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COMPLETE SURFACES WITH FINITE TOTAL CURVATURE

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0. Introduction

The goal of this project is to verify a conjecture of Yau and the authors stated in [12] for dimension 2. The conjecture asserts that:

Conjecture. *Let M be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. Assume that there exists a point $p \in M$ such that the volume of geodesic balls $B_p(r)$ centered at p with radius r satisfies*

$$\mathrm{Vol}(B_p(r)) = O(r^\alpha), \tag{0.1}$$

as $r \to \infty$ for some integer $\alpha \geq 1$. Let k be a nonnegative integer and $r(x)$ be the distance from p to x, and define

$$H_k(M) = \{f \mid \Delta f \equiv 0 \text{ and } |f|(x) = O(r^k(x))\}$$

to be the space of harmonic functions on M which do not grow faster than $r^k(x)$. Then the dimension of $H_k(M)$ must be at most the dimension of that in $\mathbf{R}^\alpha$, i.e.,

$$\dim(H_k(M)) \leq \dim(H_k(\mathbf{R}^\alpha)). \tag{0.2}$$

Yau originally conjectured that $H_k(M)$ must be of finite dimension and its dimension is bounded by $\dim(H_k(\mathbf{R}^n))$, where $n = \dim M$. In 1989, the authors proved [12] that $H_1(M)$ has an estimate of the form $\dim(H_1(M)) \leq \alpha + 1$, where α is defined by (0.1). This lead us to the refinement of Yau's conjecture in the above form.

In this work, we will verify the conjecture (Theorem 4.6) for 2-dimensional manifolds with nonnegative curvature. In fact, it turns out that if we only assume the negative part of the Gaussian curvature is integrable, then there are rigid and powerful geometric and analytic consequences which are special because of the fact that we are dealing with surfaces. We

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will refer to this special class of complete surfaces as surfaces with finite total curvature because it follows that the absolute value of the Gaussian curvature is also integrable. We would like to point out that surfaces with finite total curvature were studied rather extensively in [2]–[4], [6]–[9], [15]. We refer the reader to §1 for the essential preliminaries on the subject.

The main geometric result of this paper is to obtain control of how the geodesic distance behaves at infinity when compared to the background flat metric. Specifically, we will show in §§2 and 3 that at a given end E, if we represent the metric by a conformal metric on $\mathbf{R}^2$ with a disk removed, then the geodesic distance r and the Euclidean distance r_0 must satisfy

$$\lim_{xz\to\infty} \frac{\log r(x)}{\log r_0(x)} = 1 - \alpha,$$

where α is related to the area growth of the intersection of geodesic balls with E. It is given by the formula

$$1 - \alpha = \lim_{r\to\infty} \frac{A(B_p(r)\cap E)}{\pi r^2}.$$

The analytic results are consequences of the asymptotic behavior of the distance function. In fact, one can provide sharp upper and lower bounds (see §4, Theorems 4.2 and 4.5) on the dimension of $H_k(M)$ in terms of the area growth of each end for these surfaces of finite total curvature. More precisely, if the ends of M are given by $\{E_1, \cdots, E_m\}$, and the $\{\alpha_1, \cdots, \alpha_m\}$ are defined by

$$1 - \alpha_i = \lim_{r\to\infty} \frac{A(B_p(r)\cap E_i)}{\pi r^2},$$

then for any positive real number k, the space $H_k(M)$ satisfies

$$\sum_{i=1}^m N_i + m \ge \dim H_k(M) \ge \min\left\{1, \sum_{i=1}^m N_i' + m'\right\}.$$

The number m' denotes the number of ends which has quadratic area growth. The number N_i is the dimension of the space of nonconstant harmonic polynomials in $\mathbf{R}^2$ of degree less than or equal to $k(1-\alpha_i)$. Finally, N_i' denotes the supremum over all $\varepsilon > 0$ of the dimension of the space of nonconstant harmonic polynomials in $\mathbf{R}^2$ of degree less than or equal to $k(1-\alpha_i) - \varepsilon$.

In §5, Theorem 5.2, we will prove an isoperimetric inequality for those surfaces whose ends all have quadratic area growth. In this case, the theorem asserts that there exists a constant C_{22} depending only on M such that for any compact subdomain D of M, the length of its boundary $L(\partial D)$ and its area $A(D)$ must satisfy the inequality

$$L^2(\partial D) \geq C_{22} A(D).$$

In fact, this can be viewed as the infinitesimal version of the quadratic area growth condition. A Poincaré inequality at infinity will also be proved in §6, Theorem 6.1. Together with the isoperimetric inequality, one can use Moser's argument to prove a Harnack inequality for uniformly elliptic operator with measurable coefficients on M. Finally, we will discuss some examples in §7 for further understanding of our results.

The second author would like to thank R. Finn for providing reference [9].

1. Geometric preliminaries

This section is devoted to recalling known results on complete surfaces with finite total curvature, which will be used in the course of this article. Let M be a complete noncompact surface with finite total curvature, i.e., $\int_M |K|\, dA < \infty$, where K is the Gaussian curvature of M. Let $p \in M$ be a fixed point. Let us denote the geodesic ball of radius r with center at p by $B_p(r)$, and its boundary by $\partial B_p(r)$. For simplicity, when the center point is p, we set $A(r) = A(B_p(r))$ and $L(r) = L(\partial B_p(r))$.

The well-known theorems of Cohn-Vossen [2] and Huber [8] assert that:

Proposition 1.1. *Let M be a complete surface with the negative part of its Gaussian curvature integrable, i.e., $\int_M K_-\, dA < \infty$, where*

$$K_- = \begin{cases} 0 & \text{if } K > 0, \\ -K & \text{if } K \leq 0. \end{cases}$$

Then M must be conformally equivalent to a compact Riemann surface with finitely many points deleted. Moreover

$$\int_M K\, dA \leq 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. In particular, $\int_M |K|\, dA < \infty$. If in addition M is simply connected, then M must be conformally equivalent to the complex plane.

The following proposition was proved by Hartman in [6] for simply connected surfaces. It was later generalized by Shiohama in [15] to arbitrary complete surfaces with finite total curvature.

Proposition 1.2. *Let M be a complete surface with finite total curvature. If K is the Gaussian curvature of M, then we have*

$$2\pi\chi(M) - \int_M K\,dA = \lim_{r\to\infty} \frac{L(r)}{r} = \lim_{r\to\infty} \frac{2A(r)}{r^2}.$$

The third result, which was proved by Hartman in [6], asserts an upper bound of the area growth of a complete surface with finite total curvature. For a higher-dimensional generalization, we would like to refer the reader to [13].

Proposition 1.3. *Let M be a complete surface with finite total curvature. Then there exists a constant C_1 depending only on $\int_M |K|\,dA$, such that*

$$L(\partial B_x(r)) \le C_1 r \quad \text{and} \quad A(B_x(r)) \le \frac{C_1}{2} r^2,$$

for all $x \in M$ and for all $r > 0$.

The next proposition was also proved by Hartman in [6].

Proposition 1.4. *Let M be a simply connected complete noncompact surface with finite total curvature. Then there exists $R_0 > 0$, such that for all $r > R_0$, the boundary of the geodesic ball of radius r centered at p must be homeomorphic to the circle. In particular, the geodesic ball $B_p(r)$ is homeomorphic to the disk.*

2. Lower bound for the Green's function

In this section, we would like to obtain a lower estimate for the Green's function on a simply connected noncompact surface with finite total curvature. For a fixed point $p \in M$, we will denote by $B(r) = B_p(r)$ the geodesic ball centered at p with radius r. Following the notation of §1, let us establish the following lemma.

Lemma 2.1. *Let $R_2 > R_1 > 0$. Suppose g is a subharmonic function on $B(R_2) - \overline{B(R_1)}$ which is smooth on an open set containing $\overline{B(R_2) - B(R_1)}$. Let $s(r) = \sup_{\partial B(r)} g$ and $i(r) = \inf_{\partial B(r)} g$. If $s(R_2) > i(R_1)$, then*

$$\left(\int_{\partial B(R_1)} \frac{\partial g}{\partial r}\right)\left(\int_{R_1}^{R_2} \frac{dt}{L(t)}\right) \le s(R_2) - i(R_1).$$

Proof. Let f be the harmonic function on $B(R_2) - \overline{B(R_1)}$ such that $f = g$ on $\partial B(R_1)$ and $f = s(R_2)$ on $\partial B(R_2)$. Then $g \le f$ on $B(R_2) - \overline{B(R_1)}$. Hence

$$\frac{\partial f}{\partial r} \ge \frac{\partial g}{\partial r} \quad \text{on } \partial B(R_1). \tag{2.1}$$

Let h be the harmonic function on $B(R_2) - \overline{B(R_1)}$ such that $h = i(R_1)$ on $\partial B(R_1)$ and $h = s(R_2)$ on $\partial B(R_2)$. Then $h \leq f$ on $B(R_2) - \overline{B(R_1)}$. Hence

$$\frac{\partial f}{\partial r} \leq \frac{\partial h}{\partial r} \quad \text{on } \partial B(R_2). \tag{2.2}$$

Since both f and h are harmonic on $B(R_2) - \overline{B(R_1)}$ it is easy to see that

$$\int_{\partial B(R_2)} \frac{\partial f}{\partial r} = \int_{\partial B(R_1)} \frac{\partial f}{\partial r}, \qquad \int_{\partial B(R_2)} \frac{\partial h}{\partial r} = \int_{\partial B(R_1)} \frac{\partial h}{\partial r}.$$

