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EXPONENTIAL SUMS WITH MULTIPLICATIVE COEFFICIENTS OVER SMOOTH INTEGERS

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Dedicated to Professor Eduard Wirsing
for his 75th birthday

Abstract: In a recent paper A. Sankaranarayanan and the author using a novel method prove a special case of a recent result of G. Bachmann on exponential sums with multiplicative coefficients. Here we apply this method to the case in which the exponential sum is extended over smooth numbers only.
Keywords: Exponential sum, multiplicative function, smooth numbers, Dickman function.

1. Introduction

Let $\mathfrak{F}$ be the class of complex-valued multiplicative functions f with $|f| \leqslant 1$. Let $\mathbf{e}(t)$ denote the complex number $e^{2\pi i t}$ throughout the paper. For any real numbers $x \geqslant 3$, and α and for $f \in \mathfrak{F}$, we write the general exponential sum as

$$F(x,\alpha) = \sum_{n \leqslant x} f(n)\mathbf{e}(n\alpha). \tag{1.1}$$

The problem of obtaining bounds for $F(x,\alpha)$ uniform in $f \in \mathfrak{F}$ has been first considered by H. Daboussi [2]. He showed that if $\left|\alpha - \frac{s}{r}\right| \leqslant \frac{1}{r^2}$ and $3 \leqslant r \leqslant (\frac{x}{\log x})^{\frac{1}{2}}$, for some coprime integers s and r, then uniformly for all $f \in \mathfrak{F}$, we have

$$F(x,\alpha) \ll \frac{x}{(\log\log r)^{\frac{1}{2}}}.$$

From this estimate, one observes that for every irrational α, we have

$$\lim_{x\to\infty} \frac{1}{x}\, F(x,\alpha) = 0$$

uniformly for all $f \in \mathfrak{F}$.

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The question of characterizing those functions f such that for every irrational α having the property

$$\frac{1}{x}\ F(x,\alpha) = o\left(\frac{1}{x}\left|\sum_{n\leqslant x} f(n)\right|\right), \tag{1.2}$$

was considered first by Dupain, Hall and Tenenbaum in [4]. An interesting special case is when f is a characteristic function of integers free of prime factors greater than $y \geqslant 2$. Fouvry and Tenenbaum (see [5]) obtained sharp estimates for the corresponding exponential sum providing a quantitative version of (1.2) for a wide range of parameters x and y. On the other hand, an important advance was established by Montgomery and Vaughan (see [8]) who improved the original estimate of Daboussi. If $\left|\alpha - \frac{s}{r}\right| \leqslant \frac{1}{r^2}$ and $2 \leqslant R \leqslant r \leqslant \frac{x}{R}$, for some coprime integers s and r, then uniformly for all $f \in \mathfrak{F}$, they proved that

$$F(x,\alpha) \ll \frac{x}{\log x} + \frac{x}{\sqrt{R}}(\log R)^{\frac{3}{2}}. \tag{1.3}$$

In addition, it was shown that apart from the logarithmic factor, the above estimate is sharp. Indeed, they established that

(i) For any real $x \geqslant 3$ and any α, there is an $f \in \mathfrak{F}$ such that $|F(x,\alpha)| \gg \frac{x}{\log x}$.

(ii) If $r \leqslant x^{\frac{1}{2}}$ and $(s,r) = 1$, then there is an $f \in \mathfrak{F}$ such that $\left|F(x, \frac{s}{r})\right| \gg \frac{x}{\sqrt{r}}$.

(iii) If $\frac{x}{(\log x)^3} \leqslant T \leqslant x$, then there exist coprime integers s and r and $f \in \mathfrak{F}$ such that $T - \frac{3x}{T} \leqslant r \leqslant T$ and $\left|F(x, \frac{s}{r})\right| \gg (xT)^{\frac{1}{2}}$.

Recently, G. Bachman proved several interesting upper bounds (see [1]) for $|F(x,\alpha)|$ at various contexts. In particular, one of his results (see Theorem 5, page 46 of [1]) improves the factor $(\log R)^{\frac{3}{2}}$ in (1.3) into $(\log R\ \log\log R)^{\frac{1}{2}}$. For more information on the history of the problem see the introduction of the paper [7] by A. Sankaranarayanan and the author. More recently progress on the problem has been achieved by G. Bachmann [1].

