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GLOBAL ISOMETRIC EMBEDDING OF A RIEMANNIAN 2-MANIFOLD WITH NONNEGATIVE CURVATURE INTO A EUCLIDEAN 3-SPACE

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1. Introduction

The isometric embedding problem of a 2-dimensional Riemannian manifold M^2 with Gaussian curvature $K \geq 0$ into 3-dimensional Euclidean space $\mathbb{R}^3$ is one of many difficult problems. In fact, it is quite hard to show certain a priori estimates in a neighborhood of zero points of K and to verify the convergence of the Nash-Moser type interaction scheme, since the linearization operators are degenerating on $\{K = 0\}$. Lin [5] studied a local problem and solved it. Naturally, the next subject is a global problem, which we shall study in this paper. In a global case, Lin's method does not work well, though it is quite suggestive, since his technicalities are particularly adapted to the local situation. For instance, his ingenious parametrization would not lead to success in the global case. What we need are a new type of implicit function theorem and global a priori estimates for degenerating linearized operators.

Let $g = g_{ij}\,dx^i\,dx^j$ be a $C^{r,\alpha}$ Riemannian metric defined in $\mathbb{R}^2$, where $r \geq 2$ and $0 < \alpha < 1$ (actually, C^r smoothness will suffice for our purpose (cf. §5)). We assume that

$$(1.1) \qquad |g_{ij} - \delta_{ij}|_r \ll 1 \qquad (1 \leq i, j \leq 2),$$

where δ_{ij} stands for Kronecker's delta, $|\cdot|_r$ is the C^r supremum norm, and $A \ll 1$ means that A is sufficiently small. K denotes the Gaussian curvature of the Riemannian manifold $(\mathbb{R}^2, g)$. We assume

$$(1.2) \qquad K \geq 0,$$

and put $f = K\det(g_{ij})$. It is to be noted that (1.1) and (1.2) imply $0 \leq f \ll 1$. Let D be a bounded convex domain in $\mathbb{R}^2$ such that there exists a convex function $\phi \in C^\infty(\mathbb{R}^2)$ satisfying $\phi < 0$ in D and $\phi \geq 0$

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in $\mathbb{R}\setminus D$. For a small $\rho > 0$, D_ρ denotes a bounded convex domain $\{x \in \mathbb{R}^2 : \phi(x) < \rho\}$. We define a nonlinear operator $F[\mu]$ by

$$F[\mu] = \det(\nabla_i \nabla_j u) + f(g^{ij}u_i u_j - 1) - \det((\mu_\rho)_{ij}), \tag{1.3}$$

where $(g^{ij}) = (g_{kl})^{-1}$, ∇_i are covariant differentials, $u_i = \partial u/\partial x^i$,

$$\mu_\rho(x) = \begin{cases} \delta(x^1)^2 & (\phi(x) \le \rho/2), \\ \delta(x^1)^2 + \exp\{1/\rho/2 - \phi(x))\} & (\phi(x) > \rho/2), \end{cases}$$

$\delta > 0$ is a small constant, and $(\mu_\rho)_{ij} = \partial\mu_\rho/\partial x^i \partial x^j$.

We shall prove the following.

Theorem 1.1. *Assume* (1.1), (1.2), *and* $r \ge 14 + 2\kappa$ *for some* $0 < \kappa < 1$. *Then for a small* $\rho > 0$ *there exists a function* $u_\infty \in w^{r/2-2-\kappa}(D_\rho)$ *such that*

$$F[\mu_\rho + u_\infty] = 0 \quad \text{in } D_\rho, \qquad u_\infty = 0 \quad \text{on } \partial D_\rho \tag{1.4}$$

and

$$|u_\infty|_{r/2-2-\kappa} \ll 1. \tag{1.5}$$

Theorem 1.2. *Assume* (1.1), (1.2), *and* $r > 15$. *Then there exists a global* $C^{[(r-11)/2]}$ *isometric embedding of* $(\overline{D}, g)$ *into* 3*-dimensional Euclidean space* $\mathbb{R}^3$.

Here $W^s(\cdot)$ denotes the Sobolev space with norm $\|\cdot\|_S$, $[\cdot]$ is Gauss's symbol, i.e., $[(r-11)/2]$ is the largest integer $\le (r-11)/2$.

In Theorem 1.2, assumption (1.1) and the convexity of D are essential. If we remove one of those assumptions from Theorem 1.2, then it is no longer true; we will be able to find counterexamples. It is to be noted that (1.1) and convexity are not necessary in the local case (cf. Lin [5]). (1.2) ensures that the nonlinear equation which we study later is of elliptic type.

We first shall show that Theorem 1.2 follows from Theorem 1.1 (§2) and second, establish an iteration scheme of Nash-Moser type for the nonlinear operator (1.3) and prove Theorem 1.1 (§§3 and 4). We also prove that Theorems 1.1 and 1.2 remain true for a C^r Riemannian metric g (§5).

2. Proof of Theorem 1.2

We shall show that Theorem 1.2 follows from Theorem 1.1.

Proof of Theorem 1.2. Since $r \ge 14 + 2/4$, and Sobolev's lemma gives $|u|_{[(r-7)/2]} \le C\|u\|_{r/2-2-1/4}$ and $W^{r/2-2-1/4}(\Omega) \subset C^{[(r-7)/2]}(\overline{\Omega})$, Theorem 1.1 implies that for a small $\rho > 0$ there exists a function $u_\infty \in C^{[(r-7)/2]}(\overline{D}_\rho)$ such that

$$F[\mu_\rho + u_\infty] = 0 \quad \text{in } D_\rho, \qquad u_\infty = 0 \quad \text{on } \partial D_\rho \tag{2.1}$$

and

$$|u_\infty|_{[(r-7)/2]} \ll 1. \tag{2.2}$$

It is easy to show that $u = \mu_\rho + u_\infty$ satisfies

$$\det(\nabla_i \nabla_j u) + f(g^{ij} u_i u_j - 1) = 0 \quad \text{in } D_{\rho/2} \tag{2.3}$$

and

$$\det(g_{ij} - u_i u_j) > 0 \quad \text{in } D_{\rho/2} \tag{2.4}$$

when $(2\delta x^1)^2 < 1$ in $D_{\rho/2}$. By brute force computation, it follows from (2.3) and (2.4) that

$$K[g_{ij} - u_i u_j] = 0 \quad \text{in } D_{\rho/2}, \tag{2.5}$$

where

$$K[\gamma_{ij}] = \frac{1}{\det(\gamma_{ij})} \left[\partial_1 \left(\frac{\gamma_{12}\partial_2\gamma_{11} - \gamma_{11}\partial_1\gamma_{22}}{2\gamma_{11}\sqrt{\det(\gamma_{ij})}} \right) + \partial_2 \left(\frac{2\gamma_{11}\partial_1\gamma_{12} - \gamma_{12}\partial_1\gamma_{11} - \gamma_{11}\partial_2\gamma_{11}}{2\gamma_{11}\sqrt{\det(\gamma_{ij})}} \right) \right].$$

Gauss's *Theorema egregium* shows that $K[g_{ij} - u_i u_j]$ is the Gaussian curvature of the Riemannian manifold $(D_{\rho/2}, (g_{ij} - u_i u_j)\, dx^i\, dx^j)$. Hence, the $C^{[(r-7)/2]}$ Riemannian manifold $(D_{\rho/2}, (g_{ij} - u_i u_j)\, dx^i\, dx^j)$ is flat.

It is clear that we have only to prove the existence of a $C^{[(r-7)/2]-2}$ coordinate system (y^1, y^2) defined in $\overline{D}$ satisfying

$$(g_{ij} - u_i u_j)\, dx^i\, dx^j = (dy^1)^2 + (dy^2)^2, \qquad dy^1 \wedge dy^2 \neq 0. \tag{2.6}$$

In fact, (2.6) implies

$$g^{ij}\, dx^i\, dx^j = (dy^1)^2 + (dy^2)^2 + (du)^2, \qquad dy^1 \wedge dy^2 \neq 0,$$

i.e., the map $(y^1, y^2, u)\colon \overline{D} \to \mathbb{R}^3$ is a $C^{[(r-7)/2]-2}$ isometric embedding. In order to prove (2.6), we shall show two lemmas.

For the sake of simplicity we put $q = [(r-7)/2] - 1$, $\tilde{g}_{ij} = g_{ij} - u_i u_j \in C^q(\overline{D}_\rho)$, $\tilde{g} = \tilde{g}_{ij}\, dx^i\, dx^j$, and $\widetilde{K} = K[\tilde{g}_{ij}]$.

Lemma 2.1. *For any point* $x \in \overline{D}$ *and any unit vector* $\xi \in T_x(\overline{D})$ *there is a geodesic curve* $c(t; x, \xi)$, $0 \le t \le t_0$, *on* $(\overline{D}_\rho, \tilde{g})$ *such that* $c(0; x, \xi) = x$, $\dot{c}(0; x, \xi) = \xi$, $c(t; x, \xi) \in D_\rho$ *for* $0 \le t < t_0$, $c(t_0; x, \xi) \in \partial D_\rho$, *and* $c(t; x, \xi)$ *is a* C^{q-1} *function with respect to* t, x, *and* ξ.

Proof. The geodesic curve $c(t; x, \xi) = (c^1(t; x, \xi), c^2(t; x, \xi))$ is defined by

$$\ddot{c}^k + \tilde{\Gamma}^k_{ij}(c)\dot{c}^i\dot{c}^j = 0, \qquad c(0) = x, \quad \dot{c}(0) = \xi, \tag{2.7}$$

where $\tilde{\Gamma}^k_{ij} = \tilde{g}^{kl}((\tilde{g}_{lj})_i + (\tilde{g}_{il})_j - (\tilde{g}_{ij})_l)1/2$. Since $\tilde{\Gamma}^k_{ij} \in C^{q-1}(\overline{D}_\rho)$ and $|\tilde{\Gamma}^k_{ij}|_{q-1} \ll 1$, the C^{q-1} regularity of $c(t; x, \xi)$ follows from a well-known fundamental theorem of ordinary differential equations (cf., for example, [4]). The remaining part of Lemma 2.1 is clear.

Remark. We may regard $\gamma(t; x, \xi) = \dot{c}(t; x, \xi)$ as a solution of the initial value problem

$$\dot{\gamma}^k + \tilde{\Gamma}^k_{ij}(c)\gamma^i\gamma^j = 0, \qquad \gamma(0) = \xi,$$

where $\tilde{\Gamma}^k_{ij}(c) = \tilde{\Gamma}^k_{ij}(c(t; x, \xi))$ is a C^{q-1} function of t, x, and ξ. Hence, $\gamma(t; x, \xi) = \dot{c}(t; x, \xi)$ is C^{q-1} smooth with respect to t, x, and ξ. Furthermore, using (2.7), it is clear that $\ddot{c}(t; x, \xi)$ is also C^{q-1} smooth with respect to t, x, and ξ.

Lemma 2.2. *There exist a global geodesic parallel coordinate system* (y^1, y^2) *on* $\overline{D}$ *and a positive function* $h(y^1, y^2)$ *defined on* $\overline{D}$ *such that* $y^i \in C^{q-1}(\overline{D})$, $h \in C^{q-1}(\overline{D})$, *and*

$$\tilde{g} = (dy^1)^2 + h(y^1, y^2)(dy^2)^2. \tag{2.8}$$

In particular, if $\tilde{K} \equiv 0$ *and* $q \ge 3$, *then* $h \equiv 1$.

When $(\overline{D}, \tilde{g})$ is embedded in $\mathbb{R}^3$ and $q = \infty$, we can find the proof of Lemma 2.2 in many textbooks. However, $(\overline{D}, \tilde{g})$ is not yet embedded in $\mathbb{R}^3$ and, furthermore, we are interested in the case $q < \infty$ and the loss of regularity. Thus, we have to prove it here.

Proof of Lemma 2.2. We construct a global geodesic parallel coordinate system (y^1, y^2) on $\overline{D}$ as follows: First, we fix a point $p \in D$ and a unit vector $v \in T_p(\overline{D})$, and take a geodesic curve $c(t_2) = (c^1(t_2), c^2(t_2))$ satisfying

$$\ddot{c}^k(t_2) + \tilde{\Gamma}^k_{ij}(c)\dot{c}^i(t_2)\dot{c}^j(t_2) = 0, \qquad c(0) = p, \quad \dot{c}(0) = v. \tag{2.9}$$

Second, we define a family of geodesic curves $c(t_1\,;\,t_2)$ by

$$\ddot{c}^k(t_1\,;\,t_2) + \widetilde{\Gamma}^k_{ij}(c)\dot{c}^i(t_1\,;\,t_2)\dot{c}^j(t_1\,;\,t_2) = 0\,, \qquad c(0\,;\,t_2) = c(t_2)\,, \qquad \dot{c}(0\,;\,t_2) = v(t_2)\,, \tag{2.10}$$

where

$$v^i(t_2) = g^{ij}(c(t_2))\dot{c}_j(t_2) \Big/ \sqrt{g^{kl}(c(t_2))\dot{c}_k(t_2)\dot{c}_l(t_2))}\,,$$

and $(\dot{c}_1(t_2)\,,\,\dot{c}_1(t_2)) = (\dot{c}^2(t_2)\,,\,-\dot{c}^1(t_2))$. Here we note that $|v(t_2)| = \sqrt{g_{ij}v^i(t_2)v^j(t_2)} \equiv 1$ and $\langle\dot{c}(t_2)\,,\,v(t_2)\rangle = g_{ij}\dot{c}^i(t_2)v^j(t_2) \equiv 0$. Lemma 2.1 shows that the family of geodesic curves $\{c(t_1\,;\,t_2)\}$ covers $\overline{D}$ and, furthermore, by Lemma 2.1 and the remark following it, $c(t_1\,;\,t_2)$ is C^{q-1} smooth with respect to t_1 and t_2. Taylor expansion

$$c(t_1\,;\,t_2) = c(t_2) + v(t_2)_1 + \left\{\int_0^1 (1-t)\ddot{c}(t_1\,;\,t_2)\,dt\right\}(t_1)^2 \tag{2.11}$$

enables us to compute $\partial c^i(t_1\,;\,t_2)/\partial t_j$. Combining (2.11) with (1.1), we obtain $\det(\partial c^i(t_1\,;\,t_2)/\partial t_j) \neq 0$.