Therefore by (2.1) and (2.2), we have

$$\int_{q\partial B(R_1)} \frac{\partial g}{\partial r} \leq \int_{\partial B(R_1)} \frac{\partial h}{\partial r}. \tag{2.3}$$

Let us define the function $\phi(x)$ by

$$\phi(r(x)) = \frac{(s(R_2) - i(R_1)) \int_{R_1}^{r(x)} dt/L(t)}{\int_{R_1}^{R_2} dt/L(t)} + i(R_1).$$

Note that $\phi = h$ on $\partial B(R_1)$ and $\partial B(R_2)$. By the fact that harmonic functions minimize Dirichlet integrals, we conclude that

$$\begin{aligned} \int_{B(R_2)-B(R_1)} |\nabla h|^2 &\leq \int_{B(R_2)-B(R_1)} |\nabla \phi|^2 \\ &= \int_{R_1}^{R_2} \left(\int_{\partial B(r)} \frac{(s(R_2) - i(R_1))^2}{L^2(r) \int_{R_1}^{R_2} (dt/L(t))^2} \right) dr \\ &= \frac{(s(R_2) - i(R_1))^2}{\int_{R_1}^{R_2} dt/L(t)}. \end{aligned} \tag{2.4}$$

On the other hand,

$$\begin{aligned} \int_{B(R_2)-B(R_1)} |\nabla h|^2 &= -\int_{B(R_2)-B(R_1)} h\Delta h + \int_{\partial B(R_2)} h\frac{\partial h}{\partial r} - \int_{\partial B(R_1)} h\frac{\partial h}{\partial r} \\ &= s(R_2) \int_{\partial B(R_2)} \frac{\partial h}{\partial r} - i(R_1) \int_{\partial B(R_1)} \frac{\partial h}{\partial r} \\ &= (s(R_2) - i(R_1)) \int_{\partial B(R_1)} \frac{\partial h}{\partial r}. \end{aligned}$$

Combining this with (2.3), (2.4), and the assumption that $s(R_2) - i(R_1) > 0$, we have

$$\left(\int_{\partial B(R_1)} \frac{\partial g}{\partial r} \right) \left(\int_{R_1}^{R_2} \frac{dt}{L(t)} \right) \leq s(R_2) - i(R_1).$$

By setting the function g in Lemma 2.1 to be the Green's function, $g(x) = G(p, x)$, with the pole at p, and the convention that $g(x) \to -\infty$ as $x \to p$, we conclude the following corollary.

Corollary 2.2. *Let M satisfy the assumption of Lemma 2.1, let g be a Green's function with a pole at p, and take the value of $-\infty$ at p. Then*

$$\int_{R_1}^{R_2} \frac{dt}{L(t)} \leq s(R_2) - i(R_1)$$

for all $R_2 > R_1 > 0$.

Proof. By the maximum principle, we have $s(R_2) - i(R_1) > 0$. Also, by the assumption that g is the Green's function, we have

$$1 = \int_{B(R_1)} \Delta g = \int_{\partial B(R_1)} \frac{\partial g}{\partial r}.$$

We would like to point out that the proof of Lemma 2.1 and hence of Corollary 2.2 is valid on any complete manifold with arbitrary dimension. In general we will interpret the integral $\int_{R_1}^{R_2} dt/L(t)$ to have

$$L(t) = A_{n-1}(t) = (n-1)\text{-measure of } \partial B_p(t).$$

Corollary 2.3. *Let M be a complete manifold of dimension n (not necessarily 2). Let g be a Green's function with a pole at a fixed point $p \in M$, which takes the value of $-\infty$ at p. Then*

$$\int_{R_1}^{R_2} \frac{dt}{A_{n-1}(t)} \leq s(R_2) - i(R_1)$$

for all $R_2 > R_1 > 0$. Here we denote $A_{n-1}(t) = (n-1)$-measure of $\partial B_p(t)$, $s(R_2) = \sup_{\partial B_p(R_2)} g$, and $i(R_1) = \inf_{\partial B_p(R_1)} g$. In particular, if M admits a negative Green's function then

$$\int_1^{\infty} \frac{dt}{A_{n-1}(t)} < \infty.$$

This estimate of $s(r)$ of the Green's function is sharp. In fact, when M has a rotationally symmetric metric around a point p, one checks easily that the function

$$g(x) = \int_1^{r(x)} \frac{dt}{A_{n-1}(t)}$$

is a Green's function with a pole at p.

Corollary 2.4. *Let M be a complete manifold of dimension n. Suppose $R_2 > 2R_1 > 0$, and g is a subharmonic function on $B(R_2) - \overline{B(R_1)}$ which is smooth on an open set containing $\overline{B(R_2) - B(R_1)}$. Let $s(r) = \sup_{\partial B(r)} g$*

and $i(r) = \inf_{\partial B(r)} g$. *If* $s(R_2) > i(R_1)$, *then there exists a constant* $C_2 > 0$ *depending only on* M *and* R_1, *such that*

$$C_2\left(\int_{\partial B(R_1)} \frac{\partial g}{\partial r}\right)\left(\int_{R_1}^{R_2} \frac{t\,dt}{V(t)}\right) \le s(R_2) - i(R_1),$$

where the quantity $V(t)$ *denotes the* n*-dimensional volume of the geodesic ball of radius* t *centered at* p.

Proof. In the proof of Lemma 2.1, if we set the function ϕ to be

$$\phi(x) = \frac{(s(R_2) - i(R_1)) \int_{R_1}^{r(x)} (t/V(t))\,dt}{\int_{R_1}^{R_2} (t/V(t))\,dt} + i(R_1),$$

then the same argument will imply the desired estimate, providing we can show

$$\int_{R_1}^{R_2} \frac{r^2 A_{n-1}(r)}{V^2(r)} dr \le C_2 \int_{R_1}^{R_2} \frac{r\,dr}{V(r)}. \tag{2.5}$$

Indeed,

$$\begin{aligned}\int_{R_1}^{R_2} \frac{r^2 A_{n-1}(r)}{V^2(r)} dr &= \int_{R_1}^{R_2} r^2 d\left(\frac{-1}{V(r)}\right) \\ &= -\frac{R_2^2}{V(R_2)} + \frac{R_1^2}{V(R_1)} + \int_{R_1}^{R_2} \frac{2r\,dr}{V(r)}.\end{aligned}$$

For a fixed R_1, we can find a constant $C_3 > 0$ depending only on R_1 and M, such that

$$\frac{R_1^2}{V(R_1)} \le C_3 \int_{R_1}^{2R_1} \frac{2r\,dr}{V(r)}.$$

Hence if $R_2 > 2R_1$ then (2.5) follows.

When M is a simply connected complete noncompact surface with finite total curvature, by Huber's theorem [8] M is conformally equivalent to the complex plane. Let $p \in M$ be a fixed point which is identified as the origin of the plane. Let $r(x)$ and $r_0(x)$ be the geodesic distance and the Euclidean distance between the points p and x, respectively. By using Corollary 2.2, we will derive a sharp lower bound for r_0 in terms of r. We should point out a lower bound was first proved by Finn in [4] for surfaces with nonpositive curvature near infinity. Huber in [9] later generalized Finn's argument to the general finite total curvature metric. In both cases, they utilized the existence of a normal metric near infinity of M, and their estimates agree with ours when the area growth of M is quadratic. When M does not have quadratic area growth, our estimate which depends on the area growth is still sharp while theirs does not yield a sharp bound. Moreover, the method which we use is rather general (Corollary 2.3) and more direct.

Theorem 2.5. *Let M be a simply connected complete noncompact surface with finite total curvature. Then for any $\varepsilon > 0$, there exist $R_0 > 0$ and a constant C_4, depending only on M, such that*

$$2\pi \int_1^{r(x)} \frac{dt}{L(t)} - C_4 \leq (1+\varepsilon) \log r_0(x)$$

for all $x \in M \backslash B_p(R_0)$. In particular, if

$$\begin{aligned} \frac{1}{2\pi} \int_M K\, dA &= 1 - \lim_{r\to\infty} \frac{A(r)}{\pi r^2} \\ &= 1 - \lim_{r\to\infty} \frac{L(r)}{2\pi r} \quad \textit{(by Proposition 1.2)} \\ &= \alpha, \end{aligned}$$

where K is the Gaussian curvature of M, $A(r) = A(B_p(r))$, and $L(r) = L(\partial B_p(r))$, then

$$\limsup_{x\to\infty} \frac{\log r(x)}{\log r_0(x)} \leq 1 - \alpha. \tag{2.6}$$

Proof. By the fact that M is conformally equivalent to the complex plane, the function $\frac{1}{2\pi}\log r_0$ is the Green's function with a pole at p. Hence Corollary 2.2 implies that

$$2\pi \int_1^r \frac{dt}{L(t)} \leq s(r) - i(1) \tag{2.7}$$

for all $r > 1 > 0$. Here $s(r) = \sup_{\partial B_p(r)} \log r_0$, $i(r) = \inf_{\partial B_p(r)} \log r_0$, and we denote the length of $\partial B_p(t)$ by $L(t)$.