In the paper [7], A. Sankaranarayanan and the author give a new proof

Theorem 1.2. *Let $x \geqslant 3$, $1 \leqslant r \leqslant x(\log x)^{-2}(\log\log x)^{-1}(\log\log\log x)^{-1}$. We assume that r is a prime number and that $(r,s) = 1$. Then uniformly for $f \in \mathfrak{F}$ we have*

$$F(x, \frac{s}{r}) = \sum_{n\leqslant x} f(n)\mathbf{e}(n \cdot \frac{s}{r}) \leqslant \frac{x}{\log(x)} + \frac{x}{\sqrt{r}}\ . \tag{1.4}$$

Crucial for their proof is an evaluation of exponential sums of the form

$$\sum_{\substack{a \bmod r\\(a,r)=1}} e(ak\alpha)\ ,$$

which is simplest under the assumption $\alpha = \frac{s}{r}$, r prime. It certainly can be carried out under more general assumptions on α. In this paper we apply the ideas of the

paper [6] to investigate a new version of the problem. In (1.1) we restrict the range of summation to y*-smooth values* of n, i. e. integers all of whose prime factors are $\leqslant y$.
We set

$$S(x,y) = \left\{n \leqslant x : P^+(n) \leqslant y\right\}$$

where $P^+(n)$ denotes the largest prime factor of n. The case

$$f(n) = 1 \ , \ \alpha = 0$$

has first been investigated by Dickman in 1930. We obtain the counting function for smooth integers

$$\psi(x,y) = \left|\left\{n \leqslant x : P^+(n) \leqslant y\right\}\right| \ .$$

For wide ranges of y the asymptotics of ψ is determined by Dickman's function ϱ via

$$\psi(x,y) \sim x\varrho\left(\frac{\log x}{\log y}\right) \ .$$

ϱ is defined by the differential-difference equation

$$u\varrho'(u) = -\varrho(u-1) \quad (u > 1)$$

with the initial condition

$$\varrho(u) = 1 \quad (0 \leqslant u \leqslant 1) \ .$$

For an overview of the entire topic see [6]. In [5] (Theorem 10) the case $f = 1$ has been considered for general α. The purpose of this paper is to prove

Theorem 1. *Let* $f \in \mathfrak{F}$. *Let* $\varepsilon_0 > 0$ *be arbitrarily small,* $A > 0$ *be arbitrarily large,* $\exp((\log x)^{\varepsilon_0}) < y \leqslant x$, $r \leqslant (\log x)^A$ *be a prime number,* $(s,r) = 1$. *Then we have*

$$\left|\sum_{n \in S(x,y)} f(n)\mathbf{e}(n \cdot \frac{s}{r})\right| \ll_{\varepsilon_0, A} \psi(x,y) \cdot r^{-\frac{1}{2}}$$

uniformly for $f \in \mathfrak{F}$.

2. Notation and Preliminaries

1. We write $\log_k(x) = \log(\log_{k-1}(x))$ for any integer $k \geqslant 2$.
2. We define α, β by $\varepsilon_0 = 2\beta = 4\alpha$.
   The following consideration hold, if $B = B(\varepsilon_0)$ is chosen sufficiently large.
3. $y^{(1)} = \exp((\log x)^\alpha)$.

