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CORRESPONDENCE OF MODULAR FORMS TO CYCLES ASSOCIATED TO $O(p,q)$

S. P. WANG

Introduction

In their study of Hilbert modular surfaces, Hirzebruch and Zagier [13] have discovered a striking connection between geometry and number theory. It was established that intersection numbers of cycles are Fourier coefficients of modular forms for Hilbert modular surfaces. Since then, the study of certain liftings of automorphic forms and their relation to geodesic cycles in quotients of symmetric spaces has been of great interest. The first subsequent big advance was made by Kudla and Millson [23] for their work on $SO(p,1)$ which offers a systematic and fruitful approach to the general case. They took the reductive pair $O(p,1) \times \mathrm{Sp}(2r,\mathbf{R})$ as their framework and used Weil representation to construct a theta function which has a geometric realization. Besides technical problems, they presented a feasible scheme for the general case.

In [33], [34], the analogous problems for $SU(p,1)$ were solved by Y. L. Tong and the author. In [35], we gave a correspondence, in the form of a geometric lifting, from Hermitian cusp forms of weight $p+2$ to certain harmonic differential forms of degree $(2,2)$ in compact quotients of $SU(p,2)$. This is the first example for symmetric spaces of higher rank. In [36], we returned to $SU(p,1)$ to discuss the case of noncompact quotients. In these studies, we witnessed tremendous technical complexity and gradually shifted our reliance on invariant theory. It should be mentioned here that we were inspired by Howe's recent effort [14]–[16] to emphasize the importance of classical invariant theory.

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In this paper, we give a satisfactory presentation of the geometric lifting for the reductive pairs $O(p,q) \times \mathrm{Sp}(2r, \mathbf{R})$ ($r \leqslant p/2$). Our approach here, following [36], is to pair directly a geometric theta function and a cusp differential harmonic form to yield a cusp form with period integrals as Fourier coefficients. this is analogous to Shintani's original construction [30] which was later generalized by Oda for $SO(2, n-2)$ in [26]. Note also that our result is closely related to those of Siegel [31] that volumes of analogous cycles in arithmetic quotients of domains associated to indefinite quadratic forms are Fourier coefficients of Eisenstein series. Our technical tool for computation is the representation theory of $O(p)$. In the following, we state our main results and content organization. Let $\mathscr{D}$ be the symmetric space associated to $O(p,q)$, $j(g,Z)$ the automorphic factor given by (1.3) and G the subgroup of $O(p,q)$ consisting of those g such that the function $\det(j(g,Z))$ ($z \in \mathscr{D}$) is positive. Each $Z \in \mathscr{D}$ can be identified with a maximal negative subspace $\langle Z \rangle$ of $\mathbf{R}^n$ ($n = p + q$). Let K_Z be the subgroup of G of elements which fix elements of $\langle Z \rangle$. It is easy to see that K_Z is isomorphic to $O(p)$ and acts on the space $\bigwedge^{rq}(\mathscr{D})_Z$ of differential forms of $\mathscr{D}$ of degree rq at Z. Thus representation theory of $O(p)$ is applicable and we have the notion of a differential form of degree rq of highest signature (defined in §4.9). The analysis of the pairing rests on the invariant theorem (Theorem 4.9) of harmonic forms of $\mathscr{D}$ of degree rq of highest signature invariant under certain subgroup G_M of G. Technical matters are discussed in geometric, algebraic and analytical aspects. §1 deals with geometric preliminaries. Here we apply Flander's result [7] to construct dual forms for cycles of quotients of $\mathscr{D}$. In §3, we study some representation and invariant theorems of $O(p)$ for algebraic preparation, and §5 handles the deep involvement of analysis of period integrals. To construct the correct geometric theta function, we set it up as follows. First we translate the construction of dual forms of cycles into polynomials $f(Z,M)$ ($M \in M_{nr}(\mathbf{R})$) (Definition 2.9) of M with differential forms of $\mathscr{D}$ as values. With the aid of Theorem 3.11, we modify $f(Z,M)$ to obtain spherical polynomials $F(Z,M)$ (Definition 4.6) which yield the desired Schwartz function $f_{\tau,Z}(M)$ ((6.27)) by coupling with an exponential function $e_{\tau,Z}$ ((6.18)). Finally summing $f_{\tau,Z}(M)$ over certain lattice points M we arrive at the geometric theta functions $\theta(\tau,h,Z)$ ((6.30)). The geometric interpretation of the lifting map is given in §7 and the main result of this paper is Theorem 7.10.

