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HECKE'S THEORY AND THE SELBERG CLASS

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Dedicated to Professor Eduard Wirsing
on the occasion of his 75th birthday

Abstract: Roughly speaking, we prove that the Hecke L-functions associated with the cusp forms of the Hecke groups $\mathsf{G}(\lambda)$ belong to the extended Selberg class, and for $\lambda \leqslant 2$ we characterize the Hecke L-functions belonging to the Selberg class.
Keywords: Selberg class, Hecke theory.

1. Hecke theory

In 1936, Hecke proved in his famous work [5] a bijection between modular forms and Dirichlet series satisfying a functional equation of Riemann type. We first recall some of the basic facts of Hecke's classical theory.

For a positive real number λ, the Hecke group $\mathsf{G}(\lambda)$ is defined as the subgroup of $\mathsf{PSL}_2(\mathbb{R})$ given by

$$\mathsf{G}(\lambda) = \left\langle \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle,$$

i.e. $\mathsf{G}(\lambda)$ is generated by the fractional linear transformations $\tau \to \tau + \lambda$ and $\tau \to -\frac{1}{\tau}$. A modular form of $\mathsf{G}(\lambda)$ of weight k and multiplier $\varepsilon \in \{\pm 1\}$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, where $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$ is the upper half-plane, satisfying

$$f(\tau + \lambda) = f(\tau) \qquad \text{and} \qquad f(-1/\tau) = \varepsilon (i/\tau)^{-k} f(\tau),$$

and having a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n \tau/\lambda} \tag{1}$$

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for all $\tau \in \mathbb{H}$. The latter series representation includes the λ-periodicity and shows that $f(\tau)$ is holomorphic at ∞. The complex vector space of such modular forms, satisfying in addition the growth condition $c(n) = O(n^c)$ for some constant c, is denoted by $\mathfrak{M}_0(\lambda, k, \varepsilon)$; a modular form of $\mathfrak{M}_0(\lambda, k, \varepsilon)$ is a cusp form if $c(0) = 0$.

Hecke proved a one-to-one correspondence between the elements of $\mathfrak{M}_0(\lambda, k, \varepsilon)$ and the Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

satisfying $c(n) = O(n^c)$ for some constant c and having meromorphic continuation to the whole complex plane such that

$$\Phi(s) + \frac{c(0)}{s} + \frac{\varepsilon c(0)}{k-s},$$

where

$$\Phi(s) := \left(\frac{\lambda}{2\pi}\right)^s \Gamma(s)L(s),$$

is entire and bounded on every vertical strip and satisfies the functional equation

$$\Phi(s) = \varepsilon\Phi(k-s). \tag{2}$$

Hecke's theorem includes the case of the Riemann zeta function $\zeta(s)$ as $L(s) = \zeta(2s)$, $f(\tau)$ being in this case the classical theta-function, $\lambda = 2$, $\varepsilon = 1$ and $k = 1/2$. The Dirichlet series associated as above with the modular forms of $\mathfrak{M}_0(\lambda, k, \varepsilon)$ are called the Hecke L-functions, and clearly form a vector space isomorphic to $\mathfrak{M}_0(\lambda, k, \varepsilon)$.

The groups $\mathsf{G}(\lambda)$ operate discontinuously as groups of fractional linear transformations on $\mathbb{H}$ if and only if either $\lambda > 2$ or

$$\lambda = \lambda_m := 2\cos\frac{\pi}{m} \qquad \text{with } 3 \leqslant m \in \mathbb{N} \cup \{\infty\}.$$

Note, for example, that the space $\mathfrak{M}_0(\lambda_m, k, \varepsilon)$ with $\lambda_m < 2$ is non-trivial, i.e. $\neq \{0\}$, if and only if $k = 4\ell/(m-2) + 1 - \varepsilon$ for some positive integer ℓ. In this case

$$\dim \mathfrak{M}_0(\lambda_m, k, \varepsilon) = 1 + \left[\frac{\ell + (\varepsilon - 1)/2}{m}\right]. \tag{3}$$

The space of cusp forms is non-trivial if and only if $\dim \mathfrak{M}_0(\lambda_m, k, \varepsilon) \geqslant 2$; in view of (3) this condition holds when k is suitably large (see also [20]).