4. We set
$$m_+(n) = \prod_{\substack{p \leqslant y^{(1)} \\ p^\nu \| n}} p^\nu .$$
5. We partition the set
$$S'(x,y) = \{n \leqslant x : P^+(n) \leqslant y ,\ r \nmid n\}$$
as follows:
$$\mathfrak{m}_1 = \left\{n \in S'(x,y) : m_+(n) \leqslant \exp\left((\log x)^\beta\right)\right\} \text{ and}$$
$$\mathfrak{m}_2 = \left\{n \in S'(x,y) : m_+(n) > \exp\left((\log x)^\beta\right)\right\} .$$
6. Refinement of the partitions of $\mathfrak{m}_1$ and $\mathfrak{m}_2$:
We partition the interval $[y^{(1)}, y]$ as follows. We let
$$I_l = [y_l, y_{l+1}] \text{ with}$$
$$\frac{1}{2} y_l (\log y_l)^{-B} < y_{l+1} - y_l \leqslant y_l (\log y_l)^{-B}$$
so that
$$[y^{(1)}, y] = \bigcup_l I_l .$$
7. The partition of the set $\mathfrak{m}_1$:
For fixed $m_0 \leqslant \exp((\log(x))^\beta)$ and an L-tuplet $\bar{l} = (l_1, l_2, \dots, l_L)$ we set
$$\mathfrak{m}_{1,\bar{l},m_0} = \left\{n \in \mathfrak{m}_1 : m_+(n) = m_0 ,\ p_j \in I_{l_j}\right\} .$$
Here $n = m_+(n) p_1 p_2 \cdots p_L$ with $y^{(1)} < p_1 < p_2 < \cdots < p_L$.
8. The approximation of $\mathfrak{m}_1$ by a disjoint union of cartesian products:

Definition 2.1. For $n \in \mathfrak{m}_1$, we define
$$\omega(n,l) = \sum_{\substack{(p,\nu),\, p^\nu \| n \\ p \in I_l}} \nu .$$

Definition 2.2. We call $\mathfrak{m}_{1,\bar{l},m_0}$ *proper* if
$$\mathfrak{m}_{1,\bar{l},m_0} = \{m_0 p_1 p_2 \cdots p_L : p_1 \in I_{l_1} , \dots , p_L \in I_{l_L}\} ,$$
otherwise *improper*.

Remark 2.3. The point of the definition of proper $\mathfrak{m}_{1,\bar{l},m_0}$ is that for all possible choices of $p_j \in I_{l_j}$ we have $m_0 p_1 p_2 \cdots p_L \leqslant x$.

We set $\mathfrak{m}_1^{(1)} = \{n \in \mathfrak{m}_1 : \omega(n,l) > 1 \text{ for at least one } l\}$.

Definition 2.4. The number $n \notin \mathfrak{m}_1^{(1)}$ is called *proper* if $n \in \mathfrak{m}_{1,\bar{l},m_0}$ for a *proper* $\mathfrak{m}_{1,\bar{l},m_0}$, otherwise *improper*. We set $\mathfrak{m}_1^{(2)} = \{n \in \mathfrak{m}_1 : n \notin \mathfrak{m}_1^{(1)} ,\ n \text{ is improper}\}$.

Definition 2.5. We define

$$\mathfrak{m}_1^{(*)} = \mathfrak{m}_1 - \left(\mathfrak{m}_1^{(1)} \cup \mathfrak{m}_1^{(2)}\right) .$$

9. The decomposition of n into partial products:

Definition 2.6. Let $J = \{1, 2, \ldots, L\}$ and $J = J_1 \cup J_2$ be a partition of J in two disjoint subsets J_1 and J_2 and $n = m_+(n)p_1p_2 \cdots p_L$ with $p_j \in I_{l_j}$. Then we set

$$n_1 = n_1(n, J_1) = \prod_{j \in J_1} p_j \ , \ n_2 = n_2(n, J_2) = \prod_{j \in J_2} p_j \ .$$

This implies $n = m_+(n)n_1n_2$. We work with the following notation in the sequel:

$$\sum^{(1)} = \sum_{n \in \mathfrak{m}_1} f(n)\mathbf{e}(n\alpha) \ ,$$
$$\sum^{(2)} = \sum_{n \in \mathfrak{m}_2} f(n)\mathbf{e}(n\alpha) \ .$$

3. Some Lemmas

Lemma 3.1. *For fixed $\varepsilon > 0$ we have*

$$\psi(x, y) = x\varrho(u) \left\{1 + O\left(\frac{\log(u+1)}{\log y}\right)\right\}$$

uniformly in the range

$$y \geqslant 2 \ , \ 1 \leqslant u \leqslant \exp\left((\log y)^{\frac{3}{5}-\varepsilon}\right) \ .$$

Proof. This result is due to Hildebrand and is Theorem 1.1 in [6]. ■

Lemma 3.2. *We have*

$$\varrho(u) = \exp\left\{-u(\log u + \log_2(u+2) - 1 + O\left(\frac{\log_2(u+2)}{\log(u+2)}\right)\right\} \ .$$