For $q = 1$, our result coincides with that of [23] with improved range of r and additional coverage of noncompact quotients of $\mathscr{D}$. For Weil representation, we follow the setup of Shintani [30] and use the results in [23]. It is clear now that our method is suitable for all reductive pairs. We hope to discuss this matter elsewhere. Obviously we are indebted to the papers [19], [20], [23] which built up some technical background needed for the present approach. The author would like to thank R. Howe and Y. L. Tong for various valuable conversations.

1. Geometric preliminaries

In this section, we recall some of the geometric preliminaries, needed for our discussion, of the symmetric space associated to the group $O(p,q)$ and present a procedure to construct singular forms for cycles of quotients to $O(p,q)$ which will yield dual forms of cycles.

1.1. Let $n = p + q$ and let Q be the symmetric matrix

$$Q = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix},$$

where E_p and E_q are the identity matrices of order p and q respectively. Let $O(p,q)$ be the group given by

$$O(p,q) = \left\{ g \in \mathrm{GL}(n, \mathbf{R}) \middle| {}^t gQg = Q \right\}.$$

Here ${}^t g$ denotes the transpose of g. For $g \in \mathrm{GL}(n, \mathbf{R})$, let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be the block form with $A \in M_{pp}(\mathbf{R})$, $B \in M_{pq}(\mathbf{R})$, $C \in M_{qp}(\mathbf{R})$ and $D \in M_{qq}(\mathbf{R})$. Then the condition ${}^t gQg = Q = Q$ is equivalent to

$$g^{-1} = \begin{pmatrix} {}^t A & -{}^t C \\ -{}^t B & {}^t D \end{pmatrix}. \tag{1.1}$$

1.2. Let $G = O(p,q)$ and let $\mathscr{D}$ be the symmetric space associated to G. We realize $\mathscr{D}$ as the bounded domain

$$\mathscr{D} = \left\{ Z \in M_{pq}(\mathbf{R}) \middle| {}^t ZZ < E_q \right\}.$$

For $g \in G$, the translation of g on $\mathscr{D}$ is given by the fractional transformation

$$gZ = (AZ + B)(CZ + D)^{-1}. \tag{1.2}$$

With this action, we have an automorphic factor

$$J(g, Z) = \begin{pmatrix} \tau(g, Z) & 0 \\ 0 & j(g, Z) \end{pmatrix},$$

where

$$\tau(g, Z) = A - (gZ)C, \qquad j(g, Z) = CZ + D. \tag{1.3}$$

The action of g on $\mathscr{D}$ can also be expressed in terms of the automorphic factor $J(g, Z)$ and the linear action of G,

$$g\begin{pmatrix} Z \\ E_q \end{pmatrix} = \begin{pmatrix} gZ \\ E \end{pmatrix} j(g, Z), \tag{1.4}$$
$$(E_p, Z)^t g = \tau(g, Z)^{-1}(E, gZ).$$

By (1.4), one checks readily that $\mathscr{D}$ has a G-invariant metric given by

$$ds^2 = \operatorname{tr}\left((E - Z{}^tZ)^{-1} dZ(E - {}^tZZ)^{-1} d{}^tZ\right). \tag{1.5}$$

1.3. Definition. Let V be a positive subspace of $\mathbf{R}^n$ with respect to Q. Denote by G_V and $\mathscr{D}_V$ the subgroup and submanifold given by

$$G_V = \{g \in G \mid gV = V\}, \tag{1.6}$$
$$D_V = \left\{Z \in \mathscr{D} \middle| \left({}^tZE_q\right)Qv = 0,\ v \in V\right\}.$$

1.4. Lemma. *The group G_V and the submanifold $\mathscr{D}_V$ satisfy the following conditions*:

(i) *For $g \in G$, $g\mathscr{D}_V = \mathscr{D}_{gV}$.*

(ii) *The identity component of the subgroup of G_V leaving elements of V fixed acts transitively on $\mathscr{D}_V$.*

(iii) *$\mathscr{D}_V$ is a totally geodesic subsymmetric space of $\mathscr{D}$ of dimension $(p - r)q$ with $r = \dim_{\mathbf{R}}(V)$.*

Proof. Same as [33, Lemma 1.2].