For $\lambda_m \in \{1, \sqrt{2}, \sqrt{3}, 2\}$ (i.e. $m \in \{3, 4, 6, \infty\}$), the Hecke group $\mathsf{G}(\lambda_m)$ can be defined arithmetically and in these cases $\mathsf{G}(\lambda_m)$ holds a structure comparable to the full modular group $\Gamma := \mathsf{G}(1) = \mathsf{PSL}_2(\mathbb{Z})$. We briefly recall the main properties of the Hecke L-functions associated with the cusp forms of Γ; such properties depend heavily on the theory of the Hecke operators. First of all, a Hecke L-function has a degree 2 Euler product if and only if the associated cusp form is a normalized (i.e. $c(1) = 1$) eigenfunction for the Hecke operators. Moreover, the space of the cusp forms has a basis of normalized eigenfunctions, and such eigenfunctions have real coefficients and satisfy the Ramanujan conjecture. The situation is similar in the case of $\lambda_m \in \{\sqrt{2}, \sqrt{3}, 2\}$, thanks to the fact that the groups $\mathsf{G}(\lambda_m)$ are conjugate to index 2 extensions of the congruence subgroups $\Gamma_0(N)$ of levels $N = 2, 3, 4$ respectively. However, in these cases only the newforms (i.e. the cusp forms not induced by cusp forms of lower level) have a basis of normalized eigenfunctions (for the Hecke and Atkin-Lehner operators), and the space of cusp forms splits as a direct sum of the newforms and the oldforms (having $c(1) = 0$).

For the theory outlined above, and much more, we refer to Hecke's original paper [5], to the papers by Atkin-Lehner [1] and Leutbecher [13] and to the monographs by Hecke [6], Ogg [17], Berndt [3], Miyake [14] and Iwaniec [7].

Wolfart [19] has shown that every space $\mathfrak{M}_0(\lambda_m, k, \varepsilon)$ with $\lambda_m \notin \{1, \sqrt{2}, \sqrt{3}, 2\}$ has a basis consisting of modular forms of type

$$f(\tau) = \sum_{n=0}^{\infty} r(n) a^n e^{2\pi i n\tau/\lambda}, \tag{4}$$

where

$$r(n) \in \mathbb{Q} \text{ and } a \text{ is transcendental;} \tag{5}$$

moreover, a depends only on the space $\mathfrak{M}_0(\lambda_m, k, \varepsilon)$ (and not on the modular form $f(\tau)$). Clearly, the same statement holds for the cusp forms.

In this note we are interested in the class of the L-functions associated with Hecke's cusp forms $f \in \mathfrak{M}_0(\lambda, k, \varepsilon)$, mainly with $\lambda = \lambda_m$, with respect to the Selberg class. It will be shown that according to the arithmetic nature of the Fourier coefficients, indicated by (5), such Dirichlet series have rather different properties. *In the sequel we will always assume that $f(\tau)$ is a non-trivial cusp form.*

2. The Selberg class

Selberg [18] defined a general class $\mathcal{S}$ of Dirichlet series having analytic continuation, a functional equation of Riemann type and an Euler product, and formulated some fundamental conjectures. More precisely, the Selberg class $\mathcal{S}$ consists of the functions $F(s)$ satisfying the following axioms:

(i) *Dirichlet series*:
$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$
the series being absolutely convergent for $\sigma > 1$;

(ii) *analytic continuation*: there exists a non-negative integer c such that $(s-1)^c F(s)$ is an entire function of finite order;

(iii) *functional equation*:

$$\Phi(s) = \omega\overline{\Phi(1-\overline{s})},$$

where

$$\Phi(s) := Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

with $|\omega| = 1$, $Q > 0$, $\lambda_j > 0$ and $\operatorname{Re}\mu_j \geqslant 0$;

(iv) *Ramanujan conjecture*: $a(n) = O(n^{\varepsilon})$;

(v) *Euler product*:

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

with $b(n) = 0$ unless $n = p^k$, and $b(n) = O(n^{\theta})$ for some $\theta < \frac{1}{2}$.