Proof. This is Corollary 2.3 of [6]. ■

Lemma 3.3. *For any fixed $\varepsilon > 0$, uniformly in the range*

$$y \geqslant 2 \ , \ 1 \leqslant u \leqslant \exp\left\{(\log y)^{\frac{3}{5}-\varepsilon}\right\},$$

and for $xy^{-\frac{5}{12}} \leqslant z \leqslant x$, we have

$$\psi(x+z, y) - \psi(x, y) = z\varrho(u) \cdot \left\{1 + O\left(\frac{\log(u+1)}{\log y}\right)\right\} \ . \qquad (3.1)$$

Proof. This is Theorem 5.1 of [6]. ■

Lemma 3.4. *For $u > 2$ and $|v| \leqslant \frac{u}{2}$ we have*

$$\varrho(u-v) \ll \varrho(u)e^{v\xi(u)} . \tag{3.2}$$

Here $\xi = \xi(u)$ is the unique positive solution of the equation $e^{\xi} = 1 + u\xi$ an satisfies

$$\xi(u) = \log u + \log_2(u+2) + O\left(\frac{\log_2(u+2)}{\log(u+2)}\right) . \tag{3.3}$$

Proof. (3.2) is Corollary 2.4 of [6] and (3.3) is Lemma 2.2 of [6]. ∎

Lemma 3.5. *We have*

$$\left|\mathfrak{m}_1 - \mathfrak{m}_1^{(*)}\right| \ll \psi(x,y)\cdot(\log y)^{-A} .$$

Proof. Let $p_1, \ldots, p_\mu \leqslant y$. Then by lemma 3.3 and 3.4 we have:

$$|\{n \in S(x,y) : n \equiv 0 \bmod (p_1\cdots p_\mu)\}| \ll \frac{1}{p_1\cdots p_\mu}\cdot\varrho\left(\frac{\log x}{\log y} - \frac{\log(p_1\cdots p_\mu)}{\log y}\right)$$
$$\ll \frac{1}{p_1\cdots p_\mu}\cdot\psi(x,y)\cdot\exp\left(\mu\,\xi\left(\frac{\log x}{\log y}\right)\right) \ll \frac{1}{p_1\cdots p_\mu}\cdot\psi(x,y)\cdot(\log x)^{\mu} .$$

For any i with $2^{i+1} \geqslant y^{(1)}$ we have

$$\left|\left\{l : 2^i < y_l \leqslant 2^{i+1}\right\}\right| \ll (\log 2^i)^B \ll i^B \text{ since}$$

by section 2(6) we have

$$y_{l+k} \geqslant \frac{1}{2}k\left(y_l(\log y_l)^{-B}\right) .$$

We observe that

$$\sum_{p\in I_l}\frac{1}{p} \ll \frac{1}{y_l}(\pi(y_{l+1}) - \pi(y_l)) \ll (\log y_l)^{-(B+1)} .$$

Thus,

$$|\mathfrak{m}_1^{(1)}| \ll \psi(x,y)\sum_{\substack{i\\ \frac{1}{2}y^{(1)}\leqslant 2^i\leqslant 2y}}\ \sum_{\substack{l\\ 2^i\leqslant y_l\leqslant 2^{i+1}}}\ \sum_{\mu=2}^{\infty}(\log y)^{\mu}\left(\sum_{p\in I_l}\frac{1}{p}\right)^{\mu}$$
$$\ll \psi(x,y)\sum_{\frac{1}{2}y^{(1)}\leqslant 2^i\leqslant 2y}(\log 2^i)^{-B} \ll \psi(x,y)\cdot(\log y)^{-A} .$$

A nonempty set $\mathfrak{m}_{1,\bar{l},m_0}$ is proper if and only if

$$m_0 y_{l_1}\cdots y_{l_L} \leqslant x \leqslant m_0 y_{l_1}\cdots y_{l_{L+1}} .$$