Third, for $x = c(t_1\,;\,t_2) \in \overline{D}$, we define $(y^1(x)\,,\,y^2(x))$ by

$$y^1 = \int_0^{t_1} |\dot{c}(t\,;\,t_2)|\,dt\,, \qquad y^2 = \int_0^{t_2} |\dot{c}(t)|\,dt\,, \tag{2.12}$$

where $|\gamma| = \sqrt{g_{ij}\gamma^i\gamma^j}$. Hence, we get a global C^{q-1} coordinate system $(y^1\,,\,y^2)$ defined on $\overline{D}$. It is to be noted that, using the construction procedure of $(y^1\,,\,y^2)$, we may assume that $(y^1\,,\,y^2)$ is defined in a neighborhood of $\overline{D}$.

We shall show that, in the new coordinate system $(y^1\,,\,y^2)$, g has a simple expression of the form

$$\tilde{g}(\,dy^1)^2 + h(y)(\,dy^2))^2. \tag{2.13}$$

In order to prove (2.13), we abandon the original coordinate system $(x^1\,,\,x^2)$ temporarily, and write $\tilde{g}$ as $\tilde{g} = h_{ij}\,dy^i\,dy^j$. Let us fix a point $y_0 = (y_0^1\,,\,y_0^2) \in \overline{D}$ arbitrarily and take a curve $c(t) = (c^1(t)\,,\,c^2(t)) = (y_0^1 + t\,,\,y_0^2)$. Then, for all sufficiently small $t > 0$,

$$\operatorname{dist}(c(0)\,,\,c(t)) = c^1(t) - c^1(0) = t$$

and

$$\operatorname{dist}(c(0)\,,\,c(t) = \int_0^t \sqrt{h_{ij}(c)\dot{c}^i(\tau)\dot{c}^j(\tau)}\,d\tau = \int_0^t \sqrt{h_{11}(c(\tau))}\,d\tau.$$

Hence, we have $h_{11}(y_0) = 1$; this implies $h_{11} \equiv 1$ in $\overline{D}$. Next, let us fix a point $y_0 = (0, y_0^2) \in \overline{D}$ arbitrarily and take another curve $c(t) = (c^1(t), c^2(t)) = (t, y_0^2)$. As is well known, for a vector field $Y = \eta^i \partial/\partial y^i$ the covariant derivative $\nabla_{\dot{c}} Y$ is defined by

$$\nabla_{\dot{c}} Y = (\eta^i_j \dot{c}^j + \Gamma^i_{jk} \eta^j \dot{c}^k) \frac{\partial}{\partial y^i},$$

where $\eta^i_j = \partial \eta^i / \partial y^j$ and $\Gamma^k_{ij} = h^{kl}((h^i_{lj} + (h_{il})_j - (h_{ij})_l)/2$. Direct computation gives

$$\nabla_{\dot{c}} \frac{\partial}{\partial y^1} = \Gamma^i_{11} \frac{\partial}{\partial y^i}, \qquad \nabla_{\dot{c}} \frac{\partial}{\partial y^2} = \Gamma^i_{12} \frac{\partial}{\partial y^i}. \tag{2.14}$$

Since $c(t)$ is a geodesic curve, we have $\Gamma^i_{11} = \ddot{c}^i + \Gamma^i_{jk} \dot{c}^j \dot{c}^k = 0$ which implies, by (2.14),

$$\left\langle \nabla_{\dot{c}} \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right\rangle = 0. \tag{2.15}$$

(2.14) and $h_{11} \equiv 1$ give

$$\left\langle \frac{\partial}{\partial y^1}, \nabla_{\dot{c}} \frac{\partial}{\partial y^2} \right\rangle = \Gamma^i_{12} h_{1i} = h_{1i} h^{ij} ((h_{j2})_1 + (h_{1j})_2 - \tfrac{1}{2}(h_{12})_j) = 0. \tag{2.16}$$

Hence, we have, by (2.15) and (2.16),

$$\nabla_{\dot{c}} \left\langle \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right\rangle = \left\langle \nabla_{\dot{c}} \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right\rangle + \left\langle \frac{\partial}{\partial y^1}, \nabla_{\dot{c}} \frac{\partial}{\partial y^2} \right\rangle = 0. \tag{2.17}$$

Using the definition of the coordinate system (y^1, y^2), we obtain

$$\left\langle \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right\rangle = 0 \quad \text{at } c(0). \tag{2.18}$$

(2.17) and (1.18) show that

$$\left\langle \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right\rangle \equiv 0 \quad \text{on } c(t),$$

which implies $h_{12} = h_{21} \equiv 0$ in $\overline{D}$. We have only to put $h(y) = h_{22}(y)$. Since $\tilde{g}_{ij}$ and x^i are C^q smooth with respect to y^i by virtue of the inverse function theorem, C^{q-1} smoothness of $h(y)$ follows from

$$h(y) = \left\langle \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^2} \right\rangle = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \frac{\partial x^i}{\partial y^2} \frac{\partial x^j}{\partial y^2} = \tilde{g}_{ij} \frac{\partial x^i}{\partial y^2} \frac{\partial x^j}{\partial y^2}.$$

In the case $K \equiv 0$, Gauss's *theorema egregium* shows

$$(\partial/\partial y^1)^2\sqrt{h} = 0. \tag{2.19}$$

Here we used the assumption $q \geq 3$ which implies $h \in C^2(\overline{D})$. Let us fix a point $y_0 = (0, y_0^2) \in \overline{D}$ arbitrarily and take a curve $c(t) = (c^1(t), c^2(t)) = (0, y_0^2 + t)$. Then, for all sufficiently small $t > 0$,

$$\operatorname{dist}(c(0), c(t)) = t$$

and

$$\operatorname{dist}(c(0), c(t)) = \int_0^t \sqrt{h(c(\tau))}\, d\tau.$$

Hence, we have $h(y_0) = 1$ which implies $h|_{y^1=0} = 1$. Since

$$-\frac{1}{2}\frac{\partial h}{\partial y^1} = \Gamma^1_{22} = \ddot{c}^1 + \Gamma^1_{ij}\dot{c}^i\dot{c}^j = 0,$$

we obtain $\partial h/\partial y^1|_{y^1=0} = 0$. Therefore, we have

$$\sqrt{h}|_{y^1=0} = 1, \qquad \left.\frac{\partial\sqrt{h}}{\partial y^1}\right|_{y^1=0} = 0, \tag{2.20}$$

which together with (2.19) gives $\sqrt{h} \equiv 1$, i.e., $h \equiv 1$ in $\overline{D}$. q.e.d.

Consequently, (2.6) is proved; that is to say, the proof of Theorem 1.2 is complete.

3. Proof of Theorem 1.1 (Part 1)

The purpose of this section is to establish an implicit function theorem of Nash-Moser type. Though there already exist numerous implicit function theorems, none of them is applicable to our problem. In fact, if we used them, we would end up with linearized operators of mixed type which we cannot solve so far. However, repeated use of elliptic regularization, or elliptic singular perturbation for linearized operators, enables us to overcome the difficulty. By virtue of our implicit function theorem we can prove an important part of Theorem 1.1.

For the sake of simplicity, we put $\mu = \mu_\rho$, $s_0 = r - 2$,

$$\varepsilon = \max\{|F[\mu]|_0^{2/3}, \|F[\mu]\|_{s_0}^{2/(s_0+2)}\} \quad \text{and} \quad \theta = \varepsilon^{-1/2}. \tag{3.1}$$

It is to be noted that we may assume that $\varepsilon > 0$. In fact, if $\varepsilon = 0$, then Theorem 1.1 is trivial; we have only to take $u_\infty = 0$. (1.1) implies that

$\varepsilon > 0$ is sufficiently small. Thus we may assume that $0 < \varepsilon \leq 1$ and $\theta \geq 1$. Throughout this section, $C > 0$ denotes a certain large constant which is independent of ε and n. Actually, C will be determined as a large positive number satisfying (3.2), (3.10)–(3.12), and (3.23)–(3.24).

Lemma 3.1. *We have*

$$|\mu|_s \leq C \qquad (0 \leq s \leq s_0 + 4), \tag{3.2}$$

$$|F[\mu]|_0 \leq \sqrt{\varepsilon}\theta^{-2}, \qquad \|F[\mu]\|_s \leq \varepsilon\theta^{-s_0} \quad (0 \leq s \leq s_0), \tag{3.3}$$

where $C > 0$ is a constant independent of ε and ρ.

Proof. (3.2) is clear. Direct computation gives

$$|F[\mu]|_0 \leq \varepsilon^{3/2} = \sqrt{\varepsilon}\theta^{-2}$$

and

$$\|F[\mu]\|_s \leq \varepsilon^{(s_0+2)/2} = \varepsilon\theta^{-s_0} \qquad (0 \leq s \leq s_0).$$

We define linear operators $L[u]$ and $L_{\varepsilon_*}[u]$, $\varepsilon_* > 0$, by

$$L[u]v = \partial_t F[u + tv]|_{t=0} = a^{ij}[u]v_{ij} + a^i[u]v_i, \tag{3.4}$$

$$L_{\varepsilon_*}[u]v = L[u]v + \varepsilon_* Lv = a^{ij}_{\varepsilon_*}[u]v_{ij} + a^i_{\varepsilon_*}[u]v_i, \tag{3.5}$$

where $L = ag^{ij}\nabla_i\nabla_j$ and $a > 0$ is a constant which will be determined later.

Lemma 3.2. *If $\varepsilon_* = |F[\mu]|_0$, $g^{ij}u_iu_j \leq 1/2$, and $Lu \geq 1$, then*

$$\det(a^{ij}_{\varepsilon_*}[u]) \geq \frac{f}{2} + \det(\mu_{ij}) + \varepsilon_*^2 a^2 \det(g^{ij}). \tag{3.6}$$

Proof. (3.4) and (3.5) give

$$\begin{aligned} a^{11}_{\varepsilon_*}[u] &= u_{22} + \Gamma^k_{22}u_k + \varepsilon_* ag^{11}, \\ a^{12}_{\varepsilon_*}[u] &= -(u_{21} + \Gamma^k_{21}u_k) + \varepsilon_* ag^{12}, \\ a^{21}_{\varepsilon_*}[u] &= -(u_{12} + \Gamma^k_{12}u_k) + \varepsilon_* ag^{21}, \\ a^{22}_{\varepsilon_*}[u] &= u_{11} + \Gamma^k_{11}u_k + \varepsilon_* ag^{22}, \end{aligned}$$

which implies

$$\begin{aligned} \det(a^{ij}_{\varepsilon_*}[u]) &= \det(\nabla_i\nabla_j u) + \varepsilon_* Lu + \varepsilon_*^2 a^2 \det(g^{ij}) \\ &= F[\mu] + f(1 - g^{ij}u_iu_j) + \det(\mu_{ij}) + \varepsilon_* Lu + \varepsilon_*^2 a^2 \det(g^{ij}) \\ &= (F[\mu] + \varepsilon_*) + f(1 - g^{ij}u_iu_j) + \det(\mu_{ij}) \\ &\quad + \varepsilon_*(Lu - 1) + \varepsilon_*^2 a^2 \det\det(g^{ij}). \end{aligned}$$

Since, by the assumption, $F[\mu]+\varepsilon_* \geq 0$, $f(1-g^{ij}u_iu_j) \geq f/2$, and $\varepsilon_*(L[u]-1) \geq 0$, we obtain (3.6).

Remark. It is to be noted that direct computation gives

$$\det(\mu_{ij}) \geq (\phi - \rho/2)^{-4}\exp(2/(\rho/2-\phi)) \cdot \det(\phi_{ij}) > 0 \quad \text{in } \overline{D}_\rho \setminus \overline{D}_{\rho/2}.$$

Here the convexity of $\phi(x)$ is essential.

In this section, we repeatedly use the Sobolev inequality

$$|u|_i \leq C(i)\|u\|_{i+1+\kappa} \tag{3.7}$$

and smoothing operators $S_\theta\colon w^i(D_\rho) \to W^j(D_\rho)$, $\theta \geq 1$, defined by

$$S_\theta u(x) = \theta^2 \int_{D_\rho} \psi(\theta(x-y))u(y)\,dy,$$

where $C(i)$ are positive constants, $0 < \kappa < 1$ is a fixed sufficiently small constant, and $\psi \in C_0^\infty(\mathbb{R}^2)$ is a nonnegative function satisfying $\int \psi(x)\,dx = 1$. It is easy to show the inequalities

$$\|S_\theta u\|_i \leq C(i, j)\theta^{i-j}\|u\|_j, \tag{3.8}$$

$$\|(I-S_\theta)u\|_i \leq c(i, j)\theta^{i-j}\|u\|_j \qquad (i<j), \tag{3.9}$$

where $C(i, j)$ are positive constants and, in particular, $C(i, i) = 1$.

Lemma 3.3. *There is a constant* $C > 0$ *such that*

$$|F[\mu]-F[v]|_0 \leq C(|u|_2+|v|_2)|u-v|_2, \tag{3.10}$$

(3.11)

$$\|\partial_t L[u+tv]w\|_s \leq C\Bigg(\sum_{\substack{i+j=s\\ i\leq s/2}} \|v\|_{i+3+\kappa}\|w\|_{j+2} + \sum_{\substack{i+j=s\\ i\leq s/2}} \|v\|_{i+2}\|u\|_{j+3+\kappa}\Bigg),$$

and

$$\|Lu\|_s \leq C\|u\|_{s+2} \tag{3.12}$$

for $0 \leq s \leq s_0 - 2$ *and* $0 \leq t \leq 1$.