On the other hand, let us define the function $f(x) = \frac{1}{2\pi}\log r_0(x)$. Computing the Dirichlet integral of $\log f$ with respect to the Euclidean metric over the compliment of the Euclidean disk of radius e, we have

$$\begin{aligned} \int_{M\backslash B^*(e)} |\nabla_0 \log f|^2\, dA_0 &= \int_e^\infty \frac{2\pi}{r_0 \log^2 r_0} dr_0 \\ &= 2\pi \int_1^\infty \frac{du}{u^2} = 2\pi. \end{aligned}$$

Hence by the invariance of the Dirichlet integral under a conformal change of metric, we derive that there exists a function $\eta(r)$, for r sufficiently large, with $0 \le \eta(r) \le 2\pi$, and $\eta(r) \to 0$ as $r \to \infty$, such that

$$\int_{M \setminus B_p(r)} |\nabla \log f|^2 \, dA \le \eta(r).$$

For any pair of sufficiently large $R < r$, by the Schwartz inequality, we have

$$(2.8) \quad \int_R^r \left(\int_{\partial B_p(t)} |\nabla \log f| \, dL \right)^2 \frac{dt}{L(t)} \le \int_R^r \int_{\partial B_p(t)} |\nabla \log f|^2 \, dL \, dt \le \eta(R).$$

Proposition 1.4 implies that if t is sufficiently large, then the set $\partial B_p(t)$ is connected and is homeomorphic to a circle. If x and y are points in $\partial B_p(t)$ such that $\log r_0(x) = s(t)$ and $\log r_0(y) = i(t)$, then they must divide $\partial B_p(t)$ into two connected curves. Integrating the function $|\nabla \log f|$ along the two curves gives

$$2 \log \left(\frac{s(t)}{i(t)} \right) \le \int_{\partial B_p(t)} |\nabla \log f| \, dL.$$

Hence, combining with (2.8) yields

$$4 \inf_{R \le t \le r} \left(\log \frac{s(t)}{i(t)} \right)^2 \int_R^r \frac{dt}{L(t)} \le \eta(R).$$

On the other hand, the maximum principle and the facts that $f \to -\infty$ as $x \to p$ and $f \to \infty$ as $x \to \infty$ imply that both $s(t)$ and $i(t)$ are monotonic increasing functions of t. Therefore, we conclude that

$$(2.9) \qquad s(R) \le i(r) \exp \left(\frac{\eta^{1/2}(R)}{2(\int_R^r dt/L(t))^{1/2}} \right).$$

Inequality (2.7) now implies

$$2\pi \int_1^r \frac{dt}{L(t)} + i(1) - 2\pi \int_R^r \frac{dt}{L(t)} \le s(R) \le i(r) \exp \left(\frac{\eta^{1/2}(R)}{2(\int_R^r dt/L(t))^{1/2}} \right).$$

Since $\eta(R) \to 0$ as $R \to \infty$, to prove the theorem for r sufficiently large, we need to show that we can find a value $R < r$ such that it satisfies

$$\int_R^r \frac{dt}{L(t)} = 1$$

and $R \to \infty$ as $r \to \infty$. If not, there exist sequences $\{r_i\}$ and $\{R_i\}$ such that

$$\int_{R_i}^{r_i} \frac{dt}{L(t)} = 1$$

with $r_i \to \infty$ but $R_i \to \overline{R}$. However, this contradicts the fact that

$$\int_{\overline{R}}^{\infty} \frac{dt}{L(t)} = \infty,$$

because $L(t)$ cannot grow faster than linearly.

If we set

$$1 - \alpha = \lim_{r\to\infty} \frac{A(r)}{\pi r^2} = \lim_{r\to\infty} \frac{L(r)}{2\pi r},$$

then to prove (2.6), it suffices to show that for any $\varepsilon > 0$,

$$\log r \leq 2\pi(1 - \alpha + \varepsilon) \int_1^r \frac{dt}{L(t)}.$$

This is a direct consequence of the definition of α.

3. Upper bound for the Green's function

For our special class of surfaces, one can also derive an upper bound for the Green's function. Following the assumption of §2, M is a simply connected complete noncompact surface with finite total curvature. We consider M to be $\mathbf{R}^2$ with a complete metric of the form $ds^2 = e^{2u} ds_0^2$, with ds_0^2 being the Euclidean metric. Let $p \in M$ be a fixed point which can be chosen as the origin of $\mathbf{R}^2$, and let $B_p(r)$ and $B_p^*(r_0)$ be the geodesic balls centered at p with radii r and r_0 with respect to the metrics ds^2 and ds_0^2 respectively. For any domain $D \subseteq M$, we will denote $L(\partial D)$ and $A(D)$ to be the length of ∂D and the area of D with respect to ds^2 respectively.

Lemma 3.1. *Let M be a simply connected complete noncompact surface with finite total curvature. Let K be the Gaussian curvature of M, $\alpha = \frac{1}{2\pi}\int_M K\, dA$, and $i(r) = \inf_{\partial B_p(r)} \log r_0$, where $r_0(x)$ is the Euclidean distance from x to p. Then*

$$\liminf_{r\to\infty} \frac{\log r}{i(r)} \geq 1 - \alpha. \tag{3.1}$$

Proof. Let $\Delta_0 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ be the Euclidean Laplacian. Then

$$\Delta_0 u + K e^{2u} = 0. \tag{3.2}$$

Also

$$\iint_{\mathbf{R}^2} K e^{2u}\, dx\, dy = \int_M K\, dA = 2\pi\alpha.$$

Given $\varepsilon > 0$, there exists R_0 such that $r_0 > R_0$ implies

$$\iint_{B_p^*(r_0)} Ke^{2u}\,dx\,dy < 2\pi(\alpha + \varepsilon).$$

Hence by (3.2), if $r_0 > R_0$, then we have

$$\begin{aligned}
-2\pi(\alpha+\varepsilon) &< -\iint_{B_p^*(r_0)} Ke^{2u}\,dx\,dy \\
&= \int_{B_p^*(r_0)} \Delta_0 u\,dx\,dy = \int_{\partial B_p^*(r_0)} \frac{\partial u}{\partial r_0}\,dx\,dy \\
&= \frac{d}{dr_0}\left(\int_{\partial B_p^*(r_0)} u\,ds_0\right) - \frac{1}{r_0}\int_{\partial B_p^*(r_0)} u\,ds_0,
\end{aligned}$$

which implies

$$-\frac{\alpha+\varepsilon}{r_0} < \frac{d}{dr_0}\left(\frac{1}{2\pi r_0}\int_{\partial B_p^*(r_0)} u\,ds_0\right).$$

By integrating, we conclude that there exists a constant C_5 depending on R_0 and M, such that

$$-(\alpha+\varepsilon)\log r_0 - C_5 \le \frac{1}{2\pi r_0}\int_{\partial B_p^*(r_0)} u\,ds_0. \tag{3.3}$$

However, by Jensen's inequality,

$$\begin{aligned}
\exp\left(\frac{1}{2\pi r_0}\int_{\partial B_p^*(r_0)} u\,ds_0\right) &\le \frac{1}{2\pi r_0}\int_{\partial B_p^*(r_0)} e^u\,ds_0 \\
&\le \frac{1}{2\pi r_0}\left(\int_{\partial B_p^*(r_0)} e^{2u}\,ds_0\right)^{1/2}(2\pi r_0)^{1/2}.
\end{aligned}$$

Combining this with (3.3), we conclude that

$$2\pi e^{-2C_5} r_0^{1-2\alpha-2\varepsilon} \le \int_{\partial B_p^*(r_0)} e^{2u}\,ds_0 \tag{3.4}$$

for all $r_0 > R_0$.

Note that $\alpha \le 1$. If $\alpha = 1$ then (3.1) is clearly true. Hence we may assume that $\alpha < 1$, and by choosing ε sufficiently small we may also assume that $1-\alpha-\varepsilon > 0$. Integrating inequality (3.4) from R_0 to r_0 for $r_0 > R_0$, we obtain

$$\begin{aligned}
&\frac{2\pi e^{-2C_5}}{2(1-\alpha-\varepsilon)}\left(r_0^{2(1-\alpha-\varepsilon)} - R_0^{2(1-\alpha-\varepsilon)}\right) \\
&\qquad \le \iint_{B_p^*(r_0)} e^{2u}\,dx\,dy - \iint_{B_p^*(R_0)} e^{2u}\,dx\,dy.
\end{aligned}$$

Hence there exist constants $C_6 > 0$ and $C_7 > 0$ which depend only on R_0 and M such that if ε is sufficiently small, then for all $r_0 > R_0$ we have

$$C_6 r_0^{2(1-\alpha-\varepsilon)} - C_7 \le A(B_p^*(r_0)). \tag{3.5}$$

Now let us choose R_1 sufficiently large so that $\inf_{\partial B_p(r)} r_0 > R_0$ for all $r > R_1$. Let $B_p^*(\rho)$ be the largest disk which is contained in $B_p(r)$. Then $\rho = \rho(r) = \inf_{\partial B_p(r)} r_0 > R_0$ if $r > R_1$. By Proposition 1.3, there exists a constant C_1 depending on M such that

$$A(B_p(r)) \le \frac{C_1}{2} r^2$$

for all $r > 0$. By setting $r_0 = \rho$ in (3.5), we have

$$C_6 \rho^{2(1-\alpha-\varepsilon)} \le A(B_p^*(\rho)) + C_7 \le A(B_p(r)) + C_7 \le \frac{C_1}{2} r^2 + C_7$$

for all $r > R_1$. Taking logarithms of both sides and dividing the resulting inequality by $2\log\rho = 2i(r)$ we obtain

$$\begin{aligned} \frac{\log C_6}{2i(r)} + (1-\alpha-\varepsilon) &\le \frac{\log(C_1 r^2/2 + C_7)}{2i(r)} \\ &= \frac{\log r}{i(r)} \frac{\log(C_1 r^2/2 + C_7)}{\log r^2}. \end{aligned}$$

Using the fact that $i(r) \to 0$, and letting $r \to \infty$ and then $\varepsilon \to 0$, the inequality becomes

$$1 - \alpha \le \liminf_{r\to\infty} \frac{\log r}{i(r)},$$

which was to be proved.

