{2\pi}\int_{\partial R^+} \gamma(\zeta)d\zeta = \frac{2}{\pi}\int_{R^+} f(\zeta)d\xi d\eta. \tag{3.10}$$

The solution is unique and given as

$$
\begin{aligned}
w(z) = c &+ \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \Big\{ \log \frac{|z|^2}{r^2} - \log |(1-z\zeta)(1-z\overline{\zeta})|^2 \\
&- \sum_{n=1}^{\infty} \log \Big| \Big(1-\frac{r^{2n}\zeta}{z}\Big)\Big(1-\frac{r^{2n}\overline{\zeta}}{z}\Big)\, (1-r^{2n}z\zeta)(1-r^{2n}z\overline{\zeta}) \Big|^2 \Big\} \frac{d\zeta}{\zeta} \\
&+ \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \big\{ \log |(1-z\zeta)(1-z\overline{\zeta})|^2 - \log |z\zeta|^2 \\
&+ \sum_{n=1}^{\infty} \log \Big| \Big(1-\frac{r^{2n}}{z\zeta}\Big)\Big(1-\frac{r^{2n}}{z\overline{\zeta}}\Big)\, (1-r^{2n}z\zeta)(1-r^{2n}z\overline{\zeta}) \Big|^2 \Big\} \frac{d\zeta}{\zeta} \\
&+ \frac{1}{2\pi} \int\limits_{\partial R^+\cap\mathbb{R}} \gamma(t) \Big\{ \log \frac{|z|^2}{r^2} - \log \Big| \Big(1-\frac{z}{t}\Big)(1-zt) \Big|^2 \\
&- \sum_{n=1}^{\infty} \log \Big| \Big(1-\frac{r^{2n}t}{z}\Big)\Big(1-\frac{r^{2n}}{zt}\Big)\Big(1-\frac{r^{2n}z}{t}\big)\, (1-r^{2n}zt) \Big|^2 \Big\} dt \\
&- \frac{1}{\pi} \int_{R^+} f(\zeta) N_1(z,\zeta) d\xi d\eta.
\end{aligned}
\tag{3.11}
$$

Proof. The normalization condition

$$
\frac{1}{\pi} \int\limits_0^{\pi} w(re^{i\varphi}) d\varphi = c
$$

follows from the normalization condition of $N_1(z,\zeta)$.

Denote by w_0 the sum of boundary integrals on the right-hand side of (3.11). Then for $|z_0| = 1$, $0 < \operatorname{Im} z_0$, $z \in R^+$,

$$
\begin{aligned}
\partial_\nu w_0(z_0) &= \lim_{z\to z_0} \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta) \Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 + \frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\zeta}{\zeta - \overline{z}} - 1 \Big] \frac{d\zeta}{\zeta} \\
&= \lim_{z\to z_0} \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Gamma(\zeta) \Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} = 1 \Big] \frac{d\zeta}{\zeta} = \Gamma(z_0) = \gamma(z_0)
\end{aligned}
$$

with

$$
\Gamma(\zeta) = \begin{cases} \gamma(\zeta), & |\zeta| = 1,\ 0 \leqslant \operatorname{Im}\zeta, \\ -\gamma(\overline{\zeta}), & |\zeta| = 1,\ \operatorname{Im}\zeta < 0. \end{cases}
$$

For $|z_0| = r$, $0 < \operatorname{Im} z_0$, $z \in R^+$,

$$\begin{aligned}
&\partial_\nu w_0(z_0) \\
&= \frac{1}{\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta)\frac{d\zeta}{\zeta} - \lim_{z\to z_0} \frac{1}{2\pi i} \int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta)\left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} + \frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\zeta}{\zeta - \overline{z}}\right]\frac{d\zeta}{\zeta} \\
&\quad + \frac{1}{\pi}\int\limits_{\partial R^+\cap\mathbb{R}} \gamma(t)dt = \frac{1}{\pi}\int\limits_{\partial R^+} \gamma(\zeta)ds_\zeta - \lim_{z\to z_0}\frac{1}{2\pi i}\int\limits_{|\zeta|=r} \Gamma(\zeta)\left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\right]\frac{d\zeta}{\zeta} \\
&= \gamma(z_0) + \frac{1}{\pi}\int\limits_{\partial R^+} \gamma(\zeta)ds_\zeta
\end{aligned}$$

with

$$\Gamma(\zeta) = \begin{cases} \gamma(\zeta), & |\zeta| = r,\ \ 0 \leqslant \operatorname{Im}\zeta, \\ -\gamma(\overline{\zeta}), & |\zeta| = r,\ \ \operatorname{Im}\zeta < 0. \end{cases}$$