Proof. (3.10) and (3.12) are clear. (3.11) follows immediately from (3.4) and (3.7). q.e.d.

We construct a sequence $\{u_n\}$ as follows: We define u_0 and u_{n+1}, $n \geq 0$, by

$$u_0 = 0, \qquad u_{n+1} = u_n + v_n, \tag{3.13}$$

where $v_n \in C^{s_0+2,\alpha}(\overline{D}_\rho)$ is a solution of the Dirichlet problem

$$L_{\varepsilon_n}[\mu + \tilde{u}_n]v_n = f_n \quad \text{in } D_\rho, \qquad v_n = 0 \quad \text{in } \partial D_\rho, \tag{3.14}$$

$$\varepsilon_n = |F[\mu + \tilde{u}_n]|_0, \tag{3.15}$$

$$\tilde{u}_n = S_n u_n, \tag{3.16}$$

$$\begin{aligned} f_0 &= -S_0 F[\mu], \\ f_n &= S_{n-1}R_{n-1} - S_n R_n + S_{n-1}F[\mu] - S_n F[\mu], \end{aligned} \tag{3.17}$$

$$R_0 = 0, \qquad R_n = \sum_{j=0}^{n-1} r_j, \tag{3.18}$$

$$r_j = (L_{\varepsilon_j}[\mu + u_j] - L_{\varepsilon_j}[\mu + \tilde{u}_j])v_j - \varepsilon_j L v_j + Q_j \qquad (0 \le j \le n-1), \tag{3.19}$$

$$Q_j = F[\mu + u_{j+1}] - F[\mu + u_j] - L[\mu + u_j]v_j \qquad (0 \le j \le n-1). \tag{3.20}$$

Here $S_n = S_{\theta_n}$ and $\theta_n = \theta^n = \varepsilon^{-n/2}$.

The sequence $\{u_n\}$ is well defined and convergent if we assume the following.

Assumption 3.4.

$$g^{ij}(\mu + \tilde{u}_n)_i(\mu + \tilde{u}_n)_j \le \tfrac{1}{2}, \tag{3.21}$$

$$L(\mu + \tilde{u}_n) \ge 1. \tag{3.12}$$

Assumption 3.5.

$$\|v_n\|_0 \le C\|f_n\|_0, \tag{3.23}$$

$$\|v_n\|_s \le C\Big(\|f_n\|_s + \sum_{\substack{i+j\le s \\ j<s}} |\mu + \tilde{u}_n|_{i+4}\|v_n\|_j\Big) \qquad (0 < s \le s_0), \tag{3.24}$$

where $C > 0$ is a constant independent of ε and n.

Assumption 3.4 shows, by Lemma 3.2, that $L_{\varepsilon_n}[\mu + \tilde{u}]$ is an elliptic operator with real $C^{s_0,\alpha}$ coefficients defined in $\overline{D}_\rho$; this implies that we can solve the Dirichlet problem (3.14) in $C^{s_0+2,\alpha}(\overline{D}_\rho)$ (although $C^{r,\alpha}$ regularity of the metric g played an important role here, C^r smoothness will suffice for any other argument in this paper). Assumption 3.5 ensures the convergence of $\{u_n\}$ (cf. Proposition 3.6 and 3.7).

Based on Assumptions 3.4 and 3.5, we have the following Propositions 3.6, 3.7 and Theorem 3.8 which were originally proved by Amano [1]. (Unfortunately, the author happened to make a mistake in [1]. In fact, though the propositions of [1] are true, one of their assumptions which corresponds to (3.24) in this paper is too strong for practical applications.)

Proposition 3.6. *On Assumptions* 3.4 *and* 3.5, *if*

$$0 < \varepsilon < \min((4C)^{-2}, (2C)^{-4}), \tag{3.25}$$

$$\theta \geq 2, \tag{3.26}$$

$$s_0 \geq 4 + 2\kappa, \tag{3.27}$$

$$5 + \kappa \leq \sigma \leq s_0, \tag{3.28}$$

then we obtain

$$\|v_0\|_s \leq \frac{\sqrt{\varepsilon}}{2}\theta^{s-\sigma} \qquad (0 \leq s \leq s_0), \tag{3.29}$$

$$\|r_0\|_s \leq C_1\varepsilon\theta^{s-\sigma} \qquad (0 \leq s \leq s_0 - 2), \tag{3.30}$$

where $C_1 = \frac{1}{4}(s_0 + 1)C$.

Proposition 3.7. *On Assumptions* 3.4 *and* 3.5, *if*

$$0 < \varepsilon < \min\Bigg[(4C)^{-2}, (2C)^{-4}, (2CC_2)^{-2}, \left\{C\left(2C_2 + 2C + \sum_{i=1}^{s_0} C(i+2)C(i+3+\kappa, \sigma+\kappa)\right)\right\}^{-2}, \{CC(0)(2c(1+\kappa, s_0 - 2) + 1)\}^{-2}\Bigg].$$

$$\theta \geq 2^{1/\kappa}, \tag{3.32}$$

$$\max(3 + \kappa, \tfrac{1}{2}(\sigma + 1 + \kappa)) \leq \tau \leq \sigma - 2, \tag{3.33}$$

$$5 + \kappa \leq \sigma \leq s_0/2 - 1, \tag{3.34}$$

$$s_0 \geq 4 + 2\kappa, \tag{3.35}$$

then we obtain

$$(3.36)_j \qquad \|v_{j-1}\|_s \le \frac{\sqrt{\varepsilon}}{2}\theta_{j-1}^{s-\sigma} \qquad (0 \le s \le s_0),$$

$$(3.37)_j \qquad \|u_j\|_s \le \begin{cases} \sqrt{\varepsilon} & (s \le \sigma - \kappa), \\ \sqrt{\varepsilon}\theta_j^{s-\sigma} & (\sigma \ge \sigma + \kappa), \end{cases}$$

$$(3.38)_j \qquad \|u_j - \tilde{u}_j\|_s \le C_0\sqrt{\varepsilon}\theta_j^{s-\sigma} \qquad (0 \le s \le s_0),$$

$$(3.39)_j \qquad \|r_{j-1}\|_s \le C_1\varepsilon\theta_{j-1}^{s-\sigma} \qquad (0 \le s \le s_0 - 2),$$

$$(3.40)_j \qquad \|f_j\|_s \le C_2\varepsilon\theta_j^{s-\theta} \qquad (0 \le s \le s_0),$$

$$(3.41)_j \qquad \varepsilon_j \le C_3\sqrt{\varepsilon}\theta_j^{\tau-\sigma},$$

where $v_{-1} = 0$, $r_{-1} = 0$, *and each constant* $C_i \ge 0$ *depends only on* κ, s_0, *and* C.

Remark. By virtue of the interpolation inequality, (3.29)–(3.30) and $(3.36)_j$–$(3.41)_j$ remain valid for real s, if we modify the constants C_0—C_3 appropriately.

Theorem 3.8. *On Assumptions* 3.4 *and* 3.5, *if* $r \ge 14 + 2\kappa$ *for some* $0 < \kappa < 1$, *then for a given small* $\rho > 0$ *there exists a function* $u_\infty \in W^{r/2-2-\kappa}(D_\rho)$ *such that*

$$(3.42) \qquad F[\mu + u_\infty] = 0 \quad in\ D_\rho, \qquad u_\infty = 0 \quad on\ \partial D_\rho$$

and

$$(3.43) \qquad \|u_\infty\|_{r/2-2-\kappa} \ll \sqrt{\varepsilon}.$$

Proof of Proposition 3.6. By (3.23), (3.17), (3.8), (3.3), $\sigma_0 \ge \sigma$, and $2C\sqrt{\varepsilon} \le 1$, we have

$$(3.44) \qquad \|v_0\|_0 \le \frac{\sqrt{\varepsilon}}{2}\theta^{0-\sigma}.$$

If we assume that

$$(3.45) \qquad \|v_0\|_j \le \frac{\sqrt{\varepsilon}}{2}\theta^{j-\sigma} \qquad (0 \le j \le s),$$

then, by (3.24), (3.17), (3.2), (3.45), (3.8), (3.3), $\sqrt{\varepsilon}\theta = 1$, $s_0 \ge \sigma$, $\theta \ge 2$, and $(2C + 2C^2)\sqrt{\varepsilon} \le 1$, we obtain

$$(3.46) \qquad \|v_0\|_s \le \frac{\sqrt{\varepsilon}}{2}\theta^{s-\sigma}.$$

Hence (3.29) is proved.

Next, we note that (3.19) gives

$$(3.47) \qquad r_0 = -\varepsilon_0 L v_0 + Q_0$$

and that (3.12), (3.15), (3.3), and (3.29) imply

$$\|\varepsilon_0 L v_0\|_s \le \tfrac{1}{2} C\varepsilon\theta^{s-\sigma}. \tag{3.48}$$

Since (3.20) and direct computation yield

$$Q_0 = \int_0^1 \left\{ \int_0^t \frac{\partial}{\partial\tau} L[\mu + \tau v_0] v_0 \, d\tau \right\} dt,$$

we obtain by (3.11), $(s_0 - 2)/2 + 3 + \kappa \le s_0$, and $\sigma \ge 5 + \kappa$,

$$\|Q_0\|_s \le \tfrac{1}{4}(s+1) C\varepsilon\theta^{s-\sigma}. \tag{3.49}$$

By combining (3.47)–(3.49), we have (3.30).

Proof of Proposition 3.7. $(3.36)_0$–$(3.41)_0$. Since $u_0 = 0$, $v_{-1} = 0$, and $r_{-1} = 0$, $(3.36)_0$–$(3.39)_0$ are clear. $(3.40)_0$ follows from (3.17), (3.8), and (3.3) when $C_2 \ge 1$. By (3.15) and (3.3), $(3.41)_0$ is valid when $C_3 \ge 1$. The constants C_2 and C_3 will be determined precisely in the following part of the proof.

$(3.36)_{j\le n}$–$(3.41)_{j\le n} \Rightarrow (3.36)_{n+1}$. (3.23), $(3.40)_n$, and $2CC_2\sqrt{\varepsilon} \le 1$ give

$$\|v_n\|_0 \le \frac{\sqrt{\varepsilon}}{2} \theta_n^{0-\sigma}. \tag{3.50}$$

If we assume that

$$\|v_n\|_j \le \frac{\sqrt{\varepsilon}}{2} \theta_n^{j-\sigma} \qquad (0 \le j < s), \tag{3.51}$$

then, by (3.24), $(3.40)_n$, (3.2), (3.7), (3.8), $(3.37)_{j\le n}$, (3.51), $5+\kappa-\sigma \le 0$, $\sqrt{\varepsilon}\theta = 1$, $\theta \ge 2$, and

$$C\left(2C_2 + 2C + \sum_{i=1}^{s} C(i+2)C(i+5+\kappa, \sigma+\kappa)\right)\sqrt{\varepsilon} \le 1,$$

we have

$$\|v_n\|_s \le \frac{\sqrt{\varepsilon}}{2} \theta_n^{s-\sigma}. \tag{3.52}$$

$(3.36)_{j\le n}$–$(3.45)_{j\le n} \Rightarrow (3.37)_{n+1}$. Since (3.13) gives $u_{n+1} = \sum_{j=0}^{n} v_j$, by $(3.36)_{j\le n+1}$ we obtain

$$\|u_{n+1}\|_s \le \frac{\sqrt{\varepsilon}}{2} \sum_{j=0}^{n} \theta_j^{s-\sigma}. \tag{3.53}$$

Direct computation shows that, by $\theta \ge 2^{1/\kappa}$,

$$\frac{\sqrt{\varepsilon}}{2} \sum_{j=0}^{n} \theta_j^{s-\sigma} \le \sqrt{\varepsilon} \quad \text{when } s \le \sigma - \kappa \tag{3.54}$$

and

$$\frac{\sqrt{\varepsilon}}{2}\sum_{j=0}^{n}\theta_j^{s-\sigma} \le \sqrt{\varepsilon}\theta_{n+1}^{s-\sigma} \quad \text{when } s \ge \sigma+\kappa. \tag{3.55}$$

By combining (3.53)–(3.55), we have $(3.37)_{n+1}$.