Theorem 3.2. *Let M be a simply connected complete noncompact surface with finite total curvature. Let $r(x)$ and $r_0(x)$ be defined as above. Then*

$$\liminf_{x\to\infty} \frac{\log r(x)}{\log r_0(x)} \ge 1 - \alpha. \tag{3.6}$$

Proof. Inequality (3.6) is obvious if $\alpha = 1$, hence we may assume that $\alpha < 1$. Let $\varepsilon > 0$ be any constant such that $1 - \alpha - \varepsilon > 0$. Then Lemma 3.1 implies that there exists $R_0 > 0$ such that

$$\log R \ge (1-\alpha-\varepsilon) i(R) \tag{3.7}$$

for all $R > R_0$. By the fact that $\int_1^\infty dt/L(t) = \infty$, for any given $r > 0$ we can find $R > r$ such that $\int_r^R dt/L(t) = 1$. Applying (2.9) to (3.7) with the roles of r and R reversed, we conclude that

$$(1-\alpha-\varepsilon)s(r) \le (\log R)\exp\left(\frac{\eta^{1/2}(r)}{2(\int_r^R dt/L(t))^{1/2}}\right).$$

The theorem now follows from the facts that

$$1 = \int_r^R \frac{dt}{L(t)} \ge C_8 \int_r^R \frac{dt}{t} = C_8 \log R - C_8 \log r$$

and $\eta(r) \to 0$ as $r \to \infty$.

Theorems 2.5 and 3.2 can be combined to be:

Corollary 3.3. *With the assumptions and notation as in Lemma* 3.1, *we have*

$$\lim_{x\to\infty} \frac{\log r(x)}{\log r_0(x)} = 1 - \alpha. \tag{3.8}$$

Let us point out that (3.6) is equivalent to saying that for any $\varepsilon > 0$, there exists $R_0 > 0$ such that for $r(x) > R_0$,

$$r_0^{1-\alpha-\varepsilon}(x) \le r(x).$$

In the event that M is a complete surface with nonnegative Gaussian curvature outside a compact set, then Huber's theorem in [8] implies that M must be of finite total curvature. In this case we can sharpen the above estimate as follows.

Corollary 3.4. *Let M be a complete simply connected surface with nonnegative Gaussian curvature outside a compact set. Then following the notation of Lemma* 3.1, *there exist a constant $C_9 > 0$ and $R_0 > 0$ such that for $r(x) > R_0$, we have*

$$r_0^{1-\alpha}(x) \le C_9 r(x) \tag{3.9}$$

and

$$\log r_0(x) \le C_9 r(x). \tag{3.10}$$

Proof. Suppose $K \ge 0$ on $M \backslash B_p(R_1)$. By enlarging R_1 if necessary, we may assume that $r_0(x) > 1$ if $r(x) > R_1$. By [16], we have

$$\Delta \log(r - R_1) = \frac{\Delta r}{r - R_1} - \frac{1}{(r-R_1)^2} \le 0, \tag{3.11}$$

in the sense of distribution for $r(x) > R_1$.

On the other hand, Theorem 3.2 implies that for any $\varepsilon > 0$, there exists $R_2 > 2R_1$ such that if $r(x) > R_2$ then

$$(1 - \alpha - \varepsilon)\log r_0(x) \leq \log(r(x) - R_1). \tag{3.12}$$

This follows from the fact that the function $\log r_0(x) > 0$ if $r(x) > R_2 > 2R_1$ and that

$$\lim_{x\to\infty} \frac{\log(r(x) - R_1)}{\log r(x)} = 1.$$

Combining (3.11), (3.12), and the maximum principle, we have

$$\begin{aligned}(1 - \alpha - \varepsilon)\log r_0(x) &\leq \log(r(x) - R_1) + (1 - \alpha)\sup_{\partial B_p(2R_1)} \log r_0 \\ &= \log(r(x) - R_1) + (1 - \alpha)s(2R_1)\end{aligned}$$

for all $x \in B_p(r)\backslash B_p(2R_1)$. However, r can be taken to be arbitrarily large, so the above inequality is valid on $M\backslash B_p(2R_1)$. Note that R_1 is independent of ε. Hence by taking $\varepsilon \to 0$, (4.9) is valid with $R_0 = 2R_1$ and $C_9 = \frac{1}{R_1}\exp((1 - \alpha)s(2R_1))$.

Inequality (3.10) is a consequence of Theorems 4 and 5 in [11] and the facts that $\frac{1}{2\pi}\log r_0$ is a Green's function on M and M has at least linear area growth.

We would like point out that (3.9) was also proved in [4], [9].

4. Polynomial growth harmonic functions

In this section we would like to study the space of polynomial growth harmonic functions on a complete noncompact (not necessarily simply connected) surface with finite total curvature. More specifically, we will give detailed descriptions on the space of harmonic functions which grow at most like r^k in terms of k and the geometry of M.

Due to the fact that each end of a complete surface with finite total curvature is conformally equivalent to a punctured disk, which in terms is conformally equivalent to $\mathbf{R}^2\backslash$ disk, we will prove the following lemma.

Lemma 4.1. *Let h be a harmonic function on $E = \{z \in \mathbf{C} \mid |z| > R\}$, which is smooth up to the boundary $\partial E = \{z \in \mathbf{C} \mid |z| = R\}$. Suppose that there are constants $k > 0$ and $C_{10} > 0$ such that*

$$|h(z)| \leq C_{10}(1 + |z|)^k \quad \text{on } E.$$

Then h can be expressed uniquely in the form

$$h(z) = \widetilde{h}(z) + h^*(z) + \beta\log|z| \tag{4.1}$$

for $|z| > R$, *where* $\widetilde{h}$ *is a harmonic polynomial of degree* $\leq k$ *with zero constant term,* $h^*(z)$ *is a bounded harmonic function on* E, *and* β *is a constant.*

Proof. Let $B^*(R) = \{z \in \mathbf{C} \mid |z| < R\}$. Define the number β as

$$\beta = \frac{1}{2\pi} \int_{\partial B^*(R)} \frac{\partial h}{\partial r_0} ds_0,$$

where ds_0^2 is the Euclidean metric, and r_0 is the distance function with respect to the origin. Set $u(z) = h(z) - \beta \log|z|$. Then for any simple closed curve γ in E, we claim that

$$\int_\gamma \frac{\partial u}{\partial \nu} ds_0 = 0,$$

where ν is the unit normal vector of γ. Indeed, if γ is homotopically trivial in E, then by Stoke's theorem and the fact that u is harmonic the claim is obvious. On the other hand if γ is not homotopically trivial in E, then γ and $\partial B^*(R)$ must bound a topological annulus. Applying Stokes's theorem again, we have

$$\begin{aligned}\int_\gamma \frac{\partial u}{\partial \nu} ds_0 &= \int_{\partial B^*(R)} \frac{\partial u}{\partial r_0} ds_0 \\ &= 2\pi\beta - \beta \int_{\partial B^*(R)} \log|z|\, ds_0 = 0.\end{aligned}$$

This justifies the claim, and implies that $u = \Re(f)$ for some analytic function f defined on E. Moreover, the growth assumption on h yields that $|f(z)| \leq C_{11}(1+|z|)^k$ for some constant $C_{11} > 0$. Hence the Laurent's series expansion of f is of the form

$$f(z) = \sum_{j=0}^{[k]} a_j z^j + \sum_{j=1}^{\infty} b_j z^{-j},$$

where $[k]$ is the integral part of k. Therefore

$$h(z) = \Re(f(z)) + \beta \log|z|,$$

which can be expressed in the form (4.1), and this expression is clearly unique.

Before we state the main theorems, let us fix our notation. Let (M, ds^2) be a complete noncompact surface with finite total curvature. By [8], M is conformally equivalent to $\widetilde{M}\backslash\{p_1, p_2, \cdots, p_m\}$, where $\widetilde{M}$ is a compact Riemann surface and the $\{p_i\}$ are points in $\widetilde{M}$. Denote the ends of M by $\{E_1, E_2, \cdots, E_m\}$, such that p_i is the point at ∞ of E_i, for

$1 \le i \le m$. Hence each E_i is conformally equivalent to the exterior of a disk in $\mathbf{C}$. We may assume that there are complete conformal metrics $ds_i^2 = e^{2u_i} ds_0^2$ on $\mathbf{C}$ so that $ds_i^2 = ds^2$ on $\mathbf{C} \backslash B^*(1)$, where we identify E_i with $\mathbf{C} \backslash B^*(1)$. Let p be a fixed point in M. For all $x \in M$, let $r(x)$ be the distance from p to x with respect to the metric ds^2. Also if $x \in E_i$, let $r_0(x)$ and $r_i(x)$ be the distances from the origin to x with respect to the metrics ds_0^2 and ds_i^2, respectively. Denote by K_i and dA_i the Gaussian curvature and the area element of ds_i^2, respectively. Then it follows from the assumption on M that $\mathbf{C}$ with the metrics ds_i^2 has finite total curvatures. Define

$$\alpha_i = \frac{1}{2\pi} \int_{\mathbf{C}} K_i \, dA_i$$

for $1 \le i \le m$. Clearly, we have

$$\lim_{\substack{x \to \infty \\ x \in E_i}} \frac{r_i(x)}{r(x)} = 1.$$

Hence by Proposition 1.2, we conclude that

$$1 - \alpha_i = \lim_{r \to \infty} \frac{A(B_p(r) \cap E_i)}{\pi r^2}$$

for $1 \le i \le m$. In particular, $\alpha_i < 1$ if and only if E_i has quadratic area growth.