Since $y_{l_j+1} - y_{l_j} \leqslant y_{l_j}(1+(\log y_{l_j})^{-B})$ we obtain

$$\begin{aligned} Q_1 &:= m_0 y_{l_1+1} \cdots y_{l_L+1} - m_0 y_{l_1} \cdots y_{l_L} \\ &\leqslant m_0 \left(\prod y_{l_j}\right) \cdot \left\{ \left(1+(\log y^{(1)})^{-B}\right)^{(\log x)^{1-\varepsilon_0}} - 1 \right\} \leqslant x(\log y)^{-A} . \end{aligned}$$

The result follows from lemma 3.3. ■

Definition 3.1. Let χ be a Dirichlet character, Λ the Mangoldt function. We set

$$\begin{aligned} \psi(x,\chi) &= \sum_{n \leqslant x} \chi(n)\Lambda(n) \ , \ \psi(x,r,a) = \sum_{\substack{n\leqslant x \\ n \equiv a \bmod r}} \Lambda(n) \, , \\ \vartheta(x,r,a) &= \sum_{\substack{p\leqslant x \\ p \equiv a \bmod r}} \log p \ , \ \pi(x,r,a) = \sum_{\substack{p\leqslant x \\ p \equiv a \bmod r}} 1 \, . \end{aligned}$$

The following is a simple consequence of the theorem of Siegel-Walfisz [3]:

Lemma 3.6. *Let $\varepsilon > 0$ be arbitrarily small, r a prime number with $r \geqslant r_0(\varepsilon)$, where $r_0(\varepsilon)$ is sufficiently large. If χ is not the principal character modulo r, we have for $x \geqslant r$:*

$$\psi(x,\chi) \ll x^{1-r^{-\varepsilon}} .$$

Lemma 3.7. *For $x \geqslant r$ we have*

$$\pi(x,r,a) = \frac{\operatorname{li} x}{r-1} + O\left(x^{1-r^{-\varepsilon}}\right) .$$

Proof. We have

$$\psi(x,r,a) = \frac{1}{r-1} \sum_{\chi \bmod r} \overline{\chi(a)} \psi(x,\chi) .$$

From lemma 3.6 we obtain

$$\psi(x,r,a) = \frac{x}{r-1} + O\left(x^{1-r^{-\varepsilon}}\right) \text{ and } \vartheta(x,r,a) = \frac{x}{r-1} + O\left(x^{1-r^{-\varepsilon}}\right) .$$

From this lemma 3.7 follows by partial summation. ■

Lemma 3.8. *Let $\alpha = \frac{r}{s}$ with $(r,s)=1$, r a prime number with $r \leqslant (\log x)^A$. Then we have*

$$\sum^{(1)} \ll \psi(x,y) r^{-\frac{1}{2}} .$$

Proof. Let

$$S_i = \left\{ n_i : n_i = \prod_{j \in J_i} p_k \ , \ p_j \in I_{l_j} \right\} \text{ for } i = 1,2 .$$

By Cauchy's inequality we get

$$Q_2 := \sum_{n\in\mathfrak{m}_{1,\bar{l},m_0}} f(n)\mathbf{e}(n\frac{s}{r}) = f(m_0)\sum_{n_1\in S_1} f(n_1)\sum_{n_2\in S_2} f(n_2)\mathbf{e}(n_1n_2\frac{m_0s}{r})$$

$$\ll \left(\sum_{n_1\in S_1}|f(n_1)|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{n_2^{(1)},n_2^{(2)}\in S_2} f(n_2^{(1)})f(n_2^{(2)})\sum_{n_1\in S_1}\mathbf{e}((n_2^{(1)}-n_2^{(2)})\frac{m_0s}{r}n_1)\right)^{\frac{1}{2}}.$$

We notice that (for $i=1,2$)

$$\sum_{n_i\in S_i}\chi(n_i) = \prod_{j\in J_i}\left(\sum_{p_j\in I_{l_j}}\chi(p_j)\right) .$$