$(3.36)_{j\le n}$–$(3.41)_{j\le n} \Rightarrow (3.38)_{n+1}$. In the case $s < \sigma+\kappa$, we obtain

$$\|u_{n+1}-\tilde{u}_{n+1}\|_s \le C(s, \sigma+\kappa)\sqrt{\varepsilon}\theta_{n+1}^{s-\sigma} \tag{3.56}$$

by (3.9) and (3.37). In the case $s \ge \sigma+\kappa$, we get

$$\|U_{n+1}-\tilde{u}_{n+1}\|_s \le 2\sqrt{\varepsilon}\theta_{n+1}^{s-\sigma} \tag{3.57}$$

by (3.38) and $(3.37)_{n+1}$. Hence, we need only to set

$$C_0 = \max\left\{\max_{0\le s<\sigma+\kappa} C(s, \sigma+\kappa), 2\right\}. \tag{3.58}$$

$(3.36)_{j\le n}$–$(3.41)_{j\le n} \Rightarrow (3.39)_{n+1}$. Since

$$L_{\varepsilon_n}[\mu+u_n]v_n - L_{\varepsilon_n}[\mu+\tilde{u}_n]v_n = \int_0^1 \frac{\partial}{\partial\tau}L_{\varepsilon_n}[\mu+\tilde{u}_n+\tau(u_n-\tilde{u}_n)]v_n\,d\tau,$$

(3.11), $(s_0-2)/2+3+\kappa \le s_0$, $(3.38)_n$, $(3.36)_{n+1}$, and $5+\kappa-\sigma\le 0$ give

$$\begin{aligned}\|L_{\varepsilon_n}[\mu+u_n]v_n - L_{\varepsilon_n}[\mu+\tilde{u}_n]v_n\|_s \\ \le \tfrac{1}{2}(s_0-1)CC_0\varepsilon\theta_n^{s-\sigma} \qquad (0\le s\le s_0-2).\end{aligned} \tag{3.59}$$

(3.12), $(3.41)_n$, $(3.36)_{n+1}$, and $\tau\le\sigma-2$ show

$$\|-\varepsilon_n L v_n\|_s \le \tfrac{1}{2}CC_3\varepsilon\theta_n^{s-\sigma} \qquad (0\le s\le s_0-2). \tag{3.60}$$

Direct calculation gives

$$\begin{aligned}Q_n &= F[\mu+u_{n+1}] - F[\mu+u_n] - L[\mu+u_n]v_n \\ &= \int_0^1\left\{\int_0^t \frac{\partial}{\partial\tau}L[\mu+u_n+\tau v_n]v_n\,d\tau\right\}dt.\end{aligned} \tag{3.61}$$

By combining (3.61) with (3.11), $(s_0-2)/2+3+\kappa\le s_0$, $(3.36)_{n+1}$, and $5+\kappa-\sigma\le 0$, we obtain

$$\|Q_n\|_s \le \tfrac{1}{4}(s_0-1)C\varepsilon\theta_n^{s-\sigma} \qquad (0\le s\le s_0-2). \tag{3.62}$$

(3.59), (3.60) and (3.62) imply, by (3.19),

$$\|r_n\|_s \le C_1\varepsilon\theta_n^{s-\sigma} \qquad (0\le s\le s_0-2),$$

where

$$C_1 = \tfrac{1}{2}(s_0 - 1)CC_0 + \tfrac{1}{2}CC_3 + \tfrac{1}{4}(s_0 - 1)C. \tag{3.63}$$

$(3.36)_{j\leq n}$–$(3.41)_{j=\leq n} \Rightarrow (3.40)_{n+1}$. We note that

$$f_{n+1} = S_nR_n - S_{n+1}R_{n+1} + S_nF[\mu] - S_{n+1}F[\mu]. \tag{3.64}$$

We shall estimate each term of (3.64) separately. (3.8),(3.18), $(3.39)_{j\leq n+1}$, $\sigma \leq s_0/2 - 1$, and $\theta \geq 2$ give

$$\|S_nR_n\|_s \leq 2C_1(s, s_0 - 2)\varepsilon g q_{n+1}^{s-\sigma} \qquad (0 \leq s \leq s_0) \tag{3.65}$$

and

$$\|S_{n+1}R_{n+1}\|_s \leq C_1C(s, s_0 - 2)\varepsilon\theta_{n+1}^{s-\sigma} \qquad (0 \leq s \leq s_0). \tag{3.66}$$

(3.8), (3.3), and $\sigma - s - s_0 \leq 0$ show

$$\|S_nF[\mu]\|_S \leq C(s, \sigma)\varepsilon\theta_{n+1}^{s-\sigma} \qquad (0 \leq s \leq s_0) \tag{3.67}$$

and

$$\|S_{n+1}F[\mu]\|_S \leq C(s, \sigma)\varepsilon\theta_{n+1}^{s-\sigma} \qquad (0 \leq s \leq s_0). \tag{3.68}$$

Hence, we obtain

$$\|f_{n+1}\|_s \leq C_2\varepsilon\theta_{n+1}^{s-\sigma} \qquad (0 \leq s \leq s_0),$$

where

$$C_2 = \max_{0\leq s\leq s_0}(3C_1C(s, s_0 - 2) + 2C(s, \sigma)) + 1. \tag{3.69}$$

$(3.36)_{j\leq n}$–$(3.41)_{j\leq n} \Rightarrow (3.41)_{n+1}$. Direct computation gives

$$F[\mu + u_{n+1}] = (I - S_n)F[\mu] + (I - S_n)R_n + r_n \tag{3.70}$$

by (3.13)–(3.20). Since $\varepsilon_{n+1} = |F[\mu + \tilde{u}_{n+1}]|_0$, we have, in consequence of (3.70),

$$\begin{aligned} \varepsilon_{n+1} \leq\ & |(I - S_n)F[\mu]|_0 + |(I - S_n)R_n|_0 + |r_n|_0 \\ & + |F[\mu + \tilde{u}_{n+1}] - F[\mu + u_{n+1}]|_0. \end{aligned} \tag{3.71}$$

We shall estimate each term of (3.71) separately. (3.7), (3.9), (3.3), and $1 + \kappa - \tau \leq 0$ show

$$|(I - S_n)F[\mu]|_0 \leq C(0)C(1 + \kappa, s_0)\varepsilon\theta_{n+1}^{\tau-\sigma}. \tag{3.72}$$

(3.7), (3.9), (3.18), $(3.39)_{j\leq n}$, $\sigma \leq s_0/2 - 1$, $\theta \geq 2$, and $1 + \kappa - \tau \leq 0$ give

$$|(I - S_n)R_n|_0 \leq 2C_1C(0)C(1 + \kappa, s_0 - 2)\varepsilon\theta_{n+1}^{\tau-\sigma}. \tag{3.73}$$

(3.7), (3.30), $C_1 \geq (s_0+1)C/4$, and $1+\kappa-\tau \leq 0$ imply

$$|r_0|_0 \leq C_1 C(0)\varepsilon\theta^{\tau-\sigma}. \tag{3.74}$$

(3.7), $(3.39)_{n+1}$, $1+\kappa-\tau \leq 0$, and $\sigma+1+\kappa-2\tau \leq 0$ show

$$|r_n|_0 \leq C_1 C(0)\varepsilon\theta_{n+1}^{\tau-\sigma} \qquad (n \geq 1). \tag{3.75}$$

(3.10), (3.2), (3.7), (3.8), $(3.37)_{n+1}$, $(3.38)_{n+1}$, and $\tau \geq 3+\kappa$ give

$$|F[\mu+\tilde{u}_{n+1}] - F[\mu+u_{n+1}]|_0 \leq 2CC_0C(2)(C+C(2)\sqrt{\varepsilon})\sqrt{\varepsilon}\theta_{n+1}^{\tau-\sigma}. \tag{3.76}$$

By combining (3.72)–(3.76), we obtain, in consequence of $\varepsilon \leq 1$,

$$\varepsilon_{n+1} \leq \{C(0)C(1+\kappa, s_0) + C_1C(0)(2C(1+\kappa, S_0-2)+1)\sqrt{\varepsilon} + 2CC_0C(2)(C+C(2))\}\sqrt{\varepsilon}\theta_{n+1}^{\tau-\sigma},$$

which implies, by $C(0)(2C(1+\kappa, s_0-2)+1)\sqrt{\varepsilon} \leq 1/C$,

$$\varepsilon_{n+1} \leq C_3\sqrt{\varepsilon}\theta_{n+1}^{\tau-\sigma} \tag{3.77}$$

if we put

$$C_3 = C(0)C(1+\kappa, s_0) + C_1/C + 2CC_0C(2)(C+C(2)) + 1. \tag{3.78}$$

Substituting (3.78) in (3.63), we can determine C_1 explicitly. Hence, by (3.69) and (3.78), we also get explicit expressions of C_2 and C_3. This completes the proof.

Proof of Theorem 3.8. If we put $\sigma = s_0/2 - 1$, then $(3.36)_j$ gives

$$\|u_i - u_j\|_{s_0/2-1-\kappa} \leq \sum_{\nu=j}^{i-1} \|v_\nu\|_{s_0/2-1-\kappa} \leq \frac{\sqrt{\varepsilon}}{2}\sum_{\nu=j}^{i-1}(\theta^{-\kappa})^\nu \to 0 \tag{3.79}$$

as $i, j \to \infty$, $i > j$. Since $s_0/2 - 2 - 2\kappa \geq 5 - 1 - \kappa \geq 2$, (3.7) shows

$$|u_i - u_j|_2 \leq C(2)\|u_i - u_j\|_{s_0/2-1-\kappa}.$$

Hence, there is a function $u_\infty \in W^{s_0/2-1-\kappa}(\Omega) \cap C^2(\overline{\Omega})$ satisfying $u_n \to u_\infty$ in $W^{s_0/2-k-\kappa}(\Omega) \cap C^2(\overline{\Omega})$.

Combining (3.70) with (3.9), $(3.39)_j$, (3.7), and $s_0/2 - 2 - 2\kappa \geq 5 - 1 - \kappa \geq 0$, we can show that $F[\mu + u_n] \to 0$ in $W^{s_0/2-3-\kappa}(\Omega) \cap C^0(\overline{\Omega})$.

Since $u_n|\partial D_\rho = \sum_{j=0}^{n-1} v_n|\partial D_\rho = 0$, we have $u_\infty|\partial D_\rho = 0$. By $(3.37)_j$, we obtain $\|u_\infty\|_{s_0/2-1-\kappa} = \lim_{n\to\infty}\|u_n\|_{s_0/2-1-\kappa} \leq \sqrt{\varepsilon}$, which completes the proof. q.e.d.

Now, for the proof of Theorem 1.1, it suffices to verify that Assumptions 3.4 and 3.5 are really fulfilled, i.e., that (3.21)–(3.24) follow from (1.1), (1.2), and $r \geq 14 + 2\kappa$.

Proof of Theorem 1.1 (*Part* 1). We shall show that (3.21) and (3.22) are valid for any n.

Step 1. (3.21) and (3.22) hold when $n = 0$. In fact, if we choose the constants $\delta > 0$ and $\rho > 0$ sufficiently small and $a > 0$ sufficiently large, then we have $g^{ij}\mu_i\mu_j < 1/2$ and $L\mu > 2$.

Step 2. If we assume that (3.21) and (3.22) are valid for $n = 1, 2, \cdots, k$, then by (3.13), we can construct a function u_{k+1}. Using the proof of Proposition 3.7 we have $\|u_j\|_s \leq \sqrt{\varepsilon}$ for $s \leq \sigma - \kappa$, $\sigma = s_0/2 - 1$ and $j = 0, 1, \cdots, k+1$. Since $r \geq 14+2\kappa$ and $s_0 = r-2$, Sobolev and Hausdorff-Young inequalities, i.e., (3.7) and (3.8), give $|\tilde{u}_{k+1}|_2 \leq C(2)\sqrt{\varepsilon}$, which shows that (3.21) and (3.22) hold for $n = k+1$. Therefore, Assumption 3.4 is fulfilled.

4. Proof of Theorem 1.1 (Part 2)

The remaining part of the proof of Theorem 1.1 follows from certain estimates which we show in this section by modifying Amano's calculation [2]. Roughly speaking, we have a strong estimate in an elliptic region (Lemma 4.4) and a weak one in a neighborhood of degenerating points (Lemma 4.3). Using a sort of patchwork technique (Lemma 4.2), we can combine them together to obtain (3.23) and (3.24).

Unless otherwise specified, P denotes a degenerate elliptic operator of the form

$$P = a^{ij}\partial_i\partial_u + a^i\partial_i \tag{4.1}$$

with real C^∞ coefficients $a^{ij} = a^{ji}$ and a^i defined in $\overline{\Omega}$, where Ω is a bounded domain in $\mathbb{R}^d$ with C^∞ boundary. Assume that there is a continuous function $\lambda(x) \geq 0$ defined in $\overline{\Omega}$ such that

$$\int_{|\xi|=1} a^{ij}(x)\xi_i\xi_j \geq \lambda(x). \tag{4.2}$$

S stands for a subset of $\overline{\Omega}$ satisfying $\{x \in \overline{\Omega}: \lambda(x) = 0\} \subset S$. For the sake of simplicity, we put

$$A_k = \max_{i,j}\left(\max_{1\leq|s|\leq k}|D^s a^{ij}|_0,\ \max_{1\leq|s|\leq k}|D^s a^i|_0\right)$$

for $k \geq 1$ and

$$B_k = \max_{i,j}(|a^{ij}|_k,\ |a^i|_k,\ 1).$$

Unless otherwise specified, C and C_i are positive constants independent of a^{ij} and a^i.