1.5. Let $e_1, \cdots, e_n$ be the standard basis of $\mathbf{R}^n$. To discuss the geometric properties of G_V and $\mathscr{D}_V$, by the Witt theorem and (i) of Lemma 1.4, we may assume that V is spanned by $e_{p-r+1}, \cdots, e_p$. For simplicity, we write G_1 and $\mathscr{D}_1$ for G_V and $\mathscr{D}_V$ in this case. For $Z \in \mathscr{D}$, we decompose the matrix Z into

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

with $Z_1 \in M_{p-r,q}(\mathbf{R})$ and $Z_2 \in M_{rq}(\mathbf{R})$. Then $\mathscr{D}_1$ is simply given by

$$\mathscr{D}_1 = \{Z \in \mathscr{D} \mid Z_2 = 0\}.$$

For $g \in G_1$, we express it in the block matrix form

$$g = \begin{pmatrix} A_1 & 0 & B_1 \\ 0 & u & 0 \\ C_1 & 0 & D_1 \end{pmatrix} \tag{1.7}$$

with $A_1 \in M_{p-r,p-r}(\mathbf{R})$, $D_1 \in M_{qq}(\mathbf{R})$ and $u \in O(r)$. From (1.2), the action of G_1 on $\mathscr{D}$ is given by

$$gZ = \begin{pmatrix} (A_1Z_1 + B_1)(C_1Z_1 + D_1)^{-1} \\ uZ_2 j^{-1}(g, Z) \end{pmatrix} \qquad (g \in G_1, Z \in \mathscr{D}). \tag{1.8}$$

1.6. Lemma. *Let* $H_Z = (E - {}^tZZ)^{-1}$, $A(Z) = \det(E - {}^tZZ)$, $L_Z = (E - {}^tZ_1Z_1)^{-1}$ *and* $B(Z) = \det(E - {}^tZ_1Z_1)$. *Then we have the following conditions*:

(i) $H_{gZ} = j(g, Z)H_Z{}^tj(g, Z)\ (g \in G)$.

(ii) $L_{gZ} = j(g, Z)L_Z{}^tj(g, Z)\ (g \in G_1)$.

(iii) B/A *is* G_1*-invariant.*

Proof. (i) and (ii) are immediate from (1.4) and (iii) is an easy consequence of (i) and (ii).

The function $B(Z)/A(Z)$ is closely related to the distance function $d(Z, \mathscr{D}_1)$ from Z to $\mathscr{D}_1$. The same assertion in [33, Proposition 1.7] also holds here. In particular we have the inequalities

$$(1.9)\qquad \begin{aligned} &\text{(i)} \quad 4^m(B/A) \geqslant e^{2d(Z,\mathscr{D}_1)},\\ &\text{(ii)} \quad e^{2\sqrt{m}\,d(Z,\mathscr{D}_1)} \geqslant B/A, \end{aligned}$$

where $m = \min\{r, q\}$. For later estimation, we also need the following inequality.

1.7. Lemma. *We have the inequality*

$$1 + r^{-1} \operatorname{tr} Z_2(E - {}^tZZ)^{-1}{}^tZ_2 \geqslant (B/A)^{1/r}.$$

Proof. We have that

$$\begin{aligned} E - {}^tZ_1Z_1 &= E - {}^tZZ + {}^tZ_2Z_2\\ &= (E - {}^tZZ)^{1/2}\left\{E + (E - {}^tZZ)^{-1/2}{}^tZ_2Z_2(E - {}^tZZ)^{-1/2}\right\}(E - {}^tZZ)^{1/2}; \end{aligned}$$

as a consequence

$$\begin{aligned} B/A &= \det\left\{E + (E - {}^tZZ)^{-1/2}{}^tZ_2Z_2(E - {}^tZZ)^{-1/2}\right\}\\ &= \det\left\{E_r + Z_2(E - {}^tZZ)^{-1}{}^tZ_2\right\}. \end{aligned}$$

The matrix $Z_2(E - {}^tZZ)^{-1}{}^tZ_2$ is semipositive definite. It has real nonnegative eigenvalues $\lambda_1, \cdots, \lambda_r$. Hence

$$\begin{aligned} &1 + \frac{1}{r}\operatorname{tr}\left(Z_2(E - {}^tZZ)^{-1}{}^tZ_2\right)\\ &\quad = \frac{(1 + \lambda_1) + \cdots + (1 + \lambda_r)}{r} \geqslant \left\{(1 + \lambda_1)\cdots(1 + \lambda_r)\right\}^{1/r}\\ &\quad = \det\left\{E_r + Z_2(E - {}^tZZ)^{-1}{}^tZ_2\right\}^{1/r} = \left(\frac{B}{A}\right)^{1/r}. \end{aligned}$$