The first three axioms are of analytic nature while the last two are more related to arithmetic. The extended Selberg class $\mathcal{S}^\sharp$ consists of the non-identically vanishing functions satisfying the first three axioms; clearly, $\mathcal{S} \subset \mathcal{S}^\sharp$. It is conjectured that the Selberg class coincides with the class of automorphic L-functions. It is also expected that for every function in the Selberg class the analogue of the Riemann Hypothesis holds, i.e. all non-trivial zeros lie on the critical line $\operatorname{Re} s = 1/2$.

The degree of $F \in \mathcal{S}^\sharp$ is defined by

$$\mathrm{d}_F = 2\sum_{j=1}^{r} \lambda_j$$

and is well-defined (although the data of the functional equation are not unique). It is conjectured that all $F \in \mathcal{S}^\sharp$ have integral degree. This conjecture is known for the interval $[0, 5/3)$ ($d_F = 0$ means that there are no Γ-factors in the functional equation):

$$F \in \mathcal{S}^\sharp \quad \text{with} \quad 0 \leqslant \mathrm{d}_F < \frac{5}{3} \qquad \Longrightarrow \qquad \mathrm{d}_F = 0, 1 \tag{6}$$

(see Conrey & Ghosh [4] for the range $0 < d_F < 1$ and Kaczorowski & Perelli [11] for $1 < d_F < 5/3$). The class of functions $F \in \mathcal{S}$ with degree d is denoted by $\mathcal{S}_d$, and analogously for $\mathcal{S}^\sharp$. Kaczorowski & Perelli [9] characterized the functions of $\mathcal{S}_1$: these are the Riemann zeta function $\zeta(s)$ and the shifted Dirichlet L-functions $L(s + i\theta, \chi)$ with $\theta \in \mathbb{R}$ and χ primitive. A characterization of the functions of $\mathcal{S}_1^\sharp$ is also given in [9]; roughly speaking, these are linear combinations of suitable Dirichlet L-functions with certain Dirichlet polynomials of degree 0 as coefficients. An important tool in such characterizations is the conductor of $F \in \mathcal{S}^\sharp$, defined as

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}. \tag{7}$$

The conductor is multiplicative, i.e. $q_{GF} = q_F q_G$, and it is conjectured that $q_F \in \mathbb{N}$ for every $F \in \mathcal{S}$; [9] shows that this is the case for every $F \in \mathcal{S}_1$ and even for

$F \in \mathcal{S}_1^\sharp$, although in general the conductor of functions in $\mathcal{S}^\sharp$ is not integer. We recall that in the case of L-functions associated with modular forms, the above defined conductor coincides with the level of the form. We also recall that $\mathcal{S}$ has the multiplicity one property, i.e. two distinct functions in $\mathcal{S}$ must have infinitely many distinct Euler factors, see Murty & Murty [16].