Lemma 4.1. *We have*

$$(4.3)\quad \sum_k |||[\partial_k, P]u|||_0^2 \le C(A_2 |||Pu|||_1 |||u|||_1 + A_2^2 |||u|||_1^2) \qquad (u \in C_0^\infty(\Omega)),$$

$$(4.4)\quad \sum_k |||[\partial_k, P]u|||_s^2 \le C\Bigl(A_2 |||Pu|||_{s+1} |||u|||_{s+1} + \sum_{\substack{i+j\le s+1\\ i+2\le s+1}} A_{i+2}^2 |||u|||_j^2\Bigr),$$

$$(u \in C_0^\infty(\Omega),\ s \ge 1).$$

Proof of Step 1. We shall prove (4.3). Lemma 1.7.1 of Oleinik and Radkevich [6] shows that

$$\sum_k (a_k^{ij} u_{ij})^2 \le CA_2 \sum_l a^{ij} u_{li} u_{lj} \qquad (u \in C_0^\infty(\Omega)),$$

which implies

$$(4.5)\quad \sum_k |||[\partial_k, P]u|||_0^2 \le C \sum \int \{(a_k^{ij} u_{ij})^2 + (a_k^i u_i)^2\}\, dx \le CA_2 \sum_l \int a^{ij} u_{li} u_{lj}\, dx + CA_1^2 |||u|||_1^2.$$

By integrating by parts, we have

$$\int a^{ij} u_{li} u_{lj}\, dx = -((Pu)_l, u_l) + ([\partial_l, P]u, u_l) + (\tfrac{1}{2}(a_{ij}^{ij} - a_j^i)u_l, u_l),$$

which means

$$(4.6)\quad \sum_k \int a^{ij} u_{ki} u_{kj}\, dx \le c\Bigl(|||Pu|||_1 |||u|||_1 + \sum_k |||[\partial_k, P]u|||_0 |||u|||_1 + A_2 |||u|||_1^2\Bigr).$$

From (4.5) and (4.6) it follows that

$$\sum_k |||[\partial_k, P]u|||_0^2 \le C(A_2 |||Pu|||_1 |||u|||_1 + A_2^2 |||u|||_1^2) \qquad (u \in C_0^\infty(\Omega)).$$

Step 2. We shall prove (4.4) for $s = 1$. (4.3) shows

$$|||\partial_l[\partial_k, P]u|||_0^2 \le C(|||[\partial_k, P]u_l|||_0^2 + |||[\partial_l, [\partial_k, P]]u|||_0^2) \le C(A_2 |||Pu|||_2 |||u|||_2 + A_2 |||[\partial_l, P]u|||_1 |||u|||_2 + A_2^2 |||u|||_2^2).$$

Hence we have

$$\sum_k |||[\partial_k, P]u|||_1^2 \le C(A_2 |||Pu|||_2 |||u|||_2 + A_2^2 |||u|||_2) \qquad (u \in C_0^\infty(\Omega)).$$

Step 3. We assume that (4.4) is valid for $1 \le s < r$. Direct computation gives

$$\|\partial_l[\partial_k, P]u\|_{r-1}^2 \le C(\|[\partial_k, P]u_l\|_{r-1}^2 + \|[\partial_l, [\partial_k, P]]u\|_{r-1}^2)$$
$$\le C\Big(A_2\|Pu\|_{r+1}\|u\|_{r+1} + a_2\|[\partial_l, P]u\|_r\|u\|_{r+1} + \sum_{\substack{i+j\le r+1 \\ i+2\le r+1}} A_{i+2}^2\|u\|_j^2\Big).$$

Therefore, we obtain

$$\sum_k \|[\partial_k, P]u\|_r^2$$
$$\le C\Big(A_2\|Pu\|_{r+1}\|u\|_{r+1} + \sum_{\substack{i+j\le r+1 \\ i+2\le r+1}} A_{i+2}^2\|u\|_j^2\Big) \qquad (u \in C_0^\infty(\Omega)),$$

and the proof of Lemma 4.1 is complete.

Lemma 4.2. *For a fixed* $\chi \in C^\infty(\Omega)$ *satisfying* $\operatorname{supp}|\nabla\chi| \subset \Omega$,

$$\|[\chi, P]u\|_0^2 \le C(B_0\|Pu\|_0\|u\|_0 + B_2^2\|u\|_0^2) \qquad (u \in C^\infty(\Omega)), \tag{4.7}$$

$$\|[\chi, P]u\|_1^2 \le C(B_2\|Pu\|_1\|u\|_1 + B_2^2\|u\|_1^2) \qquad (u \in C^\infty(r)), \tag{4.8}$$

(4.9)
$$\|[\chi, P]u\|_1^2 \le C\Big(B_2\|Pu\|_1\|u\|_1 + \sum_{\substack{i+j\le s \\ i+2\le s}} B_{i+2}^2\|u\|_j^2\Big), \qquad (u \in C^\infty(\Omega),\ s \ge 2).$$

Proof of Step 1. We shall prove (4.7). Let us consider a cut-off function $\tilde\chi \in C_0^\infty(\Omega)$ satisfying $0 \le \tilde\chi \le 1$ and $\partial_i\chi \subset\subset \tilde\chi$ for any i, and define an operator $\tilde P = \tilde a^{ij}\partial_i\partial_j + \tilde a^i\partial_i$ by $\tilde P = \tilde\chi P$. Since $[\chi, \tilde P]u = [\chi, P]u$ and $\|\tilde Pu\|_0 \le \|Pu\|_0$, it will suffice to prove

$$\|[\chi, \tilde P]u\|_0^2 \le C(B_0\|\tilde Pu\|_0\|u\|_0 + B_2^2\|u\|_0^2) \qquad (u \in C^\infty(\Omega)). \tag{4.10}$$

Corollary of Lemma 1.7.1 of Oleinik and Radkevich [6] shows that

$$\sum_+ k(\tilde a^{ij}u_j)^2 \le B_0\tilde a^{kl}u_ku_l \qquad (u \in C^\infty(\Omega)),$$

which gives

$$\|[\chi, \widetilde{P}]u\|_0^2 \le CB_0 \int \tilde{a}^{kl} u_k u_l \, dx + CB_0^2 \|u\|_0^2. \tag{4.11}$$

by integrating by parts, we have

$$\begin{aligned} \int \tilde{a}^{kl} u_k u_l \, dx &= -(\widetilde{P}u, u) + \tfrac{1}{2}((\tilde{a}^{kl}_{kl} - \tilde{a}^k_k)u, u) \\ &\le \|\widetilde{P}u\|_0 \|u\|_0 + CA_2 \|u\|_0^2, \end{aligned} \tag{4.12}$$

which together with (4.11) implies (4.10).

Step 2. We shall prove (4.8). Let us take the same cut-off function $\widetilde{\chi}$ as in Step 1 and define an operator $\widetilde{P} = \tilde{a}^{ij}\partial_i\partial_j + \tilde{a}^i\partial_i$ by $\widetilde{P} = \widetilde{\chi}P$. Since $[\chi, P]u = [\chi, \widetilde{P}]u$ and $\|\widetilde{P}u\|_1 \le C\|Pu\|_1$, we need only to prove

$$\|[\chi, \widetilde{P}]u\|_1^2 \le C(B_2\|\widetilde{P}u\|_1\|u\|_1 + B_2^2\|u\|_1^2) \qquad (u \in C^\infty(\Omega)). \tag{4.13}$$

(4.7) and Lemma 4.1 give

$$\begin{aligned} \|\partial_k[\chi, \widetilde{P}]u\|_0^2 &\le C(\|\chi, \widetilde{P}]u_k\|_0^2 + \|[\partial_k, [\chi, \widetilde{P}]]u\|_0^2) \\ &\le C(B_2\|\widetilde{P}u\|_1\|u\|_1 + B_2^2\|u\|_1^2), \end{aligned}$$

which implies (4.13).

Step 3. We prove (4.9) for $s = 2$. We need only to prove

$$\|[\chi, \widetilde{P}]u\|_2^2 \le C(B_2\|\widetilde{P}u\|_2\|u\|_2 + B_2^2\|u\|_2^2) \qquad (u \in C^\infty(\Omega)), \tag{4.14}$$

where $\widetilde{P}$ is the operator introduced in Steps 1 and 2. (4.13) and Lemma 4.1 give

$$\begin{aligned} \|\partial_k[\chi, \widetilde{P}]u\|_1^2 &\le C(\|[\chi, \widetilde{P}]u_k\|_1^2 + \|[\partial_k, [\chi, \widetilde{P}]]u\|_1^2) \\ &\le C(B_2\|\widetilde{P}u\|_2\|u\|_2 + B_2^2\|u\|_2^2), \qquad (u \in C^\infty(\Omega)). \end{aligned}$$

Hence(4.14) is proved

Step 4. We assume that

$$\begin{aligned} &\|[\chi, \widetilde{P}]u\|_s^2 \\ &\quad \le C\left(B_2\|\widetilde{P}u\|_s\|u\|_s + \sum_{\substack{i+j\le s \\ i+2\le s}} B_{i+2}^2\|u\|_j^2\right), \qquad (u \in C^\infty(\Omega),\ 2 \le s < r). \end{aligned}$$

Direct computation gives

$$\begin{aligned} &\|\partial_k[\chi, \widetilde{P}]u\|_{r-1}^2 \\ &\quad \le C(\|[\chi, \widetilde{P}]u_k\|_{r-1}^2 + \|[\partial_k, [\chi, \widetilde{P}]]u\|_{r-1}^2) \\ &\quad \le C\left(B_2\|\widetilde{P}u\|_r\|u\|_r + \sum_{\substack{i+j\le r \\ i+2\le r}} B_{i+2}^2\|u\|_j^2\right), \qquad (u \in C^\infty(\Omega)); \end{aligned}$$

which implies

$$\|[\chi, \tilde{P}]u\|_r^2 \le C\Big(B_2\|\tilde{P}u\|_r\|u\|_r + \sum_{\substack{i+j\le r\\ i+2<r}} B_{i+2}^2\|u\|_j^2\Big), \qquad (u \in C^\infty(\Omega)).$$

Hence the proof of Lemma 4.2 is complete.

Lemma 4.2. *Assume that*

$$(4.15) \qquad \|u\|_0 \le K\|Pu\|_0, \qquad (u \in C_0^\infty(\Omega)).$$

Then we have

(4.16)

$$\|u\|_1 \le KC(\|Pu\|_1 + A_2\|u\|_1), \qquad (u \in C_0^\infty(\Omega)),$$

(4.17)

$$\|u\|_s \le KC\Big(\|Pu\|_s + \sum_{\substack{i+j\le s\\ i+2\le s}} A_{i+2}\|u\|_j\Big), \qquad (u \in C_0^\infty(\Omega),\ s \ge 2).$$

Proof of Step 1. We shall prove (4.16). Since

$$\|\partial_k u\|_0 \le K\|Pu\|_1 + K\|[\partial_k, P]u\|_0$$

and, by (4.3),

$$\|[\partial_k, P]u\|_0 \le C(\|Pu\|_1 + A_2\|u\|_1),$$

we easily have (4.16).

Step 2. (4.17) is valid for $s = 2$. In fact, it follows from (4.16) and (4.4) that

$$\begin{aligned}\|\partial_k u\|_1 &\le KC(\|Pu\|_2 + \|[\partial_k, P]u\|_1 + A_2\|u\|_1)\\ &\le KC\Big(\|Pu\|_2 + \sum_{\substack{i+j\le 2\\ i+2\le 2}} A_{i+2}\|u\|_j\Big).\end{aligned}$$

Step 3. If (4.17) holds for $2 \le s \le r$, then we have

$$\|\partial_k u\|_r \le KC\Big(\|Pu\|_{r+1} + \|[\partial_k, P]u\|_r + \sum_{\substack{i+j\le r\\ i+2\le r}} A_{i=2}\|u\|_{j+1}\Big),$$

and (4.4) gives

$$\|[\partial_k, P]u\|_r \le C\Big(\|Pu\|_{r+1} + \sum_{\substack{i+j\le r+1\\ i+2\le r+1}} A_{i+2}\|u\|_j\Big).$$

Combining the above two inequalities, we can show that (4.17) is valid for $s = r + 1$. Hence Lemma 4.3 is proved.

(4.16) and (4.17) will turn out to be trivial and useless unless the crucial constant $A_2 \geq 0$ is sufficiently small. Fortunately, by virtue of the definition of μ_p, the estimate $(3.37)_j$, and the fact $\varepsilon \ll 1$ which follows from our assumption (1.1), we obtain $A_2 \ll 1$. Hence, Lemma 4.3 is not useless, and actually works very well when it is teamed with the following lemmas.

Lemma 4.4. *Assume that P is uniformly elliptic in Ω, i.e.,*

$$a^{ij}(x)\xi_i\xi_j \geq \lambda_0|\xi|^2, \qquad \lambda_0 = \text{const} > 0.$$

Then there is a constant $C_\lambda \geq 0$ of the form $C_\lambda = C \cdot |$ a polynomial of λ_0^{-1} and $B_0|$ such that

$$\|u\|_1 \leq C_\lambda \left(\|Pu\|_0 + \sqrt{\tfrac{1}{2}\sup|a_{ij}^{ij} - a_i^i|}\|u\|_0\right), \quad (u \in C^\infty(\overline{\Omega}),\ u|_{\partial\Omega} = 0), \tag{4.18}$$

$$\|u\|_s \leq C_\lambda\left(\|Pu\|_{s-1} + \sum_{\substack{i+j\leq s-1\\ j<s-1}} B_{i+1}\|u\|_{j+1}\right), \quad (u \in C^\infty(\overline{\Omega}),\ u|_{\partial\Omega} = 0,\ s \geq 2). \tag{4.19}$$

It is not difficult to prove (4.18). In fact, we need only to apply well-known standard techniques to the linear elliptic operator P and to calculate several constants precisely. By induction with respect to s and patient calculation, (4.19) follows from (4.18).

For $\delta > 0$ we define a set S_δ by $S_\delta = \{x \in \overline{\Omega} \colon \operatorname{dist}(x, S) < \delta\}$.