Let P_l be the space of harmonic polynomials in $\mathbf{R}^2$ of degree less than or equal to l with zero constant term. In particular, P_l can be viewed as the space spanned by the set of homogeneous harmonic polynomials of degree less than or equal to l, which vanishes at the origin. For a real number $k > 0$, let H_k be the space of harmonic functions defined on M which grows less than or equal to r^k. In other words,

$$H_k = \{h \mid \Delta h \equiv 0 \text{ on } M \text{ and } |h(x)| \le C(1 + r(x))^k \text{ for some } C > 0\}.$$

Theorem 4.2. *Let $k_i = k(1 - \alpha_i)$ and $N_i = \dim P_{k_i}$. Then*

$$\dim H_k \le \sum_{i=1}^{m} N_i + m.$$

Proof. Let $h \in H_k$. Corollary 3.3 implies that for any $\varepsilon > 0$ satisfying $k(1 - \alpha_i + \varepsilon) < k_i + 1$ for all $1 \le i \le m$, there exists $C_{12} > 0$ such that $|h(x)| \le C_{12}(1 + r_0(x))^{k(1-\alpha_i+\varepsilon)}$ for $x \in E_i$ and for all $1 \le i \le m$. By the fact that the harmonic equation is conformally invariant in dimension 2,

h is harmonic on $\mathbf{R}^2\backslash B^*(1)$ with respect to ds_0^2. Hence by Lemma 4.1, h can be expressed uniquely as

$$h(x) = h_i(x) + h_i^*(x) + \beta_{h,i} \log r_0(x) \tag{4.2}$$

for $x \in E_i$. The function $h_i(x)$ is a harmonic polynomial of degree less than or equal to k_i which vanishes at the origin, $h_i^*(x)$ is a bounded harmonic function on $\mathbf{R}^2\backslash B^*(1)$, and $\beta_{h,i}$ is a constant. We can define the map

$$\Phi: H_k \to P_{k_1} \times P_{k_2} \times \cdots \times P_{k_m} \times \mathbf{R}^{m-1}$$

by

$$\Phi(h) = (h_1, \cdots, h_m, \vec{\beta}_h), \tag{4.3}$$

where $\vec{\beta}_h = (\beta_{h,1}, \cdots, \beta_{h,m-1}) \in \mathbf{R}^{m-1}$. It is clear that Φ is a linear map. Also $\Phi(h) = 0$ implies that h is bounded on $E_1, \cdots, E_{m-1}$ and $h = h_m^* + \beta_{h,m} \log r_0$ on E_m. In any case, h is a harmonic function on M which is bounded either from above or from below, which means $h \equiv$ constant. Therefore the kernel of Φ is of dimension 1, and

$$\begin{aligned} \dim H_k &\leq \dim(P_{k_1} \times \cdots \times P_{k_m} \times \mathbf{R}^{m-1}) + 1 \\ &= \sum_{i=1}^m N_i + m. \end{aligned}$$

In order to obtain a lower bound of the dimension of H_k, we need to study the range of Φ defined in (4.3). Let us first establish the following two lemmas.

Lemma 4.3. *Consider any one of the ends, say E_1, which we identify as $\mathbf{R}^2\backslash B^*(1)$. Let f be a harmonic function on $\mathbf{R}^2$. Then there exists a harmonic function g on M such that $f - g$ is bounded on E_1 and g is bounded on any other end, E_i, for $i \neq 1$.*

Proof. Recall that we denote by $B^*(\rho)$ the Euclidean ball of radius ρ centered at the origin. Suppose f is harmonic on $\mathbf{R}^2$. By the compactness of $\widetilde{M}$, for $\rho > 1$ there exists a harmonic function g_ρ defined on $\widetilde{M}\backslash(E_1\backslash B^*(\rho))$ such that $g_\rho = f$ on $\partial B^*(\rho)$. We claim that

$$\inf_{\partial B^*(1)} |g_\rho - f| = 0. \tag{4.4}$$

In fact, if $g_\rho - f > 0$ on $\partial B^*(1)$, then by the strong maximum principle and the fact that $g_\rho = f$ on $\partial B^*(\rho)$, we have $g_\rho - f > 0$ on $B^*(\rho) - B^*(1)$

and also $\partial g_\rho/\partial r_0 - \partial f/\partial r_0 < 0$ on $\partial B^*(\rho)$. However Stoke's theorem implies that

$$0 = \int_{\widetilde{M}-(E_1-B^*(\rho))} \Delta g_\rho \, dA = \int_{\partial B^*(\rho)} \frac{\partial g_\rho}{\partial \nu} ds = \int_{\partial B^*(\rho)} \frac{\partial g_\rho}{\partial r_0} ds_0,$$

and

$$0 = \int_{B^*(\rho)} \Delta_0 f \, dA_0 = \int_{\partial B^*(\rho)} \frac{\partial r}{\partial r_0} ds_0,$$

which is a contradiction. The same argument also rules out the possibility that $g_\rho - f < 0$ on $\partial B^*(1)$. Hence (4.4) holds.

Let ω_ρ denote the oscillation of the function g_ρ on $\partial B^*(1)$. We assert that there exists a constant $C_{13} > 0$ such that for all $\rho > 1$, we have

$$\omega_\rho \leq C_{13}. \tag{4.5}$$

If (4.5) is not valid, then we can find a sequence $\rho_i \to \infty$ such that $\lim_{j\to\infty} \omega_{\rho_j} = \infty$. Consider the harmonic function on $\widetilde{M}\backslash(E_1\backslash B^*(\rho_j))$ defined by

$$\tilde{g}_j = \frac{g_{\rho_j}}{\omega_{\rho_j}}.$$

Clearly the oscillation of $\tilde{g}_j$ is 1 on $\partial B^*(1)$ for all j. Together with (4.4), this implies that

$$\frac{f}{\omega_{\rho_j}} - \frac{2A}{\omega_{\rho_j}} - 1 \leq \tilde{g}_j \leq \frac{f}{\omega_{\rho_j}} + \frac{2A}{\omega_{\rho_j}} + 1 \tag{4.6}$$

on $\partial B^*(1)$ for all j, where $A = \sup_{\partial B^*(1)} |f|$. Since $\tilde{g}_j = f/\omega_{\rho_j}$ on $\partial B^*(\rho_j)$, it is easy to see that (4.6) is true on $\partial B^*(\rho_j)$, and by the maximum principle (4.6) is also valid on $B^*(\rho_j)\backslash B^*(1)$ for all j. Hence for $\rho > 1$ we have

$$\frac{1}{\omega_{\rho_j}}\left(\inf_{\partial B^*(\rho)} f\right) - \frac{2A}{\omega_{\rho_j}} - 1 \leq \tilde{g}_j \leq \frac{1}{\omega_{\rho_j}}\left(\sup_{\partial B^*(\rho)} f\right) + \frac{2A}{\omega_{\rho_j}} + 1 \tag{4.7}$$

on $\partial B^*(\rho)$ for all j with $\rho_j > \rho$. The maximum principle now implies that the functions $\{\tilde{g}_j\}$ are uniformly bounded on $\widetilde{M}\backslash(E_1\backslash B^*(\rho))$. Hence by passing through a subsequence, $\tilde{g}_j$ converges uniformly on compact subsets of $\widetilde{M}\backslash\{p_1\}$ to a harmonic function $\tilde{g}$, which is defined on $\widetilde{M}\backslash\{p_1\}$. By (4.7), the maximum principle, and the fact that $\lim_{j\to\infty} \omega_{\rho_j} = \infty$, the function $\tilde{g}$ must satisfy $-1 \leq \tilde{g} \leq 1$. Therefore $\tilde{g}$

must be identically constant by the parabolicity of $\widetilde{M}\backslash\{p_1\}$. This contradicts the fact that the oscillations of the functions $\{\tilde{g}_j\}$ on $\partial B^*(1)$ are 1, thus (4.5) is valid.

Applying (4.4) and (4.5), a similar argument shows that the set of functions $\{g_\rho\}$ are uniformly bounded on compact subsets of $\widetilde{M}\backslash\{p_1\}$. So there is a sequence $\rho_i \to \infty$ such that $g_{\rho_i} \to g$, a harmonic function on $\widetilde{M}\backslash\{p_1\}$. Obviously, g is harmonic on M and is bounded on E_i for all $i \neq 1$. To see that $f - g$ is bounded on E_1, we simply observe that

$$f - (2A + C_{13}) \leq g_\rho \leq f + 2A + C_{13}$$

on $B^*(\rho)\backslash B^*(1)$ for all $\rho > 1$.

Lemma 4.4. *Let E_1 and E_2 be any two arbitrary ends of M. There exists a harmonic function g on M such that g is bounded on all other ends E_i for $i \neq 1, 2$. Moreover there are bounded harmonic functions g_1 and g_2 defined on E_1 and E_2, respectively, such that*

$$g = g_1 + \log r_0 \quad \text{on } E_1$$

and

$$g = g_2 - \log r_0 \quad \text{on } E_2.$$

Proof. Let $d\tilde{s}^2$ be a complete conformal metric on $\widetilde{M}\backslash\{p_2\}$ so that $d\tilde{s}^2 = ds_0^2$ on $E_2\backslash B^*(2)$. By the construction of a Green's function in [11, Theorem 1], there exists a harmonic function g on $\widetilde{M}\backslash\{p_2\}$ such that $g(x) \to +\infty$ as $x \to p_1$, and $g \leq 0$ on E_2. Note that g must be unbounded on E_2. Therefore $g = C_{14}\log r_0 + g_1$ on E_1 and $g = -C_{15}\log r_0 + g_2$ on E_2 for some positive constants C_{14} and C_{15} and for some bounded harmonic functions g_1 and g_2 on E_1 and E_2, respectively. Integrating Δg on $\widetilde{M}\backslash(E_1 \cup E_2)$ and applying Stoke's theorem, we conclude that $C_{14} = C_{15}$. Hence dividing g by C_{14}, we obtain the required harmonic function on M.