1.8. Now consider the trivial vector bundle $E = \mathscr{D} \times M_{qr}(\mathbf{R})$. We introduce an action of G_1 on E suggested by (1.8). For $g \in G_1$ and $(Z, Y) \in E$,

$$(1.10)\qquad g(Z, Y) = \left(gZ, {}^tj^{-1}(g, Z)Y{}^tu\right),$$

where $u \in O(r)$ is given in the block form of g ((1.7)). On E, H_Z defines a fiber metric

$$(1.11) \qquad (Y, Y)_Z = \operatorname{tr}({}^tYH_ZY).$$

We have a canonical smooth section v: $\mathscr{D} \to E$ defined by

$$(1.12) \qquad V(Z) = (Z, {}^tZ_2) \qquad (Z \in \mathscr{D}).$$

By (1.8) and (i) of Lemma 1.6, both the fiber metric and the section v are G_1-invariant; moreover

$$\mathscr{D}_1 = \{Z \in \mathscr{D} | v(Z) = 0\},$$

i.e., D_1 is the zero set of v.

The existence of such a fiber metric and section enables one to construct the dual forms of cycles associated to $\mathscr{D}_1$ in quotients of $\mathscr{D}$. For this purpose, in the following §§1.9–1.13 we sketch a procedure to construct singular forms, a real analogy of the complex transgression formula of Chern-Bott [4], discussed by Flander in [7].

1.9. Let X be a real manifold of dimension n and π: $E \to X$ a vector bundle of fiber dimension m. Suppose that E is endowed with a fiber metric $\langle\ ,\ \rangle$ and a metric connection d. For a local frame field $e = (e_1, \cdots, e_m)$, let $h_{ij} = \langle e_i, e_j \rangle$ $(1 \leqslant i, j \leqslant m)$, and $H = (h_{ij})$. Then there exist square matrices ω and Ω of order m such that

$$(1.13) \qquad de = e\omega, \qquad d^2e = e\Omega.$$

The entries of ω and Ω are 1-forms and 2-forms of X, respectively. We call ω and Ω the connection and curvature matrices, respectively. By (1.13) we have

$$(1.14) \qquad \Omega = d\omega + \omega \wedge \omega.$$

From the definition of metric connection, we have that

$$(1.15) \qquad dH = H\omega + {}^t\omega H.$$

Conversely, (1.15) and a certain transformation law among ω define a metric connection for E.

1.10. Let v: $X \to E$ be a smooth section with zero set X_v. Let $X^\times = X - X_v$. Over $X^\times$, decompose dv and d^2v in the direction of v and the component orthogonal to v. We obtain

$$(1.16) \qquad dv = \theta v + \beta, \qquad d^2v = |v|^2\gamma,$$

where

$$\theta = d \log|v|, \qquad \gamma = \frac{1}{|v|^2}(e_1, \cdots, e_m)(\Omega_{ij})\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}.$$

Here $v_1, \cdots, v_m$ are the components of v with respect to e. Set

$$K = -e\Omega H^{-1}\,{}^te. \tag{1.17}$$

By induction, one derives readily

$$\begin{aligned}(dv)^a &= a\beta^{a-1}\theta v + \beta^a,\\ K &= K_1 + 2v\gamma,\\ K^b &= K_1^b + 2b\,K_1^{b-1}v\gamma.\end{aligned} \tag{1.18}$$

Now let $l = [\frac{m}{2}]$. In the following, we introduce forms s_i and w_i whose constructions depend on the parity of m.

$$\begin{aligned}&\textit{Case } m = 2l\text{: Set}\\ &s_{2k} = v(dv)^{2k-1}K^{l-k}, \qquad 0 < k \leqslant l,\\ &w_{2k} = (dv)^{2k}K^{l-k}, \qquad 0 \leqslant k \leqslant l.\\ &\textit{Case } m = 2l+1\text{: Set}\\ &s_{2k+1} = v(dv)^{2k}K^{l-k},\\ &w_{2k+1} = (dv)^{2k+1}K^{l-k}, \qquad 0 \leqslant k \leqslant l.\end{aligned} \tag{1.19}$$

In both cases, let

$$s'_k = \frac{s_k}{|v|^k}, \qquad w'_k = \frac{w_k}{|v|^k}. \tag{1.20}$$

1.11. Lemma. *Let* $\chi = (\det H)^{-1/2}e_1 \wedge \cdots \wedge e_m$. *We have the conditions*

(i) $dK = 0$,

(ii) $d\chi = 0$.