Lemma 4.5. *Assume that S is a compact C^∞ submanifold of Ω and $\Omega \setminus S$ is connected. Then there exists a function $\gamma \in L^\infty(\Omega)$ such that $\gamma = 0$ on S, $\inf_{\Omega\setminus S_\delta} \gamma > 0$ for any sufficiently small $\delta > 0$, and*

$$\int \gamma u^2\,dx \leq C\left(\|Pu\|_0\|u\|_0 + \tfrac{1}{2}\sup|a_{ij}^{ij} - a_i^i|\|u\|_0^2\right), \quad (u \in C^\infty(\overline{\Omega}),\ u|_{\partial\Omega} = 0). \tag{4.20}$$

Proof. Standard techniques of elliptic operators give

$$\int \lambda|Du|^2\,dx \leq C\left(\|Pu\|_0\|u\|_0 + \tfrac{1}{2}\sup|a_{ij}^{ij} - a_i^i|\|u\|_0^2\right),$$

where $\lambda = \lambda(x)$ is a continuous function satisfying (4.2) and $Du = (\partial_1 u, \partial_2 u, \cdots, \partial_d u)$. Hence, it suffices for our purpose to show

$$\int \gamma u^2\, dx \le \int \lambda |Du|^2\, dx. \tag{4.21}$$

Step 1. Let us fix a point $p \in \overline{\Omega \setminus S}$ arbitrarily. By virtue of the fundamental theorem of ordinary differential equations, we can construct a family of curves $c(t; x) \in C^\infty([0, T_p] \times U_p)$ such that $c(0; x) = x$, $c(t; x) \notin S$ for $0 < t < T_p$ when $x \in \overline{\Omega \setminus S}$, $c(T_p; x) \notin \overline{\Omega}$, $|\dot{c}(t; x)| \equiv 1$, $\sup_{x \in U_p} \tau_x < \infty$, and such that $c(t; \cdot)$ is a local C^∞ diffeomorphism defined in U_p for any fixed t, where U_p is a sufficiently small open neighborhood of p, T_p is a positive constant, and $\tau_x = \inf\{t \ge 0 : c(t; x) \notin \overline{\Omega}\}$. We define a function $\mu_p(x)$ by

$$\mu_p(x) = \inf\{\lambda(c(t; x)) : 0 \le t \le \tau_x\} \tag{4.22}$$

for $x \in U_p$. For a function $u \in C^\infty(\overline{\Omega})$ satisfying $u|_{\partial\Omega} = 0$,

$$u(x) = u(c(0; x)) - u(c(\tau_x; x)) = -\int_0^{\tau_x} Du(c(t; x)) \cdot \dot{c}(t; x)\, dt$$

holds, so we have

$$|u(x)|^2 \le C \int_0^{\tau_x} |Du(c(t; x))|^2\, dt. \tag{4.23}$$

Multiplying (4.23) by $\mu_p(x)$ and using (4.22), we obtain

$$\mu_p(x)|u(x)|^2 \le C \int_0^{\tau_x} \lambda(c(t; x))|Du(c(t; x))|^2\, dt,$$

which implies

$$\int_{U_p} \mu_p u^2\, dx \le C \int_\Omega \lambda |Du|^2\, dx.$$

Step 2. Step 1 shows that there is a finite number of points $p_1, p_2, \cdots, p_N$ in $\overline{\Omega \setminus S}$ such that $\overline{\Omega \setminus S} \subset \bigcup_{i=1}^N U_{p_i}$ and

$$\int_{U_{p_i}} \mu_{p_i} u^2\, dx \le C \int_\Omega \lambda |Du|^2\, dx.$$

Therefore, we need only to define $\mu(x)$ by

$$\mu(x) = \begin{cases} \min\{\mu_{p_i}(x) : x \in U_{p_i},\ 1 \le i \le N\} & \text{if } x \in \Omega \setminus S, \\ 0 & \text{if } x \in S. \end{cases}$$

Hence (4.21) is proved. q.e.d.

Let us take a real smooth function $\Phi(x)$ defined in $\overline{\Omega}$ such that $\Phi(x)$ has at least a zero point in Ω and that $\nabla\Phi \neq 0$, and put for $t \geq 1$,

$$U(\Phi, t) = \{x \in \Omega: |\Phi(x)| < 1/t\}.$$

Lemma 4.6. *There are constants* $C_0 > 0$ *and* $C \geq 0$ *independent of* P *and* $t \geq 1$ *such that*

$$\begin{aligned} C_0 t^2 \inf_{U(\Phi, f)} (a^{ij}\Phi_i\Phi_j)\|u\|_0^2 &= t \inf_{U(\Phi,t)} (a^{ij}_{ij} - a^i_i)\|u\|_0^2 \\ &\leq C\left(\|Pu\|_0\|u\|_0 + \frac{1}{2}\sup_{U(\Phi,t)} (a^{ij}_{ij} - a^i_i)\|u\|_0^2\right), \\ &(u \in C^\infty(\overline{U(\Phi, t)}),\ u|_{\partial U(\Phi,t)} = 0,\ t \geq 1). \end{aligned} \tag{4.24}$$

Proof. For a real-valued function $u \in C^\infty(\overline{U(\Phi, t)})$ satisfying $u|_{\partial U(\Phi,t)} = 0$, we put

$$v = (T - e^{t\Phi})^{-1}u, \qquad T = \text{const} > 0.$$

Direct computation gives

$$\begin{aligned} Pu = {} & (T - e^{t\Phi})(a^{ij}v_{ij} + a^i v_i) \\ & - e^{t\Phi}\{t^2(a^{ij}\Phi_i\Phi_j)v + t(a^{ij}\Phi_{ij} + a^i\Phi_i)v + 2t(a^{ij}\Phi_i v_j)\}. \end{aligned}$$

By integrating by parts, we obtain

$$\begin{aligned} & \int (T - e^{t\Phi})^{-1} Pu \cdot v\,dx \\ & \quad = -\int a^{ij}v_i v_j\,dx + \int \left(\frac{1}{2}a^{ij}_{ij} - \frac{1}{2}a^i_i\right) v^2\,dx \\ & \quad\quad - t^2 \int e^{t\Phi}(T - e^{t\Phi})^{-1}(a^{ij}\Phi_i\Phi_j)v^2\,dx \\ & \quad\quad - t\int e^{t\Phi}(T - e^{t\Phi})^{-1}(a^{ij}\Phi_{ij} + a^i\Phi_i)v^2\,dx \\ & \quad\quad - w\int a^{ij}\{te^{t\Phi}(T - e^{t\Phi})^{-1}\Phi_i v\}v_j\,dx. \end{aligned}$$

Since

$$\begin{aligned} & \left|2\int a^{ij}\{te^{t\Phi}(T - e^{t\Phi})^{-1}\Phi_i v\}v_j\,dx\right| \\ & \quad \leq t^2 \int e^{2t\Phi}(T - e^{t\Phi})^{-2}(a^{ij}\Phi_i\Phi_j)v^2\,dx + \int a^{ij}v_i v_j\,dx \end{aligned}$$

and

$$e^{t\Phi}(T - e^{t\Phi})^{-1} - e^{2t\Phi})(T - e^{t\Phi})^{-2} = e^{t\Phi}(T - 2e^{t\Phi})(T - e^{t\Phi})^{-2},$$

we have

$$\begin{aligned}(4.25)\qquad &\int (T-e^{t\Phi})^{-2}Pu\cdot u\,dx\\ &\le -t^2\int e^{t\Phi}(T-2e^{t\Phi})(T-e^{t\Phi})^{-4}(a^{ij}\Phi_i\Phi_j)u^2\,dx\\ &\quad -t\int e^{t\Phi}(T-e^{t\Phi})^{-3}(a^{ij}\Phi_{ij}+a^i\Phi_i)u^2\,dx\\ &\quad +\int (T-e^{t\Phi})^{-2}\frac{1}{2}(a^{ij}_{ij}-a^i_i)u^2\,dx.\end{aligned}$$

Combining (4.25) with

$$e^{-1}\le e^{t\Phi}\le e,\quad (T-e^{-1})^{-1}\le (T-e^{t\Phi})^{-1}\le (T-e)^{-1},\quad (x\in U(\Phi,t)),$$

we obtain (4.24). q.e.d.

Now we can prove the remaining part of the proof of Theorem 1.1. From now on C and C_i denote positive constants which are independent of $\varepsilon>0$, and $n=0,1,2,\cdots$.

Proof of Theorem 1.1 (*Part* 2). We shall show that (3.23) and (3.24) are valid for any n. For the sake of simplicity, we put $L_n=L_{\varepsilon_n}[\mu+\tilde u_n]$ for $n=0,1,\cdots$, and use $A_k^{(n)}$ and $B_k^{(n)}$ to denote constants A_k and B_k when $a^{ij}=a^{ij}_{\varepsilon_n}[\mu+\tilde u_n]$ and $a^i=a^i_{\varepsilon_n}[\mu+\tilde u_n]$. Let us take the following cut-off functions χ, $\tilde\chi$, and $\tilde{\tilde\chi}$: $\chi\in C_0^\infty(D_\rho)$, $\tilde\chi$, $\tilde{\tilde\chi}\in C_0^\infty(\overline{D}_\rho\setminus\overline{D})$, $0\le\chi,\tilde\chi,\tilde{\tilde\chi}\le 1$, $\chi=1$ in a neighborhood of $D_\rho/2$, $\tilde\chi=1$ in a neighborhood of $\operatorname{supp}(\partial_1\chi)\cup\operatorname{supp}(\partial_2\chi)$, and $\tilde{\tilde\chi}=1$ in a neighborhood of $\sup\tilde\chi$.

Step 1 (*estimate of* $\|\chi v_0\|_s$). Applying Lemma 4.6 to a function $\Phi(x)=\Phi(x^1,x^2)=(x^2-p^2)/(\text{the diameter of } D_\rho)$, $p=(p^1,p^2)\in D$, and the operator

$$L_0=L_{\varepsilon_0}[\mu]=a^{ij}_{\varepsilon_0}[\mu]\partial_i\partial_j+a^i_{\varepsilon_0}[\mu]\partial_i,$$

we obtain

$$\begin{aligned}&C_0\inf a^{22}_{\varepsilon_0}[\mu]\|\chi v_0\|_0^2+\inf a^2_{\varepsilon_0}[\mu]\|\chi v_0\|_0^2\\ &\quad\le C(\|L_0\chi v_0\|_0\|_0\cdot\|\chi v_0\|_0\\ &\qquad+\tfrac12\operatorname{supp}((a^{ij}_{\varepsilon_0}[\mu])_{ij}-(a^i_{\varepsilon_0}[\mu])_i)\|\chi v_0\|_0^2).\end{aligned}$$

Here we note that, by (1.1) and the definition of $\mu=\mu_\rho$, we have $a^{22}_{\varepsilon_0}[\mu]\sim 2\delta$, $a^2_{\varepsilon_0}[\mu]\sim 0$, and

$$(4.26)\qquad (a^{ij}_{\varepsilon_0}[\mu])_{ij}-(a^i_{\varepsilon_0}[\mu])_i\sim(\mu_{2211}-\mu_{2112}-\mu_{1221}+\mu_{2211})=0,$$

where $A \sim B$ means that A and its derivatives are approximately equal to B and its derivatives respectively. This implies that

$$\|\chi v_0\|_0 \leq C\|L_0\chi v_0\|_0. \tag{4.27}$$

Lemma 4.3 shows that (4.27) gives

$$\|\chi v_0\|_1 \leq C(\|L_0\chi v_0\|_1 + A_2^{(0)}\|\chi v_0\|_1),$$

$$\|\chi v_0\|_s \leq C\Bigg(\|L_0\chi v_0\|_s + \sum_{\substack{i+j\leq s\\ i+2\leq s}} A_{i+2}^{(0)}\|\chi v_0\|_j\Bigg) \qquad (s \geq 2).$$

Hence, if we take $\rho > 0$ sufficiently small so that

$$CA_2^{(0)} = C\max_{i,j}\left(\max_{1\leq s\leq 2}|D^s a_{\varepsilon_0}^{ij}[\mu]|_0, \max_{1\leq s\leq 2}|D^s a_{\varepsilon_0}^{i}[\mu]|_0\right) < 1,$$

then we obtain

$$\|\chi v_0\|_1 \leq C\|L_0\chi v_0\|_1, \tag{4.28}$$

$$\|\chi v_0\|_s \leq C\Bigg(\|L_0\chi v_0\|_s + \sum_{\substack{i+j\leq s\\ i+2\leq s,\ j<s}} A_{i+2}^{(0)}\|\chi v_0\|_j\Bigg), \qquad (2 \leq s \leq s_0). \tag{4.29}$$

Step 2 (*estimates of* $\|(1-\chi)v_0\|_s$). Since

$$\det(\mu_{ij}) \geq (\phi - \rho/2)^{-4}\exp(2/(\rho/2-\rho))\cdot\det(\phi_{ij}) > 0 \quad \text{in } \overline{D}_\rho \setminus \overline{D}_{\rho/2},$$

we can show, by (1.1), that L_0 is uniformly elliptic in $\overline{D}_\rho \cap \operatorname{supp}(1-\chi)$. Applying Lemma 4.4 to the operator L_0, we obtain

$$\|(1-\chi)v_0\|_1 \leq C\Big(\|L_0(1-\chi)v_0\|_0 + \tfrac{1}{2}\sqrt{\sup|(a_{\varepsilon_0}^{ij}[\mu])_{ij} - (a_{\varepsilon_0}^{i}[\mu])_i|}\,\|(1-\chi)v_0\|_0\Big),$$

this gives, in consequence of (4.26),

$$\|(1-\chi)v_0\|_1 \leq C\|L_0(1-\chi)v_0\|_0. \tag{4.30}$$

It is easy to show, by Lemma 4.4, that

$$\|(1-\chi)v_0\|_s \leq C\Bigg(\|L_0(1-\chi)v_0\|_{s-1} + \sum_{\substack{i+j\leq s-1\\ j<s-1}} B_{i+1}^{(0)}\|(1-\chi)v_0\|_{j+1}\Bigg) \qquad (2 \leq s \leq s_0). \tag{4.31}$$

Step 3 (*estimates of* $\|v_0\|_s$). (4.27) and (4.30) imply

$$\|v_0\|_0 \le C(\|L_0 v_0\|_0 + \|[\chi, L_0]v_0\|_0 + \|[(1-\chi), L_0]v_0\|_0). \tag{4.32}$$

By virtue of Lemma 4.2 and the definition of χ, $\tilde{\chi}$, $\tilde{\tilde{\chi}}$, we have

$$\begin{aligned}\|[\chi, L_0]v_0 &= \|[\chi, \tilde{\chi}L_0]\tilde{\tilde{\chi}}v_0\|_0 \\ &\le C(\|\tilde{\chi}L_0\tilde{\tilde{\chi}}v_0\|_0 + B_2^{(0)}\|\tilde{\tilde{\chi}}v_0\|_0) \\ &\le C(\|L_0 v_0\|_0 + B_2^{(0)}\|\tilde{\tilde{\chi}}v_0\|_0),\end{aligned} \tag{4.33}$$

$$\begin{aligned}\|[(1-\chi), L_0]v_0\|_0 &= \|[(1-\chi), \tilde{\chi}L_0]\tilde{\tilde{\chi}}v_0\|_0 \\ &\le C(\|\tilde{\chi}L_0\tilde{\tilde{\chi}}v_0\|_0 + B_2^{(0)}\|\tilde{\tilde{\chi}}v_0\|_0) \\ &\le C(\|L_0 v_0\|_0 + B_2^{(0)}\|\tilde{\tilde{\chi}}v_0\|0).\end{aligned} \tag{4.34}$$