Both classes $\mathcal{S}$ and $\mathcal{S}^\sharp$ are multiplicatively closed. An element $F \in \mathcal{S}$ is called primitive if it cannot be factored as a product of two elements non-trivially, i.e. the equation $F(s) = F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}$ implies $F_1(s) = 1$ or $F_2(s) = 1$ identically. Conrey & Ghosh [4] proved that every function in $\mathcal{S}$ has a factorization into primitive functions; furthermore, they showed that this factorization is unique if the deep orthonormality conjecture of Selberg is true (see [18] or [10]). The situation for the extended Selberg class $\mathcal{S}^\sharp$ is definitely more difficult since $\mathcal{S}_0^\sharp$ is larger than $\mathcal{S}_0 = \{1\}$ ($\mathcal{S}_0^\sharp$ consists of certain Dirichlet polynomials, see [9]). An element $F \in \mathcal{S}^\sharp$ is called almost-primitive if any factorization $F(s) = F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}^\sharp$ implies $d_{F_1} = 0$ or $d_{F_2} = 0$; moreover, $F \in \mathcal{S}^\sharp$ is primitive (in $\mathcal{S}^\sharp$) if for any such factorization, $F_1(s)$ is constant or $F_2(s)$ is constant. Kaczorowski & Perelli [12] proved that almost-primitivity implies primitivity up to a factor of degree zero; more precisely, if $F \in \mathcal{S}^\sharp$ is almost-primitive, then there exist $F_1, F_2 \in \mathcal{S}^\sharp$ such that $F(s) = F_1(s)F_2(s)$ with $d_{F_1} = 0$ and $F_2(s)$ primitive. This implies that any element of the extended Selberg class $\mathcal{S}^\sharp$ can be factored into primitive functions.

It is not clear whether one can expect unique factorization in $\mathcal{S}^\sharp$; furthermore, it is not clear whether an element which is primitive in $\mathcal{S}$ is necessarily primitive in $\mathcal{S}^\sharp$ as well. The Riemann zeta function and the Dirichlet L-functions are primitive in $\mathcal{S}$ and $\mathcal{S}^\sharp$. Other examples of primitive elements in $\mathcal{S}$ are suitably normalized L-functions associated with eigenfunction newforms for congruence subgroups of $\mathsf{PSL}_2(\mathbb{Z})$, as shown by M.R. Murty [15]. On the contrary, the Dedekind zeta functions associated with cyclotomic fields $\neq \mathbb{Q}$ are not primitive.

For more details concerning the Selberg class we refer to the survey [10] of Kaczorowski & Perelli.

3. Functions of degree 2 in $\mathcal{S}$ and $\mathcal{S}^\sharp$

We first normalize the Hecke L-functions in order to satisfy a functional equation close to the one in axiom (iii). For $f \in \mathfrak{M}_0(\lambda, k, \varepsilon)$ we write

$$L_f(s) = \sum_{n=1}^{\infty} \frac{c(n)n^{\frac{1-k}{2}}}{n^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \tag{8}$$

say, thus functional equation (2) becomes

$$\big(\frac{\lambda}{2\pi}\big)^s \Gamma(s + \frac{k-1}{2}) L_f(s) = \varepsilon \big(\frac{\lambda}{2\pi}\big)^{1-s} \Gamma(1 - s + \frac{k-1}{2}) L_f(1-s). \tag{9}$$

Note that (9) is not exactly of the form required by axiom (iii) since there is no conjugation on the right hand side. Therefore, we consider the modified Selberg classes $\bar{\mathcal{S}}$ and $\bar{\mathcal{S}}^\sharp$, where the conjugation is dropped in axiom (iii). However, it is easy to see that if there exists a complex number $|\xi| = 1$ such that $\xi L_f(s)$ has real coefficients, then functional equation (9) has the form required by axiom (iii). Moreover, it is clear that the basic definitions and properties of $\mathcal{S}$ and $\mathcal{S}^\sharp$ carry over to $\bar{\mathcal{S}}$ and $\bar{\mathcal{S}}^\sharp$. For example, if $L_f \in \bar{\mathcal{S}}^\sharp$ then its degree is 2, and the multiplicity one property holds for $\bar{\mathcal{S}}$ as well (see e.g. the simple proof in [10]).

In this note we show that $L_f \in \bar{\mathcal{S}}^\sharp$ for every cusp form $f \in \mathfrak{M}_0(\lambda, k, \varepsilon)$ with any $\lambda > 0$, and we completely characterize the Hecke L-functions $L_f(s)$ belonging to $\mathcal{S}$ (or to $\bar{\mathcal{S}}$), where $f \in \mathfrak{M}_0(\lambda_m, k, \varepsilon)$ is a cusp form.