Following the notation and the assumptions of Theorem 4.2, we are now ready to prove a lower bound for $\dim H_k$.

Theorem 4.5. *Let us consider the following complimentary cases:*

(1) *If M has subquadratic area growth, i.e., $A(r) = o(r^2)$, then* $\dim H_k \geq 1$.

(2) *If M has quadratic area growth, then*

$$\dim H_k \geq \sum_{i=1}^{m} N_i' + m',$$

where $N_i' = \dim P_{k_i - \varepsilon}$ *for all* $\varepsilon > 0$, m' *is the number of ends with quadratic area growth, and we have adapted the convention that* $\dim P_l = 0$ *for* $l < 0$.

Proof. Case (1) is obvious. To prove (2), let us fix $1 \le i \le m$. Consider the case where $k(1-\alpha_i) > 0$ is not an integer. We will prove that for any $h_i \in P_{k_i}$ there exists $h \in H_k$ such that

$$\Phi(h) = (0, \cdots, \overset{i\text{th}}{h_i}, \cdots, 0),$$

which Φ is the linear map defined by (4.3). Since $h_i \in P_{k_i}$, there exists $C_{16} > 0$ such that

$$|h_i(x)| \le C_{16}(1 + r_0(x))^{k_i}$$

in $\mathbf{R}^2$. By Lemma 4.3, there exists a harmonic function h on M such that h is bounded on E_j for all $j \ne i$ and $h - h_i$ is bounded in E_i. The fact that $k(1-\alpha_i) > 0$ is not an integer implies that there is an $\varepsilon > 0$ such that $k_i' = [k(1-\alpha_i)] < k(1 - \alpha_i - \varepsilon)$. Hence for $x \in E_i$, we have

$$\begin{aligned} |h(x)| &\le |h_i(x)| + |h(x) - h_i(x)| \\ &\le C_{17}(1 - r_0(x))^{k_i'} + |h(x) - h_i(x)| \\ &\le C_{17}(1 + r_0(x))^{k(1-\alpha_i-\varepsilon)} + |h(x) - h_i(x)|. \end{aligned}$$

By Corollary 3.3 and the fact that $h - h_i$ is bounded in E_i, there is a constant $C_{18} > 0$ such that $|h(x)| \le C_{18}(1 + r(x))^k$ on E_i. Hence $h \in H_k$, and

$$\Phi(h) = (0, \cdots, \overset{i\text{th}}{h_i}, \cdots, 0).$$

In the case where $k(1-\alpha_i) > 0$ is an integer, $k_i - 1 = k(1-\alpha_i) - 1 < k(1-\alpha_i)$. A similar argument shows that for each $h_i \in P_{k_i - 1}$, there exists $h \in H_k$ such that

$$\Phi(h) = (0, \cdots, \overset{i\text{th}}{h_i}, \cdots, 0).$$

Suppose that $m' = 1$; then from the above results together with the fact that the nullity of Φ is 1, it is easy to see that $\dim H_k \ge \sum_{i=1}^m N_i' + 1$. Hence we may assume that $m' \ge 2$. We may also assume that E_m and E_i, for $1 \le i \le m' - 1$, have quadratic area growth. By Lemma 4.4, for each $1 \le i \le m' - 1$, there exists a harmonic function $g^{(i)}$ on M which is bounded on E_j for all $j \ne i$ or m. Moreover, there are bounded harmonic functions $g_m^{(i)}$ and $g_i^{(i)}$ defined on E_m and E_i, respectively, such that $g^{(i)} = -\log r_0 + g_m^{(i)}$ on E_m and $g^{(i)} = \log r_0 + g_i^{(i)}$ on E_i.

By Proposition 1.2, $1-\alpha_j > 0$ for $1 \le j \le m'-1$ or $j = m$. Hence Corollary 3.3 implies that $g^{(i)} \in H_k$ and $\Phi(g^{(i)}) = (0, \cdots, 0, \vec{\beta})$ where

$$\vec{\beta}_i = (0, \cdots, \overset{i\text{th}}{1}, \cdots, 0) \in \mathbf{R}^{m-1}$$

for $1 \le i \le m'-1$. Hence the rank of Φ must be at least $\sum_{i=1}^m N_i' + m' - 1$ and $\dim H_k \ge \sum_{i=1}^m N_i' + m'$.

When the curvature of M is nonnegative outside a compact set, by using Corollary 3.4 instead of Corollary 3.3, we obtain the following theorem.

Theorem 4.6. *With the assumptions and notation of Theorem* 4.5, *and the additional assumption that M has nonnegative curvature outside a compact set,*

$$\dim H_k = \sum_{i=1}^m N_i + m$$

for all $k \ge 1$.

5. An isoperimetric inequality

In this section we will prove that a complete surface with finite total curvature which has quadratic area growth at each end must satisfy an isoperimetric inequality. On the other hand, any surface satisfying this isoperimetric inequality must have quadratic area growth. Hence we can view the isoperimetric inequality and the area growth condition as equivalent conditions in our special class of complete surfaces.

We will first prove the isoperimetric inequality for simply connected surfaces.

Theorem 5.1. *Let M be a simply connected complete noncompact surface with finite total curvature. Suppose that*

$$2\pi - \int_M K\, dA = \alpha > 0.$$

Then there exists a constant $C_{19} > 0$ depending only on M such that for any relatively compact domain $D \subseteq M$, we have $L^2(\partial D) \ge C_{19} A(D)$.

Proof. By Proposition 1.2 and the assumption that $\alpha > 0$, for any $\varepsilon > 0$ if r is sufficiently large, then

$$\frac{L^2(r)}{A(r)} \ge (2-\varepsilon)\alpha \tag{5.1}$$

and

$$A(r) \geq \frac{(1-\varepsilon)\alpha}{2} r^2. \tag{5.2}$$

We can also choose $R_0 > 0$ such that

$$\int_{M \setminus B_p(R_0)} |K| \, dA < \pi. \tag{5.3}$$

By enlarging R_0 if necessary, we may assume that (5.1) and (5.2) hold for all $r > R_0$, and also that $\partial B_p(r)$ is homeomorphic to a circle for $r > R_0$ because of Proposition 1.4.

In order to prove the theorem, it is sufficient to consider the case where D is simply connected. Indeed, by the fact that M is homeomorphic to $\mathbf{R}^2$, D must be homeomorphic to a domain of the form

$$D = D_0 \setminus \left(\bigcup_{1 \leq i \leq k} D_i \right),$$

where the domains D_0 and D_i, for $1 \leq i \leq k$, are mutually disjoint and homeomorphic to the unit disk with $D_i \subset D_0$. Clearly, an isoperimetric inequality for D_0 will imply the same inequality for D.

For a relatively compact simply connected domain $D \subseteq M$, let us denote $\sigma = \partial D$, $\rho = \min_{x \in \sigma} r(x)$, and $R = \max_{x \in \sigma} r(x)$, where $r(x) = d(x, p)$ is the geodesic distance between x and p. By the definition of ρ and R, we have the inequality

$$L(\sigma) \geq R - \rho. \tag{5.4}$$

We will now consider the following cases:

Case 1. Suppose $R < 2R_0$. Then by the definition of R, $\sigma \subseteq B_p(2R_0)$. In fact, $D \subseteq B_p(2R_0)$, since $D \cap \{M \setminus B_p(2R_0)$ is a compact connected component of $M \setminus B_p(2R_0)$, which is impossible because $M \setminus B_p(2R_0)$ is homeomorphic to $\mathbf{R}^2 \setminus B^*(1)$. By the relative compactness of the ball $B_p(2R_0)$, there exists a constant $C_{20} > 0$ depending on R_0 such that

$$L^2(\sigma) \geq C_{20} A(D). \tag{5.5}$$

Case 2. Suppose $R \geq 2R_0$ and $R - \rho \geq R/2$. Inequality (5.4) implies

$$L^2(\sigma) \geq (R-\rho)^2 \geq \frac{R^2}{4}. \tag{5.6}$$

However Proposition 1.3 and the fact that $D \subseteq B_p(R)$ implies

$$C_1 R^2 \geq A(R) \geq A(D).$$

Hence

$$L^2(\sigma) \geq \frac{A(D)}{4C_1}.$$

Case 3. Suppose $\rho > R/2 \geq R_0$ and $p \notin D$. This shows that $\sigma \subseteq M\backslash B_p(R_0)$, and hence by the simple connectivity of both D and $B_p(R_0)$, we have $D \cap B_p(R_0) = \varnothing$. An isoperimetric inequality of Huber in [7] together with (5.3) yields

$$\begin{aligned} L^2(\sigma) &\geq 2\left(2\pi - \int_D K^+ \, dA\right) A(D) \\ &\geq 2\left(2\pi - \int_{M\backslash B_p(R_0)} |K| \, dA\right) A(D) \geq 2\pi A(D), \end{aligned} \tag{5.7}$$

where $K^+ = \max\{0, K\}$.