Since L_0 is uniformly elliptic in $\overline{D}_\rho \cap (\operatorname{supp}\tilde{\tilde{\chi}})$, by applying Lemma 4.5 to $u = v_0$, $S = \overline{D}$, and $\Omega = D_\rho$, we obtain

$$\begin{aligned}\|\tilde{\tilde{\chi}}v_0\|_0^2 \le C(&\|L_0 v_0\|_0\|v_0\|_0 \\ &+ \tfrac{1}{2}\sup|(a_{\varepsilon_0}^{ij}[\mu])_{ij} - (a_{\varepsilon_0}^{i}[\mu])_i|\|v_0\|_0^2).\end{aligned} \tag{4.35}$$

It follows from (4.32)–(4.35) that

$$\begin{aligned}\|v_0\|_0 \le C(&\|L_0 v_0\| + \sqrt{\|L_0 v_0\|_0\|v_0\|_0} \\ &+ \sqrt{\tfrac{1}{2}\sup|(a_{\varepsilon_0}^{ij}[\mu])_{ij} - (a_{\varepsilon_0}^{i}[\mu])_i|}\|v_0\|_0),\end{aligned}$$

which implies, in consequence of (4.26),

$$\|v_0\|_0 \le C\|L_0 v_0\|_0. \tag{4.36}$$

(4.28) and (4.30) give

$$\begin{aligned}\|v_0\|_1 &\le C(\|L_0\chi v_0\|_1 + \|L_0(1-\chi)v_0\|_0) \\ &\le C(\|L_0 v_0\|_1 + \|[\chi, L_0]v_0\|_1 + \|[(1-\chi), L_0]v_0\|_0).\end{aligned} \tag{4.37}$$

As in (4.33) and (4.34), by using Lemma 4.2 and cut-off functions χ, $\tilde{\chi}$, $\tilde{\tilde{\chi}}$ we have

$$\|[\chi, L_0]v_0\|_1 + \|[(1-\chi), L_0]v_0\|_0 \le C(\|L_0 v_0\|_1 + B_2^{(0)}\|\tilde{\tilde{\chi}}v_0\|_1). \tag{4.38}$$

(4.18) of Lemma 4.4 and (4.7) of Lemma 4.2 show

$$\begin{aligned}\|\tilde{\tilde{\chi}}v_0\|_1 &\le C(\|L_0\tilde{\tilde{\chi}}v_0\|_0 + B_2^{(0)}\|\tilde{\tilde{\chi}}v_0\|_0) \\ &\le C(\|L_0 v_0\|_0 + B_2^{(0)}\|v_0\|_0 + \|[\tilde{\tilde{\chi}}, L_0]v_0\|_0) \\ &\le C(\|L_0 v_0\|_0 + B_2^{(0)}\|v_0\|_0).\end{aligned} \tag{4.39}$$

Combining (4.37)–(4.39) with (4.36), we obtain

$$\|v_0\|_1 \le C\|L_0 v_0\|_1. \tag{4.40}$$

(4.29) and (4.31) give

$$\begin{aligned}\|v_0\|_s &\le C\Bigl(\|L_0\chi v_0\|_s + \|L_0(1-\chi)v_0\|_{s-1} + \sum_{\substack{i+j\le s\\ i+2\le s,\ j<s}} B_{i+2}^{(0)}\|v_0\|_j\Bigr)\\ &\le C\Bigl(\|L_0 v_0\|_s + \sum_{\substack{i+j\le s\\ i+2\le s,\ j<s}} B_{i+2}^{(0)}\|v_0\|_j\\ &\qquad + \|[\chi, L_0]v_0\|_s + \|[(1-\chi), L_0]v_0\|_{s-1}\Bigr)\end{aligned} \tag{4.41}$$

for $s \ge 2$. (4.9) of Lemma 4.2 shows

$$\|[(1-\chi), L_0]v_0\|_{s-1} \le C\Bigl(\|L_0 v_0\|_{s-1} + \sum_{\substack{i+j\le s-1\\ i+2\le s-1}} B_{i+2}^{(0)}\|v_0\|_j\Bigr) \tag{4.42}$$

and

$$\begin{aligned}\|[\chi, L_0]v_0\|_s &= \|[\chi, \tilde{\chi}L_0]\tilde{\tilde{\chi}}v_0\|_s\\ &\le C\Bigl(\|\tilde{\chi}L_0\tilde{\tilde{\chi}}v_0\|_s + \sum_{\substack{i+j\le s\\ i+2\le s}} B_{i+2}^{(0)}\|\tilde{\tilde{\chi}}v_0\|_j\Bigr)\\ &\le C\Bigl(\|L_0 v_0\|_s + \sum_{\substack{i+j\le s\\ i+2\le s,\ j<s}} B_{i+2}^{(0)}\|v_0\|_j + B_2^{(0)}\|\tilde{\tilde{\chi}}v_0\|_s\Bigr).\end{aligned} \tag{4.43}$$

Since L_0 is uniformly elliptic in $D_\rho \cap (\operatorname{supp}\tilde{\tilde{\chi}})$, we have, by (4.19) of Lemma 4.4,

$$\begin{aligned}\|\tilde{\tilde{\chi}}v_0\|_s &\le C\Bigl(\|L_0\tilde{\tilde{\chi}}v_0\|_{s-1} + \sum_{\substack{i+j\le s-1\\ j<s-1}} B_{i+1}^{(0)}\|\tilde{\tilde{\chi}}v_0\|_{j+1}\Bigr)\\ &\le C\Bigl(\|L_0 v_0\|_{s-1} + \sum_{\substack{i+j\le s\\ i+2\le s,\ j<s}} B_{i+2}^{(0)}\|v_0\|_j + \|[\tilde{\tilde{\chi}}, L_0]v_0\|_{s-1}\Bigr),\end{aligned}$$

which gives, in consequence of (4.8) and (4.9),

$$\|\tilde{\tilde{\chi}}v_0\|_s \le C\Bigl(\|L_0 v_0\|_{s-1} + \sum_{\substack{i+j\le s\\ i+2\le s,\ j<s}} B_{i+2}^{(0)}\|v_0\|_j\Bigr). \tag{4.44}$$

Hence, combining (4.41)–(4.44), we obtain

$$\|v_0\|_s \le C\Big(\|L_0 v_0\|_s + \sum_{\substack{i+j\le s\\ i+2\le s,\ j<s}} B^{(0)}_{i+2}\|v_0\|_j\Big) \tag{4.45}$$

for $s \ge 2$. Since (3.14) gives $L_0 v_0 = L_{\varepsilon_0}[\mu + \tilde{u}_o]v_0 = f_0$, and (1.1) implies $B^{(0)}_{i+2} \sim |\mu|_{i+4}$, it follows from (4.36), (4.40), and (4.45) that

$$\|v_0\|_0 \le C\|f_0\|_0,$$
$$\|v_0\|_s \le C\Big(\|f_0\|_s + \sum_{\substack{i+j\le s\\ j<s}} |\mu|_{i+4}\|v_0\|_j\Big) \qquad (0 \le s \le s_0),$$

i.e., Assumption 3.5 is fulfilled when $n = 0$.

Step 4 (*estimates of* $\|\chi v_{k+1}\|_s$). In Steps 4–6, we assume that (3.21)–(3.24) are valid for $n = 1, 2, \cdots, k$. Then, by (3.13), we can construct u_{k+1}. Since we have already proved, in Part 1, that (3.21) and (3.22) hold for $n = k+1$, by Lemma 3.2 we are able to solve the Dirichlet problem (3.14) for $n = k+1$ and to obtain v_{k+1}.

As in Step 1, we consider the same auxiliary function $\Phi(x)$ and apply Lemma 4.6 to the operator

$$L_{k+1}y = L_{\varepsilon_{k+1}y}[\mu + \tilde{u}_{k+1}] = a^{ij}_{\varepsilon_{k+1}y}[\mu + \tilde{u}_{k+1}]\partial_i\partial_j + a^{i}_{\varepsilon_{k+1}y}[\mu + \tilde{u}_{k+1}]\partial_i,$$

so that we have

$$\begin{aligned} &C_0 \inf a^{22}_{\varepsilon_{k+1}}[\mu + \tilde{u}_{k+1}]\|\chi v_{k+1}y\|_0^2 + \inf a^{2}_{\varepsilon_{k+1}y}[\mu + \tilde{u}_{k+1}]\|\chi v_{k+1}y\|_0^2 \\ &\quad \le C(\|L_{k+1}\chi v_{k+1}\|_0 \cdot \|\chi v_{k+1}y\|_0 \\ &\qquad + \tfrac{1}{2}\sup((a^{ij}_{\varepsilon_{k+1}}[\mu + \tilde{u}_{k+1}y])_{ij} - (a^{i}_{\varepsilon_{k+1}}[\mu + \tilde{u}_{k+1}y])_i)\|\chi v_{k+1}y\|_0^2]. \end{aligned}$$

By using the proof of Proposition 3.7, $(3.37)_{k+1}y$ and $(3.41)_{k+1}$ are valid. Hence, (1.1), (3.1), (3.15), and the definition of $\mu = \mu_\rho$ give $a^{22}_{\varepsilon_{k+1}}[\mu + \tilde{u}_{k+1}y] \sim 2\delta$, $a^{2}_{\varepsilon_{k+1}}[\mu + \tilde{u}_{k+1}] \sim 0$, and

$$\begin{aligned} &(a^{ij}_{\varepsilon_{k+1}y}[\mu + \tilde{u}_{k+1}y])_{ij} - (a^{i}_{\varepsilon_{k+1}y}[\mu + \tilde{u}_{k+1}y])_i \\ &\qquad \sim (\mu_{2211} - \mu_{2112} - \mu_{1211} - \mu_{2211}) = 0, \end{aligned} \tag{4.46}$$

where $A \sim B$ means that A and its derivatives are approximately equal to B and its derivatives uniformly in $k = 0, 1, 2, \cdots$. This implies

$$\|\chi v_{k+1}y\|_0 \le C\|L_{k+1}\chi v_{k+1}y\|_0. \tag{4.47}$$

By virtue of Lemma 4.3, (4.47) gives

$$\|\chi v_{k+1}\|_1 \le C(\|L_{k+1}\chi v_{k+1}y\|_1 + A_2^{(k+1)}\|\chi v_{k+1}\|_1),$$

$$\|\chi v_{k+1}y\|_2 \le C\left(\|L_{k+1}\chi v_{k+1}\|_s + \sum_{\substack{i+j\le s\\ i+2\le s}} A_{i+2}^{(k+1)}\|\chi v_{k+1}\|_j\right), \qquad (s\ge 2).$$

$(3.37)_{k+1}$ and $(3.41)_{k+1}$ show that

$$|\tilde{u}_{k+1}y|_4 \le C(4)\|\tilde{u}_{k+1}\|_{s+\kappa} \le C(4)\|u_{k+1}\|_{5+\kappa} \le \sqrt{\varepsilon}C(4),$$

when $s = 5+\kappa$ and $\sigma = s_0/2 - 1$, and also that

$$\varepsilon_{k+1} \le C_3\theta_{k+1}^{\tau-\sigma}y \le C_3\theta^{-2} = C_3\varepsilon$$

by (3.33) and (3.1). Thus, it follows from the definition of $\mu = \mu_\rho$ that

$$CA_2^{(k+1)} = \max_{i,j}\left(\max_{1\le s\le 2}|D^s a^{ij}_{\varepsilon_{k+1}y}[\mu+\tilde{u}_{k+1}y]|_0, \max_{1\le s\le 2}|D^s a^{i}_{\varepsilon_{k+1}y}[\mu+\tilde{u}_{k+1}]|_0\right)$$
$$< 1$$

if we take $\rho > 0$ sufficiently small. Here it is to be noted that the procedure to determine ρ depends neither on $\varepsilon_{k+1}y$ nor on $\tilde{u}_{k+1}y$; actually, it depends only on ε, g_{ij}, and certain constants which we have already specified. Hence, we obtain

$$\|\chi v_{k+1}\|_1 \le C\|L_{k+1}y\chi v_{k+1}y\|_1 \tag{4.48}$$

and

$$\|\chi v_{k+1}y\|_s \le C\left(\|L_{k+1}y\chi v_{k+1}y\|_s + \sum_{\substack{i+j\le s\\ i+2\le s,\ j<s}} A_{i+2}^{(k+1)}\|\chi v_{k+1}y\|_j\right) \tag{4.49}$$

for $s\ge 2$.