Theorem. (i) *Let $\lambda > 0$ and $f \in \mathfrak{M}_0(\lambda, k, \varepsilon)$ be a non-trivial cusp form; then $L_f \in \bar{\mathcal{S}}^\sharp$.*

(ii) *Let $\lambda_m \in \{1, \sqrt{2}, \sqrt{3}, 2\}$ and $f \in \mathfrak{M}_0(\lambda_m, k, \varepsilon)$ be a cusp form; if $f(\tau)$ is a normalized eigenfunction newform then $L_f \in \mathcal{S} \cap \bar{\mathcal{S}}$, otherwise $L_f \notin \mathcal{S} \cup \bar{\mathcal{S}}$.*

(iii) *Let $\lambda_m \notin \{1, \sqrt{2}, \sqrt{3}, 2\}$ and $f \in \mathfrak{M}_0(\lambda_m, k, \varepsilon)$ be a cusp form; then $L_f \notin \mathcal{S} \cup \bar{\mathcal{S}}$.*

Statement (iii) is the most interesting part of the Theorem, and is based on the Wolfart basis theorem described in Section 1. An interesting open problem raised by Peter Sarnak is characterizing the functions $L_f(s)$, $f \in \mathfrak{M}_0(\lambda, k, \varepsilon)$, belonging to $\mathcal{S}$ (or to $\bar{\mathcal{S}}$) when $\lambda > 2$. In this case the space $\mathfrak{M}_0(\lambda, k, \varepsilon)$ has an uncountable basis.

Proof. (i) In view of Hecke's correspondence theorem, see Chapter I of Hecke [6], we only need to prove that the Dirichlet series of $L_f(s)$ is absolutely convergent for $\sigma > 1$. To this end we follow a classical argument, reported in Theorem 5.1 and Corollary 5.2 of Iwaniec's book [7]. Given a cusp form $f(\tau)$ and writing $\tau = x + iy$, we consider the function $g(\tau) = y^{k/2}|f(\tau)|$. It is easy to check that $g(\tau)$ is $\mathsf{G}(\lambda)$-invariant, i.e. $g(\gamma(\tau)) = g(\tau)$ for every $\gamma \in \mathsf{G}(\lambda)$. Since $g(\tau)$ decays exponentially at the cusps, we conclude that $g(\tau)$ is bounded on the fundamental domain of $\mathsf{G}(\lambda)$, and hence on $\mathbb{H}$. Therefore, as $y \to 0^+$

$$f(x + iy) \ll y^{-k/2}$$

uniformly in x, hence by (1) and Parseval's formula we get

$$\sum_{n=1}^{\infty} |c(n)|^2 e^{-4\pi n y/\lambda} = \frac{1}{\lambda} \int_0^\lambda |f(x + iy)|^2 dx \ll y^{-k}$$

as $y \to 0^+$. Choosing $y = \frac{1}{X}$ with $X \to \infty$ we obtain

$$\sum_{n \leqslant X} |c(n)|^2 \ll X^k,$$

thus by the Cauchy-Schwarz inequality

$$\sum_{n \leqslant X} |c(n)| \ll X^{(k+1)/2}$$

and finally, in view of (8), by partial summation

$$\sum_{n \leqslant X} |a(n)| \ll X,$$

and (i) follows.

(ii) For $\lambda_m = 1$ it is clear from the description in Section 1 that if $f(\tau)$ is a normalized eigenfunction (for $\lambda_m = 1$ all cusp forms are newforms), then $L_f(s)$ has real coefficients, has an Euler product and satisfies the Ramanujan conjecture (thanks to Deligne's work), so $L_f \in \mathcal{S} \cap \bar{\mathcal{S}}$. Note that by [8] the functions in $\mathcal{S}$ are linearly independent, hence in particular there are exactly g normalized eigenfunctions cusp forms, where g is the dimension of the space of cusp forms. Note also that the only property of $\mathcal{S}$ involved in the results of [8] is the multiplicity one property, hence the same results hold for $\bar{\mathcal{S}}$ as well. Therefore $L_f \notin \mathcal{S} \cup \bar{\mathcal{S}}$ if $f(\tau)$ is not a normalized eigenfunction.