Proof. $dK = 0$ is the Bianchii identity. Observe that χ is independent of frame field. For orthonormal frame, $H = E$ and $\omega_{ii} = 0$, $1 \leqslant i \leqslant m$, which imply easily the condition $d\chi = 0$.

1.2. From (1.18) and (i) of Lemma 1.11, the following lemma is immediate.

Lemma 1.12. *We have the following conditions*:

(i) $$\frac{ds_{2k} - w_{2k}}{|v|^2} = \frac{(2k-1)}{2(l-k+1)}\left\{w_{2(k-1)} + (2k-2)s_{2(k-1)}\theta\right\}, \qquad m = 2l.$$

(ii) $$\frac{ds_{2k+1} - w_{2k+1}}{|v|^2} = \frac{k}{l-k+1}\left\{w_{2k-1} - (2k-1)s_{2k-1}\theta\right\}, \qquad m = 2l+1.$$

Now set

$$u_k = w'_k + (-1)^k k s'_k\theta.$$

The above lemma yields

$$(1.21)\qquad \begin{aligned} ds'_{2k} &= u_{2k} + \frac{2k-1}{2(l-k+1)}u_{2k-2},\\ ds'_{2k+1} &= u_{2k+1} + \frac{2k}{2(l-k+1)}u_{2k-1}, \qquad 1 \leqslant k \leqslant l. \end{aligned}$$

We also obtain from (1.18) the conditions:

$$(1.22)\qquad u_m = 0.$$

1.13. Let ψ_k be the forms given by $s'_k = c\psi_k\chi$, where $\Gamma(m/2)c = -(m-1)!2\cdot\pi^{m/2}$. Now we define the form ψ by

$$(1.23)\qquad \begin{aligned} \psi &= \psi_{2l} - \frac{(2l-1)}{2}\psi_{2(l-1)} + \frac{(2l-1)(2l-3)}{2\cdot 4}\psi_{2(l-2)}\\ &\quad - \cdots + (-1)^{l-1}\frac{(2l-1)\cdots 3}{2\cdot 4\cdots(2l-2)}\psi_2, \qquad m = 2l,\\ \psi &= \sum_{\lambda=0}^{l}(-1)^\lambda\binom{l}{\lambda}\psi_{2l+1-2\lambda}, \qquad m = 2l+1. \end{aligned}$$

From (1.21), one concludes the following proposition.

Proposition 1.13. *We have the conditions*:

(i) $$(-1)^{l-1}\frac{1\cdot 3\cdots(2l-1)}{\partial/\partial 2\cdot 4\cdots 2l}K^l = cd\psi\chi, \qquad m = 2l,$$

(ii) $$d\psi = 0, \qquad m = 2l+1.$$

1.14. Now we return to our vector bundle $E = \mathscr{D}\times M_{qr}(\mathbf{R})$ discussed in §1.8. Line up the columns in order in a $q\times r$ matrix into a single column. We shall view $M_{qr}(\mathbf{R})$ as $\mathbf{R}^{qr}$. For a square matrix A, let $A^{[r]}$ be the square matrix with a diagonal block form such that all the diagonal matrices are A. By (1.11), the fiber metric is given by the positive definite matrix $H = H_Z^{[r]}$ with respect to the standard frame fields $e = (e_1,\cdots,e_r)$. Here e_i stands for $(e_{1i}, e_{2i},\cdots, e_{qi})$. By a simple computation,

$$(1.24)\qquad dH_Z = H_Z\omega_1 + {}^t\omega_1 H_Z, \qquad \omega_1 = (d\,{}^tZ)Z(E - {}^tZZ)^{-1}.$$

Let $\omega = \omega_1^{[r]}$. One can introduce a metric connection $de = e\omega$. For any $g\in G$, we have the transformation relation

$$(1.25)\qquad g^*\omega_1 = {}^tj^{-1}(g,Z)\omega_1{}^tj(g,Z) + {}^tj^{-1}(g,Z)d\,{}^tj(g,Z).$$