Let χ not be the principal character modulo r and let $c>0$ be arbitrarily large. From lemma 3.6 we obtain by partial summation

$$\sum_{p_j\in I_{l_j}}\chi(p_j) \ll y_{l_{j-1}}^{1-r^{-\varepsilon}}$$

and thus by the inequalities for y_{l_j} and r

$$\sum_{p_j\in I_{l_j}}\chi(p_j) \ll |I_{l_j}|\cdot(\log x)^{-c} .$$

We obtain

$$\sum_{\substack{n_i\in S_i\\ n_i\equiv a \bmod r}} 1 = \frac{1}{r-1}\sum_{\chi \bmod r}\overline{\chi(a)}\sum_{n_i\in S_i}\chi(n_i) = \frac{|S_i|}{r-1} + O\left(|S_i|(\log x)^{-c}\right)$$

for $i=1,2$. Hence we obtain

$$Q_3 := \sum_{n_1\in S_1}\mathbf{e}\left((n_2^{(1)}-n_2^{(2)})\frac{m_0s}{r}n_1\right)$$

$$= \sum_{\substack{a \bmod r\\ (a,r)=1}}\mathbf{e}\left(a\left(n_2^{(1)}-n_2^{(2)}\right)\frac{m_0s}{r}\right)\sum_{\substack{n_1\in S_1\\ n_1\equiv a \bmod r}} 1 .$$

The number of pairs $(n_2^{(1)},n_2^{(2)})$ with $n_2^{(1)}\equiv n_2^{(2)} \bmod r$ is $\ll |S_2|^2r^{-1}$. Thus we have

$$Q_2 \ll |S_1|^{\frac{1}{2}}\left(\left(|S_1|r(\log x)^{-c}+|S_1|\right)|S_2|^2r^{-1}+\left(|S_1|r(\log x)^{-c}+\frac{|S_1|}{r-1}\right)|S_1|^2\right)^{\frac{1}{2}}.$$

Therefore $Q_2 \ll |\mathfrak{m}_{1,\bar{l},m_0}|r^{-\frac{1}{2}}$, this proves the lemma. ■

Lemma 3.9. *We have*

$$\sum^{(2)} \ll \psi(x,y)r^{-\frac{1}{2}} .$$

Proof. We set $M_0 = \exp((\log x)^{\beta})$ and obtain

$$\begin{aligned}|\mathfrak{m}_2| &\leqslant \sum_{\substack{M_0<m_0\leqslant x\\ P^+(m_0)\leqslant y^{(1)}}} \sum_{\substack{n\leqslant x\\ m^+(n)=m_0\\ P^+(n)\leqslant y}} 1\\ &\leqslant \sum_{\substack{M_0<m_0\leqslant x\\ P^+(m_0)\leqslant y^{(1)}}} \sum_{\substack{l\leqslant \frac{x}{m_0}\\ P^+(l)\leqslant y}} 1 = \sum_{\substack{M_0<m_0\leqslant x\\ P^+(m_0)\leqslant y^{(1)}}} \psi\left(\frac{x}{m_0}, y\right) .\end{aligned}$$

By Lemmas 3.1 and 3.4 we have

$$\psi\left(\frac{x}{m_0}, y\right) \ll \frac{x}{m_0}\varrho\left(\frac{\log x - \log m_0}{\log y}\right) \leqslant \frac{x}{m_0}\varrho\left(\frac{\log x}{\log y}\right)\exp\left(\frac{\log m_0}{\log y}\xi\left(\frac{\log x}{\log y}\right)\right).$$

Thus

$$\begin{aligned}|\mathfrak{m}_2| &\leqslant x\varrho\left(\frac{\log x}{\log y}\right)\sum_{j=0}^{\infty} \sum_{\substack{2^j M_0<m_0\leqslant 2^{j+1}M_0\\ P^+(m_0)\leqslant y^{(1)}}} m_0^{-1}\exp\left(\frac{\log m_0}{\log y}\xi\left(\frac{\log x}{\log y}\right)\right)\\ &\ll x\varrho\left(\frac{\log x}{\log y}\right) M_0^{-1}\cdot\sum_{j=0}^{\infty} 2^{-j}\exp\left(\frac{\log M_0+(j+1)\log 2}{\log y}\log\log x\right)\\ &\quad\cdot\varrho\left(\frac{\log M_0+(j+1)\log 2}{\log y^{(1)}}\right) .\end{aligned}$$

The terms in the inner sum are exponentially decreasing in j by lemma 3.1. The result follows. ■

4. Proof of Theorem 1

Lemma 3.4 shows that in Theorem 1 we may restrict the summation to integers n with $r \nmid n$. Theorem 1 follows from Lemmas 3.8 and 3.9.

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