A similar argument applies to the subspace of the newforms in the cases $\lambda_m \in \{\sqrt{2}, \sqrt{3}, 2\}$. We remark that $L_f \in \mathcal{S} \cap \bar{\mathcal{S}}$ for $f(\tau)$ normalized eigenfunction newform since the eigenfunctions of the Hecke algebra are automatically eigenfunctions for the conjugation operator K, see p.113, 114 and 118 of [7]. Moreover, the L-functions associated with the oldforms do not belong to $\mathcal{S} \cup \bar{\mathcal{S}}$ since $a(1) = 0$ and hence the coefficients are not multiplicative. The general case of linear combinations of oldforms and newforms requires the following

Lemma 1. *For $m = 4, 6, \infty$, the Hecke L-functions associated with the oldforms of the space $\mathfrak{M}_0(\lambda_m, k, \varepsilon)$ are linear combinations over the ring of Dirichlet polynomials of Hecke L-functions associated with newforms of strictly lower levels.*

Proof. This is a L-functions reformulation of parts of Theorem 5 in Atkin-Lehner [1]. Roughly speaking, the coefficients of such linear combinations are of type $c_j m_j^{-s}$ with c_j complex and m_j integer, and the underlying philosophy is as follows. The oldforms are linear combinations of newforms of strictly lower levels, and the factors m_j^{-s} come from the lifting of the newform L-functions to a higher level. ■

Denote by $M(s)$ and $N(s)$ the generic L-functions in $\mathfrak{M}_0(\lambda_m, k, \varepsilon)$ associated with oldforms and newforms, respectively, and let

$$L(s) = aM(s) + bN(s)$$

with $ab \neq 0$. Then by Lemma 1 we have

$$M(s) = \sum_j c_j(s) N_j(s)$$

with $c_j(s)$ Dirichlet polynomials and $N_j(s)$ newform L-functions of strictly lower levels. Passing to the basis of normalized eigenfunction newforms, we finally have

$$L(s) = \sum_j k_j(s) L_j(s) \tag{10}$$

with $k_j(s)$ non-trivial Dirichlet polynomials and $L_j(s)$ being L-functions associated with normalized eigenfunction newforms. Since the level of a form equals the conductor of the associated L-function, and since in (10) there are at least two $L_j(s)$ of different levels, in (10) there are at least two different $L_j(s)$. But the $L_j(s)$ belong to the Selberg class, so by the linear independence results in [8], $L(s)$ does not belong to the Selberg class (or to the modified Selberg class, for which the same results as in [8] hold). Part (ii) is therefore proved.

(iii) We show that $L_f \notin \mathcal{S} \cup \bar{\mathcal{S}}$ by proving that its coefficients are not multiplicative. We argue by contradiction, assuming that $L_f \in \mathcal{S} \cup \bar{\mathcal{S}}$. We first note that $L_f(s) \neq 1$ since the constant function 1 has degree 0. Hence, by the multiplicity one property, the Euler product of $L_f(s)$ must have infinitely many non-trivial Euler factors. Thus by multiplicativity the coefficients $a(n)$ satisfy infinitely many equations of type

$$a(nm) = a(n)a(m) \neq 0 \quad \text{with} \quad (n,m) = 1. \tag{11}$$

In view of Wolfart's basis (4) and (5), the $a(n)$'s can be expressed as a linear combination of the form

$$a(n) = a^n \sum_j c_j r_j(n)$$

with a transcendental, $r_j(n)$ rational and c_j complex. Let K be a finitely generated extension of the field $\overline{\mathbb{Q}}$ of the algebraic numbers, containing the coefficients c_j; then we can rewrite the above linear combination as

$$a(n) = R(n) a^n, \tag{12}$$

where $R(n) \in K$. More precisely, denoting by $V \subset K$ the finite dimensional $\overline{\mathbb{Q}}$-vector space generated by the coefficients c_j and their products $c_i c_j$, the coefficients $R(n)$ and their products $R(n)R(m)$, $n, m \in \mathbb{N}$, belong to V.