Case 4. Suppose $R \geq 2R_0$, $R - \rho < R/2$, and $p \in D$. As in Case 3, we conclude that $\sigma \subseteq M\backslash B_p(R_0)$. By the assumption that $p \in D$, $B_p(\rho)$ is the smallest geodesic ball centered at p which is contained in D, and $B_p(R)$ is the largest geodesic ball centered at p which contains D. We claim that

$$L(\sigma) \geq C_{21} R, \tag{5.8}$$

where $C_{21} = \min\{1/2, 4\sqrt{2(2-\varepsilon)(1-\varepsilon)}/(25C_1)\}$, with the constant C_1 as in Proposition 1.3. To prove the claim, in view of (5.4) let us assume that $R - \rho < C_{21} R$. Hence

$$\rho > (1 - C_{21})R. \tag{5.9}$$

Note that by the definition of C_{21}, $1 - C_{21} > 0$. Let x be a fixed point on σ. Suppose that (5.8) is not true; then $\sigma \subseteq B_x(C_{21}R/2)$. For any $y \in B_p((1 + C_{21})R)\backslash B_p(R)$, let γ be a minimal geodesic from p to y. We know that $\gamma \cap \sigma \neq \varnothing$, because $p \in D$ and $y \in D$ by the choice of R. Let $z \in \gamma \cap \sigma$; then

$$d(y, z) = d(p, y) - d(p, z) < (1 + C_{21})R - (1 - C_{21})R = 2C_{21}R,$$

where we have used the facts that $y \in B_p((1 + C_{21}R)$ and $z \in \sigma \subseteq M\backslash B_p(\rho) \subseteq M\backslash B_p((1 - C_{21})R)$ by (5.9). Therefore $B_p((1 + C_{21})R)\backslash B_p(R) \subseteq B_x(5C_{21}R/2)$, and

$$A((1 + C_{21})R) - A(R) \leq A\left(B_x\left(\frac{5C_{21}R}{2}\right)\right) \leq \frac{25}{8} C_1 C_{21}^2 R^2, \tag{5.10}$$

by Proposition 1.3. However, by (5.1) we have $A'(r)/\sqrt{A(r)} = L(r)/\sqrt{A(r)} \geq \sqrt{(2-\varepsilon)\alpha}$ for $r \geq R \geq 2R_0$. Integrating the inequality from R to

$(1+C_{21})R$ yields

$$2(\sqrt{A((1+C_{21})R} - \sqrt{A(R)} \geq \sqrt{(2-\varepsilon)\alpha}C_{21}R.$$

Applying inequality (5.2), we obtain

$$\begin{aligned} A((1+C_{21})R) - A(R) &\geq \frac{\sqrt{(2-\varepsilon)\alpha}}{2}C_{21}R(\sqrt{A(1+C_{21})R)} + \sqrt{A(R)} \\ &\geq \sqrt{\frac{(2-\varepsilon)(1-\varepsilon)}{2}}\alpha C_{21}R^2. \end{aligned}$$

Combining this with (5.10) gives $C_{21} \geq 4\sqrt{2(2-\varepsilon)(1-\varepsilon)}/(25C_1)$, which contradicts the definition of C_{21}. Therefore $L(\sigma) \geq R - \rho \geq C_{21}R$ and by Proposition 1.3

$$L^2(\sigma) \geq C_{21}^2R^2 \geq \frac{2C_{21}^2}{C_1}A(R) \geq \frac{2C_{21}^2}{C_1}A(D). \tag{5.11}$$

We now conclude that the theorem is valid with the choice of

$$C_{19} = \min\left\{C_{20}, \frac{1}{4C_1}, 2\pi, \frac{2C_{21}^2}{C_1}\right\}.$$

Theorem 5.2. *Let M be a complete noncompact surface with finite total curvature. Suppose that all the ends of M have quadratic area growth. Then there exists a constant $C_{22} > 0$ depending only on M such that for any relatively compact domain $D \subseteq M$, we have*

$$L^2(\partial D) \geq C_{22}A(D).$$

Proof. Let p be a fixed point in M. By [15], there exists $a > 0$, such that for $r \geq a$, the set $M \backslash B_p(r)$ can be written as $\bigcup_{i=1}^m M_i(r)$, where m is the number of ends of M. Moreover, $M_i(r)$ is homeomorphic to $S^1 \times [0, \infty)$, and $\partial M_i(r)$ is homeomorphic to S^1 for all i. By Huber's theorem in [8], each $M_i(a)$ is conformally equivalent to $\mathbf{R}^2 \backslash B^*(1)$. By arbitrarily extending the metric to $\mathbf{R}^2$, we may assume that the metric ds^2 on $M_i(a)$ from M agrees with a complete metric ds_i^2 of $\mathbf{R}^2$ on the set $\mathbf{R}^2 \backslash B^*(1)$. Let $r(x)$ denote the distance from p to x with respect to the metric ds^2, and let $r_i(x)$ denote the distance from the origin to x with respect to the metric ds_i^2. Then clearly $r_i(x)/r(x) \to 1$ as $x \to \infty$. Hence we may assume that for all $x \in \mathbf{R}^2 \backslash B^*(1)$, we have

$$2 \geq \frac{r_i(x)}{r(x)} \geq \frac{1}{2}.$$

Let D be a bounded domain in M with boundary $\sigma = \bigcup_{j=1}^{k} \sigma_j$, where each σ_j is a simple closed curve. Let $R_i = \sup_{x\in\sigma_j} r(x)$, $\rho_j = \inf_{x\in\sigma_j} r(x)$, and $R = \max_{1\le j\le k} R_j$.

Case 1. Suppose $R \le 2a$. Then, following the same argument as in the proof of Theorem 5.1, $D \subseteq B_p(2a)$. By the compactness of $B_p(2a)$ there exists a constant $C_{23} > 0$ which depends on $B_p(2a)$, such that

$$L^2(\sigma) \ge C_{23} A(D).$$

Case 2. Suppose $R > 2a$. Without loss of generality, we may assume $R = R_1 > 2a$. Then $D \subseteq B_p(R_1)$. By Proposition 1.3, $A(R_1) \le C_1 R_1^2/2$. If $R_1 - \rho_1 \ge R_1/2$, then

$$\begin{aligned} L^2(\sigma) \le L^2(\sigma_1) &\ge 4(R_1 - \rho_1)^2 \ge R_1^2 \\ &\ge \frac{2A(R_1)}{C_1} \ge \frac{2A(D)}{C_1}. \end{aligned}$$

On the other hand, if $R_1 - \rho_1 < R_1/2$, then $\rho_1 > R_1/2 > a$. Hence $\sigma_1 \subseteq M\backslash B_p(a)$. We assume $\sigma_1 \subseteq M_1(a)$, and further that $D \not\subseteq M_1(a)$. Indeed, if $D \subseteq M_1(a)$, then we can view $D \subseteq \mathbf{R}^2$ and apply Theorem 5.1 to the metric ds_1^2, and use the fact $ds^2 = ds_1^2$ to conclude that

$$L^2(\sigma) \ge C_{19} A(D).$$

Hence $D \not\subseteq M_1(a)$. We claim that $\partial M_1(a)$ must lie inside σ_1 in $\mathbf{R}^2$, after we identify $M_1(a)$ to $\mathbf{R}^2\backslash B^*(1)$. In fact, by the definition of $R = R_1$, the set $D \cap M_1$ must lie inside σ_1. Since $\sigma_1 \subseteq M_1$, we conclude that σ_1 must be homotopic to ∂M_1, otherwise $D \subseteq M_1$. Moreover, the origin of $\mathbf{R}^2$ is contained inside σ_1.

The assumption that $\rho_1 > R_1/2$ implies that for all $x \in \sigma_1$, we have

$$r_1(x) \ge \frac{r(x)}{2} > \frac{R_1}{4}.$$

Hence the set $D_1 = \{x \mid r_1(x) < R/4\}$ lies inside σ_1. Using Theorem 5.1 on the domain $\tilde{D}$ bounded by σ_1, we derive

$$L^2(\sigma_1) = L_1^2(\sigma_1) \ge C_{19} A_1(\tilde{D}) \ge C_{19} A_1(D_1) \ge C_{24} R_1^2,$$

where L_1 and A_1 are the length and the area computed with respect to the metric ds_1^2 respectively. The last inequality follows from the area growth assumption on M. Proposition 1.3 and the above inequality imply that

$$L^2(\sigma) \ge L^2(\sigma_1) \ge \frac{2C_{24}}{C_1} A(R_1) \ge \frac{2C_{24}}{C_1} A(D).$$

This completes the proof.

The fact that the isoperimetric inequality is equivalent to the Sobolev inequality (see [17]) allows us to state the theorem in the following form.

Corollary 5.3. *Let M be a complete noncompact surface with finite total curvature. Suppose that each of the ends of M has quadratic area growth. Then there exists a constant $C_{22} > 0$ (given by Theorem 5.2), such that for any compactly supported function $f \in H_1^1(M)$, we have*

$$\int_M |\nabla f|\, dA \geq C_{22}\left(\int_M |f|^2\, dA\right)^{1/2}.$$

6. A Poincaré inequality and a Harnack inequality

In this section, we will prove a Poincaré inequality for the Neumann boundary value problem, which together with Theorem 5.2 and Proposition 1.3 will imply a Harnack inequality for solutions of second-order linear elliptic partial differential equations.