For $z_0 = \overline{z_0}$, $r < |z_0| < 1$, $z \in R^+$,

$$\begin{aligned}
\partial_\nu w_0(z_0) &= \lim_{z\to z_0}\frac{1}{2\pi}\int\limits_{\partial R^+\cap\mathbb{R}} \gamma(t)\left[\frac{z-\overline{z}}{|t-z|^2} - \frac{z-\overline{z}}{|z|^2} - \frac{(z-\overline{z})(z+\overline{z})}{|1-zt|^2}\right]dt \\
&= \lim_{z\to z_0}\frac{1}{2\pi}\left[\int_{-\frac{1}{r}}^{-r} + \int_{r}^{\frac{1}{r}}\right]\widehat{\Gamma}(t)\frac{z-\overline{z}}{|t-z|^2}dt = \widehat{\Gamma}(z_0) = \gamma(z_0)
\end{aligned}$$

with

$$\widehat{\Gamma}(t) = \begin{cases} \gamma(t), & r \leqslant |t| \leqslant 1, \\ -\gamma\left(\dfrac{1}{t}\right), & 1 < |t| \leqslant \dfrac{1}{r}. \end{cases}$$

Moreover, for $|z_0| = r^2$, $0 < \operatorname{Im} z_0$

$$\partial_\nu[w(z_0) - w_0(z_0)] = -\frac{4}{\pi}\int\limits_{R^+} f(\zeta)d\xi d\eta,$$

while this is vanishing for the other parts of the boundary. Hence, on ∂R^+

$$\partial_\nu w(z) = \gamma(z) + \beta(z)$$

with β vanishing on $\partial R^+ \setminus \{|z| = r,\ 0 < \operatorname{Im} z\}$ and

$$\beta(z) = \frac{1}{\pi}\int\limits_{\partial R^+} \gamma(\zeta)ds_\zeta - \frac{4}{\pi}\int\limits_{R^+} f(\zeta)d\xi d\eta. \qquad \blacksquare$$

References

[1] S.A. Abdymanapov, H. Begehr, G. Harutyunyan, A.B. Tungatarov, *Four boundary value problems for the Cauchy-Riemann equation in a quarter palne*, More Progresses in Analysis. Proc. 5th Intern. ISAAC Congress, Catania, Italy, 2005; Eds. H. Begehr, F. Nicolosi. – Singapore: World Sci., (2009), 1137–1147.

[2] S.A. Abdymanapov, H. Begehr, A.B. Tungatarov, *Some Schwarz problems in a quarter plane*, Eurasian Math. J. **3** (2005), 22–35.

[3] H.G.W. Begehr, *Complex Analytic Methods for Partial Differential Equations: an introductory text*, Singapore: World Scientific, 1994.

[4] H. Begehr, *Boundary value problems for the Bitsadze equation*, Memoirs Diff. Eqs. Math. Phys. **33** (2004), 5–23.

[5] H. Begehr, *Boundary value problems in complex analysis, I, II*, F. Bol. Asoc. Mat. Venezolana. V, XII (2005), 65–85, 217–250.

[6] H. Begehr, *The main theorem of calculus in complex analysis*. Ann. EAS. (2005), 184–210.

[7] H. Begehr, *Six biharmonic Dirichlet problems in complex ananlysis*, Function spaces in complex and Clifford analysis; Eds. Le Hung Son, W. Tutschke. – Hanoi: National Univ. Publ. (2007), 243–252.

[8] H. Begehr, *A particular polyharmonic Dirichlet problem*, Complex Analysis and Potential Theory; Eds. T. Aliev Azeroglu, P.M. Tamrazov. Singapore: World Sci. (2007), 84–115.

[9] H. Begehr, J.Y. Du, Y.F. Wang, *A Dirichlet problem for polyharmonic functions*, Ann. Mat. Pura Appl. **187** (2008), 435–457.