Valuation theory (see Ch. V of Bachman [2]) implies the existence of a non-archimedean exponential valuation ν on the field $K(a)$ with the following properties:

i) $\nu(x) = 0$ for $x \in \overline{\mathbb{Q}}^*$

ii) $\nu(a) = -1$.

Writing $V^* := V \setminus \{0\}$ we have

Lemma 2. *The valuation ν is bounded on V^*, i.e. there exist real numbers L, M such that $L \leqslant \nu(x) \leqslant M$ for every $x \in V^*$.*

Proof. By induction on $\dim V$. For $\dim V = 1$ the claim is obvious since all $x \in V^*$ are algebraic multiples cw_1, $c \in \overline{\mathbb{Q}}$, of a fixed vector w_1, hence ν takes only the value $\nu(w_1)$ on V^*. Now suppose that $\dim V = n$ and let W be a codimension 1 subspace of V such that $L' \leqslant \nu(x) \leqslant M'$ for $x \in W \setminus \{0\}$. Suppose also that V is generated by W and a vector w_n, thus all elements of V can be written in the form $w + cw_n$, $w \in W$ and $c \in \overline{\mathbb{Q}}$. Thanks to $\nu(c) = 0$ and the fact that ν is non-archimedian we deduce the lower bound

$$\nu(w + cw_n) \geqslant \min(\nu(w), \nu(cw_n)) \geqslant \min(L', \nu(w_n)) =: L.$$

For the upper bound we distinguish two cases. If all $w_n \in V - W$ satisfy already $\nu(w_n) \leqslant M'$, set $M := M'$ and the proof is complete. Otherwise, take w_n such that $\nu(w_n) > M'$ and recall that in this case

$$\nu(w + cw_n) = \min(\nu(w), \nu(w_n)) \leqslant \nu(w_n) =: M$$

and the lemma is proved. ■

From (11) and (12) we obtain infinitely many non-vanishing equations of type

$$a^{nm-n-m} R(nm) = R(n)R(m),$$

and applying the valuation ν we get

$$n + m - nm = \nu\big(R(n)R(m)\big) - \nu\big(R(nm)\big)$$

which contradicts Lemma 2 since the left hand side is unbounded, thus proving (iii). The Theorem is therefore proved. ■

We conclude the paper by a brief and partial discussion of the primitivity problem for the Hecke L-functions $L_f(s)$. We first remark that *if* $\lambda^2 \notin \mathbb{N}$ *and* $f \in \mathfrak{M}_0(\lambda, k, \varepsilon)$ *is such that* $L_f \in \mathcal{S}^\sharp$, *then* $L_f(s)$ *is almost-primitive.* Indeed, in view of (6) we have to show that $L_f(s)$ cannot be factored into a product of elements of $\mathcal{S}_0^\sharp$ and $\mathcal{S}_1^\sharp$. Since $q_{L_f} = \lambda^2$ and the conductor of the degree 0 and 1 functions is integer, such a factorization is not possible since $\lambda^2 \notin \mathbb{N}$. Moreover, thanks to the special structure of the groups $\mathsf{G}(\lambda_m)$ for $\lambda_m \in \{1, \sqrt{2}, \sqrt{3}, 2\}$, see Section 1, we have that *if* $f \in \mathfrak{M}_0(\lambda_m, k, \varepsilon)$, $\lambda_m \in \{1, \sqrt{2}, \sqrt{3}, 2\}$, *is a normalized eigenfunction newform, then* $L_f(s)$ *is primitive in* $\mathcal{S}$. This follows from Murty's result in [15].

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