For any set $E \subseteq M$ and any point $x \in M$, let us define the set

$$\Theta_x(E) = \{v \in S_x^1 \mid \exp_x(t_0 v) \in E \text{ for some } t_0 \text{ and the geodesic } \gamma(t) = \exp_x(tv) \text{ minimizes up to } t_0\},$$

where $S_x^1 \in T_x(M)$ is the set of unit tangent vectors at $x \in M$. We also denote the one-dimensional Lebesgue measure of $\Theta_x(E)$ by

$$\omega_x(E) = \mu(\Theta_x(E)),$$

and the geodesic cone over E by

$$C_x(E) = \{y \in M \mid y = \exp_x(tv) \text{ for some } t \geq 0 \text{ and } v \in \Theta_x(E)\}.$$

For any value of $t > 0$, let

$$\Theta_x(E, t) = \{v \in \Theta_x(E) \mid \text{the geodesic } \gamma(s) = \exp_x(sv) \text{ minimizes up to } t\},$$

$$\omega_x(E, t) = \mu(\Theta_x(E, t)),$$

and

$$C_x(E, t) = \{y \in M \mid y = \exp_x(sv) \text{ for some } v \in \Theta_x(E, t) \text{ and some } s \leq t\}.$$

We also denote the area and the length respectively by

$$A_x(E, t) = A(C_x(E) \cap B_x(t))$$

and

$$L_x(E, t) = L(C_x(E) \cap \partial B_x(t)).$$

Theorem 6.1. *Let M be a complete noncompact surface with finite total curvature. Let $p \in M$ be a fixed point. Assume that there exists a constant $C_{25} > 0$ depending only on M such that for all $x \in M$ and all $r > 0$, the area of the geodesic balls of radius r, centered at x, satisfy $A(B_x(r)) \geq C_{25} r^2$. Then there exist $R_0 > 0$ and $C_{26} > 0$, such that for $R > R_0$ and $q \in \partial B_p(5R)$, the first nonzero Neumann eigenvalue, λ_1, for the Laplacian on $B_q(r)$ for $r \leq R$ must satisfy*

$$\lambda_1(B_q(r)) \geq \frac{C_{26}}{r^2}.$$

In particular, we have the inequality

$$\inf_k \int_{B_q(r)} (f-k)^2\, dA \leq \frac{R^2}{C_{26}} \int_{B_q(r)} |\nabla f|^2\, dA$$

for all $f \in H_1^2(B_q(r))$.

Proof. In view of [17], it suffices to show that for all $x \in B_q(r)$ and all $E \subseteq B_q(r)$ with $A(E) \geq \frac{1}{2} A(B_q(r))$ the quantity $\omega_x(E)$ is bounded below by a positive constant depending only on M.

Let x and E be as above. By the fact that $E \subseteq C_x(E) \cap B_x(2r)$, we have

$$\frac{1}{2} A(B_q(r)) \leq A(C_x(E) \cap B_x(2r)) = A_x(E, 2r). \tag{6.1}$$

Applying a similar argument as in [3], [6] to $C_x(E, t)$ we obtain

$$\begin{aligned} \frac{dL_x(E,t)}{dt} &\leq \omega_x(E,t) - \int_{C_x(E,t)} K\, dA \\ &\leq \omega_x(E) + \int_{C_x(E) \cap B_x(t)} |K|\, dA \end{aligned}$$

in the sense of distribution. Integrating twice from 0 to $2r$ yields

$$A_x(E, 2r) \leq 2r^2 \left(\omega_x(E) + \int_{C_x(E) \cap B_x(2r)} |K|\, dA \right).$$

Combining this with (6.1), we have

$$A(B_q(r)) \leq 4r^2 \left(\omega_x(E) + \int_{C_x(E) \cap B_x(2r)} |K|\, dA \right). \tag{6.2}$$

Suppose the theorem is not true. Then there exists a divergent sequence $\{R_i\}$ satisfying: $r_i \leq R_i$, $q_i \in \partial B_p(5R_i)$, $x_i \in B_{q_i}(r_i)$, and $E_i \subseteq B_{q_i}(r_i)$ with $A_i(E_i) \geq \frac{1}{2} A(B_{q_i}(r_i))$, such that

$$\lim_{i \to \infty} \omega_{x_i}(E_i) = 0.$$

Using (6.2), we find

$$\begin{aligned}\limsup_{i\to\infty} \frac{A(B_{q_i}(r_i))}{r_i^2} &\le \limsup_{i\to\infty} 4\left(\omega_{x_i}(E_i) + \int_{C_{x_i}(E_i)\cap B_{x_i}(2r_i)} |K|\, dA\right) \\ &\le \limsup_{i\to\infty} 4 \int_{B_{q_i}(3r_i)} |K|\, dA \\ &\le \limsup_{i\to\infty} 4 \int_{M-B_p(R_i)} |K|\, dA = 0,\end{aligned}$$

since $\int_M |K|\, dA < \infty$. However, by the assumption that $A(B_{q_i}(r_i)) \ge C_{27} r_i^2$, this is a contradiction, and the theorem is hence proved.

By applying Corollary 5.3, we conclude the following.

Corollary 6.2. *Let M be a complete noncompact surface with finite total curvature and let E be an end of M. Let $p \in M$ be a fixed point. Suppose E has quadratic area growth. Then there exist $R_0 > 0$ and $C_{27} > 0$ such that for $R > R_0$, we have*

$$\lambda_1(B_q(r)) \ge \frac{C_{27}}{r^2}$$

for all $q \in \partial B_p(5R) \cap E$ and all $r \le R$.

It is known that by a modification of Moser's method (see [1], [14]), the Poincaré inequality (Corollary 6.2), the Sobolev inequality (Theorem 5.3), and an area growth assumption imply the following Harnack inequality.

Theorem 6.3. *Let M be a complete noncompact surface with finite total curvature. Let $p \in M$ be a fixed point. Suppose E is an end of M with quadratic area growth. Then there exist $R_0 > 0$ and a constant $C_{28} > 0$ depending only on M such that for $R > R_0$, $q \in \partial B_p(5R) \cap E$, and any positive harmonic function u defined on $B_q(R)$, we have*

$$\sup_{B_q(R/2)} u \le C_{28} \inf_{B_q(R/2)} u.$$

7. Examples

In this section, we will give some examples to demonstrate some of the fine points of the previous results.

Example 1. Let $M = (\mathbf{R}^2, ds^2)$, where $ds^2 = e^{2u} ds_0^2$ for a smooth function $u = u(r_0)$ with $e^u = (r_0 \log r_0)^{-1}$ on $\mathbf{R}^2 \backslash B^*(2)$. For $x \in M$ such that $r_0(x) > 2$, we have

$$r(x) = \int_2^{r_0(x)} \frac{dr_0}{r_0 \log r_0} + \int_0^2 e^{u(r_0)} dr_0 = \log(\log r_0(x)) + C_{29}$$

for some constant C_{29} which is independent of x. Hence M is complete and for $r_0(x) > 2$, $\log r_0(x) = C_{30}e^{r(x)}$.

On $\mathbf{R}^2 \backslash B^*(2)$,

$$\Delta_0 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = (r_0 \log r_0)^{-2}.$$

Therefore M has finite total curvature and the Gaussian curvature is negative outside the set $B^*(2)$.

This example shows that:

(1) Inequality (3.10) of Corollary 3.4 is not valid without the assumption that $K \geq 0$ outside a compact set. In fact, one does not expect $\log r_0$ to be of polynomial growth.

(2) Note that $\int_M K\,dA = 2\pi$, and M is of subquadratic area growth. Also, the conclusion of Theorem 6.3 is not true on M. In fact, $\log r_0$ is positive if $r_0(x) > 1$. If R is sufficiently large and $q \in \partial B_0(5R)$, then $\log r_0(q) = C_{30}e^{5R}$. We can find a point $x \in B_q(R/2)$ so that $r(x) = (5 + \frac{1}{4})R$, and $\log r_0(x) = C_{30}e^{(5+1/4)R}$. Hence $\log r_0(x) = C_{30}e^{R/4} \log r_0(q)$, and we do not have the inequality asserted by Theorem 6.3. This implies that the assumption that M has quadratic area growth is essential for Theorem 6.3.

(3) From this example, it is easy to construct other examples to show that m' cannot be replaced by m in the statement of Theorem 4.5.

Example 2. Let $M = (\mathbf{R}^2, ds^2)$, with $ds^2 = e^{2u}ds_0^2$. Set $e^u = (\log(r_0^2 + 2))^{-1}$, check that M is complete, and compute

$$\Delta_0 u = -\frac{8}{(r_0^2+2)^2 \log(r_0^2+2)} + \frac{4r_0^2}{(r_0^2+2)^2(\log(r_0^2+2))^2}.$$

Hence M has finite total curvature. Also

$$\begin{aligned}\alpha &= \frac{1}{2\pi}\int_M K\,dA = -\frac{1}{2\pi}\iint_{\mathbf{R}^2} \Delta_0 u\,dx\,dy \\ &= -\frac{1}{2\pi}\lim_{r_0\to\infty}\int_{\partial B^*(r_0)} \frac{\partial u}{\partial r_0} ds_0 = 0.\end{aligned}$$

Since $r(x) = \int_0^{r_0(x)} (\log(t^2+1))^{-1}\,dt$, we have

$$\lim_{r_0\to\infty} \frac{r}{r_0} = \lim_{r_0\to\infty} \frac{dr}{dr_0} = \lim_{r_0\to\infty} (\log(r_0^2+1))^{-1} = 0.$$

Theorem 3.2 implies that for any $\varepsilon > 0$, $r_0^{(1-\alpha-\varepsilon)}(x) \leq r(x)$ asymptotically. This example shows that the constant $\varepsilon > 0$ cannot be removed, because $\alpha = 0$. Moreover, this example also shows that N_i' cannot be replaced by N_i in Theorem 4.5.

Example 3. Let M be the flat cylinder. Then there is a linear growth harmonic function which is positive at one end and negative at the other end. In fact it must be asymptotically $\log r_0 +$ constant at one end and asymptotically $-\log r_0 +$ constant at the other end. Hence the function $\log r_0$ must be of linear growth, and Theorem 4.6 does not hold when $k < 1$.

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