[10] H. Begehr, E.A. Gaertner, *Dirichlet problem for the inhomogeneous polyharmonic equation in the upper half plane*. Georg. Math. J., **14** (1) (2007), 33–52.

[11] H. Begehr, G. Harutyunyan, *Robin boundary value problem for the Cauchy-Riemann operator*, Complex Variables. Theory Appl. **50** (2005), 1125–1136.

[12] H. Begehr, G. Harutyunyan, *Robin boundary value problem for the Poisson equation*, J. Anal. Appl. **4** (2006), 201–213.

[13] H. Begehr, G. Harytyunyan, *Complex boundary value problems in a quarter plane*, Proc. 13th Intern. Conf. Finite Infinite Dimensional Complex Anal. Appl., Shantou, China; Eds Y. Wang et al. New Jersey: World. Sci. (2006), 1–10.

[14] H. Begehr, D. Schmersau, *The Schwarz problem for polyanalytic functions*, ZAA. **24** (2005), 341–351.

[15] H. Begehr, T. Vaitekhovich, *Complex partial differential equations in a manner of I.N. Vekua*, TICMI Lecture Notes. **8** (2007), 15–26.

[16] H. Begehr, T. Vaitekhovich, *Iterated Dirichlet problem for higher order Poisson equation*, Le Matematiche. V, LXIII (2008), 139–154.

[17] H. Begehr, T. Vaitekhovich, *Green functions in complex plane domains*, Uzbek. Math. J. **4** (2008), 29–34.

[18] H. Begehr, T. Vaitekhovich, *Polyharmonic Green functions for particular plane domains*, Proc. 6th Congress of Roman. Mathematicians, Bucharest,

2007; Eds. L. Beznea et al. Bucharest: Publ. House Roman. Acad. Sci. **1** (2009), 119–126.

[19] H. Begehr, T. Vaitekhovich, *Some harmonic Robin functions in the complex plane.* J. Analysis and Appl. **1** (2009), 1–13.

[20] H. Begehr, C.J. Vanegas, *Iterated Neumann problem for the higher order Poisson equation.* Math. Nach. **279** (2006), 38–57.

[21] H. Begehr, T.N.H. Vu, Z.X. Zhang, *Polyharmonic Dirichlet problems*, Proceedings of the Steklov Institute of Math. **255** (2006), 13–34.

[22] H. Begehr, Y.F. Wang, *A new approach for solving a Dirichlet problem for polyharmonic functions*, Complex Variables. Ell. Eqs. **52** (2007), 907–920.

[23] B. Bojarski, *Boundary value problems for elliptic systems with two independent variables*, Ph.D. thesis, MGU, Moscow, 1954 (Russian).

[24] B. Bojarski, *Geometry of the general Beltrami equations*, Complex Analysis and Potential Theory; Eds. T. Aliev Azeroglu, P.M. Tamrazov. Singapore: World Sci. (2007), 66–83.

[25] B. Bojarski, *On the Beltrami equation, once again: 54 years later*, Preprint, IMPAM, Warsaw, 2008.

[26] A. Dinghas, *Vorlesungen über Funktiontheorie*, Berlin: Springer Verlag. 1961.

[27] Z.H. Du, *Boundary value problems for higher order complex partial differential equations*, Ph.D. thesis, FU Berlin, 2008: www.diss.fu-berlin.de/diss/receive/FUDISS_thesis_000000003677

[28] E. Gaertner, *Basic complex boundary value problems in the upper half plane*, Ph.D. thesis, FU Berlin, 2006: www.diss.fu-berlin.de /diss/receive/FUDISS_thesis_000000002129

[29] T.S. Vaitekhovich, *Boundary value problems to second order complex partial differential equations in a ring domain*, Šauliai Mathematical Seminar. **2** (2007), 117–146.

[30] T.S. Vaitekhovich, *Boundary value problems to first order complex partial differential equations in a ring domain*, Integral Transf. Special Funct. **19** (2008), 211–233.

[31] T. Vaitsiakhovich, *Boundary value problems for complex partial differential equations in a ring domain*, Ph.D. thesis, FU Berlin, 2008: www.diss.fu-berlin.de/diss/receive/FUDISS_thesis_000000003859

[32] I.N. Vekua, *Generalized Analytic Functions*, Pergamon Press: Oxford, 1962.

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