Step 5 (*estimates of* $\|(1-\chi)v_{k+1}\|_s$). It is relatively easy to estimate $(1-\chi)v_{k+1}$, since $L_{k+1}y$ is uniformly elliptic in $\overline{D}_\rho \cap \operatorname{supp}(1-\chi)$, by virtue of Lemma 3.2, (3.21)–(3.22) with $n = k+1$, and the inequality $\det(\mu_{ij}) > 0$ in $\overline{D}_\rho \setminus \overline{D}_{\rho/2}$ which we have obtained in Step 2. In fact, as in Step 2, (4.18) of Lemma 4.4 and (4.46) give

$$\|(1-\chi)v_{k+1}y\|_1 \le C\|L_{k+1}y(1-\chi)v_{k+1}y\|_0 \tag{4.50}$$

and, also, (4.19) of Lemma 4.4 gives

(4.51)
$$\|(1-\chi)v_{k+1}y\|_s \le C\left(\|L_{k+1}y(1-\chi)v_{k+1}\|_{s-1}\right.$$
$$\left. + \sum_{\substack{i+j\le s-1\\ j<s-1}} B_{i+1}^{(k+1)}\|(1-\chi)v_0\|_{j+1}\right), \qquad (2\le s\le s_0).$$

Step 6 (*estimates of* $\|v_{k+1}\|_2$). (4.47) and (4.50) show

$$\begin{aligned}\|v_{k+1}y\|_0 &\le \|\chi v_{k+1}y\|_0 + \|(1-\chi)v_{k+1}y\|_0 \\ &\le C(\|L_{k+1}v_{k+1}\|_0 + \|[\chi, L_{k+1}]v_{k+1}\|_0 \\ &\qquad + \|[(1-\chi), L_{k+1}]v_{k+1}\|_0).\end{aligned} \tag{4.52}$$

As in Step 3, from Lemma 4.4 and the definition of cut-off functions χ, $\tilde{\chi}$, and $\tilde{\tilde{\chi}}$ it follows that

$$\begin{aligned}&\|[\chi, L_{k+1}]v_{k+1}\|_0 + \|[(1-\chi), L_{k+1}]v_{k+1}\|_0 \\ &= \|[\chi, \tilde{\chi}L_{k+1}]\tilde{\tilde{\chi}}v_{k+1}\|_0 + \|[(1-\chi), \tilde{\chi}L_{k+1}]\tilde{\tilde{\chi}}v_{k+1}\|_0 \\ &\le C(\|\tilde{\chi}L_{k+1}\tilde{\tilde{\chi}}v_{k+1}\|_0 + B_2^{(k+1)}\|\tilde{\tilde{\chi}}v_{k+1}\|_0) \\ &= C(\|L_{k+1}v_{k+1}\|_0 + B_2^{(k+1)}\|\tilde{\tilde{\chi}}v_{k+1}\|_0).\end{aligned} \tag{4.53}$$

Lemma 4.5 shows

$$\begin{aligned}\|\tilde{\tilde{\chi}}v_{k+1}\|_0^2 &\le C(\|L_{k+1}v_{k+1}\|_0 v_{k+1}\|_0 \\ &\quad + \tfrac{1}{2}\sup|(a^{ij}_{\varepsilon_{k+1}y}[\mu+\tilde{u}_{k+1}])_{ij} - (a^{i}_{\varepsilon_{k+1}}[\mu+\tilde{u}_{k+1}])_i|\|v_{k+1}\|_0^2.\end{aligned} \tag{4.54}$$

Combining (4.52) and (4.53) with (4.54), we obtain

$$\|v_{k+1}\|_0 \le C\|L_{k+1}v_{k+1}\|_0. \tag{4.55}$$

Here it is to be noted that, by $(3.37)_{k+1}$ and $(3.41)_{k+1}$, the constant C does not depend on $k = 0, 1, 2, \cdots$. (4.48) and (4.50) imply

$$\begin{aligned}\|v_{k+1}\|_1 &\le \|\chi v_{k+1}\|_1 + \|(1-\chi)v_{k+1}\|_1 \\ &\le C(\|L_{k+1}v_{k+1}\|_1 + \|[\chi, L_{k+1}]v_{k+1}\|_1 \\ &\qquad + \|[(1-\chi), L_{k+1}]v_{k+1}\|_0).\end{aligned} \tag{4.56}$$

By virtue of Lemma 4.2 and the definition of χ, $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$, we have

$$\begin{aligned}&\|[\chi, L_{k+1}]v_{k+1}\|_1 + \|[(1-\chi), L_{k+1}]v_{k+1}\|_0 \\ &= \|[\chi, \tilde{\chi}L_{k+1}]\tilde{\tilde{\chi}}v_{k+1}\|_1 + \|[(1-\chi), \tilde{\chi}L_{k+1}]\tilde{\tilde{\chi}}v_{k+1}\|_0 \\ &\le C(\|\tilde{\chi}L_{k+1}\tilde{\tilde{\chi}}v_{k+1}\|_1 + B_2^{(k+1)}\|\tilde{\tilde{\chi}}v_{k+1}\|_1) \\ &= C(\|L_{k+1}v_{k+1}\|_1 + B_2^{(k+1)}\|\tilde{\tilde{\chi}}v_{k+1}\|_1).\end{aligned} \tag{4.57}$$

(4.18) of Lemma 4.4 and (4.7) of Lemma 4.2 give

$$\|\tilde{\tilde{\chi}}v_{k+1}\|_1 \le C(\|L_{k+1}v_{k+1}\|_0 + B_2^{(k+1)}\|v_{k+1}\|_0). \tag{4.58}$$

Combining (4.55)–(4.57) with (4.58), we obtain

$$\|v_{k+1}\|_1 \le C\|L_{k+1}v_{k+1}\|_1. \tag{4.59}$$

Here $(3.37)_{k+1}$ and $(3.41)_{k+1}$ ensure that the constant C does not depend on $k = 0, 1, 2, \cdots$. By (4.49) and (4.51), we have

$$\begin{aligned}
\|v_{k+1}\|_s &\le \|\chi v_{k+1}\|_s + \|(1-\chi)v_{k+1}\|_s \\
&\le C\Big(\|L_{k+1}\chi v_{k+1}\|_s + \|(1-\chi)v_{k+1}\|_s \\
&\qquad + \sum_{\substack{i+j\le s \\ i+2\le s,\ j<s}} B_{i+2}^{(k+1)}\|v_{k+1}\|_j\Big) \\
&\le C\Big(\|L_{k+1}v_{k+1}\|_s + \sum_{\substack{i+j\le s \\ i+2\le s,\ j<s}} B_{i+2}^{(k+1)}\|v_{k+1}\|_j\Big) \\
&\qquad + \|[\chi, L_{k+1}]v_{k+1}\|_s + \|[(1-\chi), L_{k+1}]v_{k+1}\|_{s-1})
\end{aligned} \tag{4.60}$$

for $s \ge 2$. (4.9) of Lemma 4.2 shows

$$\begin{aligned}
&\|[(1-\chi), L_{k+1}]v_{k+1}\|_{s-1} \\
&\qquad \le C\Big(\|L_{k+1}v_{k+1}\|_{s-1} + \sum_{\substack{i+j\le s-1 \\ i+2\le s-1}} B_{i+2}^{(k+1)}\|v_{k+1}\|_j\Big)
\end{aligned} \tag{4.61}$$

and

$$\begin{aligned}
\|[\chi, L_{k+1}]v_{k+1}\|_s &= \|[\chi, \tilde{\chi}L_{k+1}]\tilde{\tilde{\chi}}v_{k+1}\|_s \\
&\le C\Big(\|L_{k+1}v_{k+1}\| - s + \sum_{\substack{i+j\le s \\ i+2\le s,\ j<s}} B_{i+2}^{(k+1)}\|v_{k+1}\|_j \\
&\qquad + B_2^{(k+1)}\|\tilde{\tilde{\chi}}v_{k+1}\|_s\Big)
\end{aligned} \tag{4.62}$$

for $s \ge 2$. Since L_{k+1} is uniformly elliptic in $D_\rho \cap (\operatorname{supp}\tilde{\tilde{\chi}})$, (4.19) of Lemma 4.4 gives

$$\begin{aligned}
\|\tilde{\tilde{\chi}}v_{k+1}\|_s &\le C\Big(\|L_{k+1}\tilde{\tilde{\chi}}v_{k+1}\|_{s-1} + \sum_{\substack{i+j\le s-1 \\ j<s-1}} B_{i+1}^{(k+1)}\|\tilde{\tilde{\chi}}v_{k+1}\|_{j+1}\Big) \\
&\le C\Big(\|L_{k+1}v_{k+1}\|_{s-1} + \sum_{\substack{i+j\le s \\ i+2\le s,\ j<s}} B_{i+2}^{(k+1)}\|v_{k+1}\|_j \\
&\qquad + \|[\tilde{\tilde{\chi}}, L_{k+1}p]v_{k+1}\|_{s-1}\Big),
\end{aligned}$$

which implies, by (4.8) and (4.9),

$$\|\tilde{\tilde{\chi}} v_{k+1}\|_{s-1} \le C\Big(\|L_{k+1}v_{k+1}\|_{s-1} + \sum_{\substack{i+j\le s\\ i+2\le s,\ j<s}} B_{i+2}^{(k+1)}\|v_{k+1}\|_j\Big). \tag{4.63}$$

Hence, combining (4.60)–(4.63), we obtain

$$\|v_{k+1}\|_s \le C\Big(\|L_{k+1}v_{k+1}\|_s + \sum_{\substack{i+j\le s\\ i+2\le s,\ j<s}} B_{i+2}^{(k+1)}\|v_{k+1}\|_j\Big) \tag{4.64}$$

for $s \ge 2$. Since (3.14) and (1.1) give, respectively, $L_{k+1}v_{k+1} = L_{\varepsilon_{k+1}}[\mu + \tilde{u}_{k+1}]v_{k+1} = f_{k+1}$ and $B_{i+2}^{(k+1)} \sim C|\mu + \tilde{u}_{k+1}|_{i+4}$, from (4.55), (4.59), and (4.64) it follows that

$$\|v_{k+1}\|_0 \le C\|f_{k+1}\|_0,$$
$$\|v_{k+1}\|_s \le C\Big(\|f_{k+1}\|_s + \sum_{\substack{i+j\le s\\ j<s}} |\mu + \tilde{u}_{k+1}|_{i+4}\|v_{k+1}\|_j\Big), \qquad (0 < s \le s_0),$$

i.e., Assumption 3.5 is fulfilled when $n = k+1$.

Therefore, by induction with respect to $n = 0, 1, 2, \cdots$, Assumptions 3.4 and 3.5 are valid for any n. Hence the proof of Theorem 1.1 is now complete.

5. Appendix

We shall prove that Theorems 1.1 and 1.2 remain valid for a C^r Riemannian metric g; that is to say, the Hölder continuity of rth-order derivatives of g_{ij} are not necessary. Throughout this section, we assume that $g = g_{ij}\,dx^i dx^j$ is a C^r Riemannian metric in $\mathbb{R}^2$ satisfying (1.1) and (1.2).

We define a metric $\gamma = \gamma_{ij}\,dx^i dx^j$ by

$$\gamma_{ij} = \gamma_{ij}(x;k) = k^2 \int_{\mathbb{R}^2} \psi(k(x-y))g_{ij}(y)\,dy,$$

where $k = 1, 2, 3, \cdots$ and $\psi \in C_0^\infty(\mathbb{R}^2)$ is a nonnegative function satisfying $\int_{\mathbb{R}^2} \psi(x)\,dx = 1$. We define the nonlinear operator $F[u]$ by replacing g_{ij}, $\nabla_i = \nabla_i^g$, and f with γ_{ij}, ∇_i^γ and

$$f(x;k) = k^2 \int_{\mathbb{R}^2} \psi(k(x-y))f(y)\,dy$$

respectively in (1.3). We put $\mu = \mu_\rho$, $s_0 = r-2$, and

$$\varepsilon(k) = \max\{|F[\mu]|_0^{2/3}, \|F[\mu]\|_{s_0}^{2/(s_0+2)}\}.$$

It is easy to show that $\varepsilon(k)$ converges to a certain number when $k \to \infty$. Since $\gamma_{ij} \in C^\infty(\mathbb{R}^2) \subset C^{r,\alpha}(\mathbb{R}^2)$, $f(\cdot\,; k) \in C\infty(\mathbb{R}^2) \subset C^{r-2,\alpha}(\mathbb{R}^2)$, and $\gamma_{ij} \to g_{ij}$ in $C^r(\mathbb{R}^2)$ as $k \to \infty$, all the results proved in §§3 and 4 are valid for the mollified metric Γ when k is sufficiently large. In particular, it is important that Assumptions 3.4 and 3.5 are both satisfied for γ^{ij}, $v_n = v_n(x; k)$, $u_n = u_n(x; k)$, $f_n = f_n(x; k)$, and $C = C(k)$. As is noted in §3, $C^{r,\alpha}$ regularity is necessary when we solve the Dirichlet problem (3.14). However, C^r regularity suffices for any other purpose. Furthermore, no constants which appeared in this paper depend on α; actually, we may assume that they are continuous functions of supremum norms $|\gamma_{ij}|_s$ $(i, j = 1, 2, 0 \le s \le r)$. This implies that any constant which depends on k converges to a certain value when k tends to ∞.

Applying Theorem 3.8 to γ and $f = f(\cdot\,; k)$, we can show that there exist a function $u_\infty = u_\infty(x; k) \in w^{r/2-2-\kappa}(D_\rho)$ and a large number N such that (3.42) and (3.43) hold for all $k \ge N$. Thus (4.43) implies that the sequence $\{u_\infty(x; k)\}_{k=N}^\infty$ is strongly bounded in the Sobolev space $w^{r/2-2-\kappa}(D_\rho)$. Hence, there is a subsequence $\{u_\infty(x; k_\nu)\}_{\nu=1}^\infty$ of $\{u_\infty(x; k)\}_{k=N}^\infty$ such that $\{u_\infty(x; k_\nu)\}_{\nu=1}^\infty$ is weakly convergent in $w^{r/2-2-\kappa}(D_\rho)$ and such that $u_\infty(x; k_\nu)$ and its derivatives converge to a function $u_\infty(x)$ and its weak derivatives respectively for almost every x in D_ρ. As is well known,

$$\|u_\infty(x)\|_{r/2-2-\kappa} \le \underline{\lim}_{\nu\to\infty} \|u_\infty(x; k_\nu)\|_{r/2-2-\kappa}.$$

Therefore, by replacing $u_\infty = u_\infty(x; k)$ with $u_\infty(x; k_\nu)$ and letting $\nu \to \infty$ in (4.42) and (4.43), we can prove that the result of Theorem 3.8 remains true for the C^r metric g. Since (1.1) implies $\lim_{\nu\to\infty} \varepsilon(k_\nu) \ll 1$, (1.5) follows from (4.43). Consequently, Theorem 1.1 remains valid for g. As is proved in §2, if Theorem 1.1 is true, then Theorem 1.2 is also true for the C^r metric g.

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