It yields that the connection is G_1-invariant. By (1.14), the curvature matrix is given by

$$\Omega = \Omega_1^{[r]}, \qquad \Omega_1 = -d\,{}^tZ(E - Z\,{}^tZ)^{-1}dZ(E - {}^tZZ)^{-1}.$$

The transformation relation of Ω_1 takes the form

$$g^*\Omega_1 = {}^tj^{-1}(g, Z)\Omega_1{}^tj(g, Z) \qquad (g \in G). \tag{1.26}$$

1.15. For simplicity of notation, let

$$\tau = \begin{pmatrix} e_{11} & \cdots & e_{q1} \\ & \vdots & \\ e_{1r} & \cdots & e_{qr} \end{pmatrix}.$$

In terms of the standard frame fields, we have

$$\begin{aligned} v &= \operatorname{tr}\big(\tau{}^tZ_2\big), \\ dv &= \operatorname{tr}\big(\tau d{}^tZ(E - Z{}^tZ)^{-1}Z{}^tZ_2 + \tau d{}^tZ_2\big), \\ K &= \operatorname{tr}\big(\tau d{}^tZ(E - Z{}^tZ)^{-1}dZ{}^t\tau\big), \\ |v|^2 &= \operatorname{tr}\big(Z_2(E - {}^tZZ)^{-1}{}^tZ_2\big). \end{aligned} \tag{1.27}$$

Now we construct s_k', ψ_k and ψ by formulas (1.19), (1.20) and (1.23).

For a fixed $g \in G$, the function $\det(j(g, Z))$ is of constant sign on $\mathscr{D}$. Denote this sign by

$$\operatorname{sgn}(g) = \operatorname{sign}(\det(j(g, Z))). \tag{1.28}$$

It is easy to see that $\operatorname{sgn}(g) = \pm 1$ and is a character of G. For $g \in G_1$, let $\alpha(g)$ be the function

$$\alpha(g) = \det(u) \qquad (g \in G_1),$$

where u is given in (1.7) of the block form of g.

Proposition 1.15. *The forms ψ_k and ψ satisfy the invariant conditions*

$$\begin{aligned} g^*\psi_k &= \operatorname{sgn}(g)^r\alpha(g)^q\psi_k, \\ g^*\psi &= \operatorname{sgn}(g)^r\alpha(g)^q\psi, \qquad g \in G_1. \end{aligned}$$

Proof. Let $v(\tau, Z)$, $dv(\tau, Z)$, $K(\tau, Z)$ and $\chi(\tau, Z)$ be the functions v, dv, K and χ with dependence on τ and Z. Clearly we have that

$$\begin{aligned} v(\tau, gZ) &= v\big({}^tu\tau{}^tj^{-1}(g, Z), Z\big), \\ \chi(\tau, gZ) &= |\det j(g, z)|^{-r}\chi(\tau, Z). \end{aligned}$$

By (1.25) and (1.26)

$$\begin{aligned} dv(\tau, gZ) &= dv\big({}^tu\tau{}^tj^{-1}(g, Z), Z\big), \\ K(\tau, gZ) &= K\big({}^tu\tau{}^tj^{-1}(g, Z), Z\big). \end{aligned}$$

Since $|v|$ is G_1-invariant, by our formula of s_k' and the above relations

$$\begin{aligned} s_k'(\tau, gZ) &= s_k'\big({}^tu\tau{}^tj^{-1}(g, Z), Z\big) \\ &= \det(u)^q\det(j(g, Z))^{-r}s_k'(\tau, Z); \end{aligned}$$

consequently

$$g^*\psi_k = \operatorname{sgn}(g)^r \alpha(g)^q \psi_k \qquad (g \in G_1). \tag{1.29}$$

From the definition of ψ ((1.23)), ψ is a linear combination of ψ_k. Thus the assertion on ψ is immediate from (1.29).

1.16. Let ϕ be a k-form on $\mathscr{D}$ and $\|\phi\|$ (resp. $\|\phi\|_0$) its pointwise norm induced by the metric (1.5) (resp. the Euclidean metric). It is easy to see that $ds^2 \geqslant \operatorname{tr}(dZd{}^tZ)$; consequently

$$\|\phi\|_0 \geqslant \|\phi\|. \tag{1.30}$$

We know that

$$A/B = \det\left\{E + Z_2(E - {}^tZZ)^{-1}{}^tZ_2\right\}^{-1},$$

and the matrix $Z_2(E - {}^tZZ)^{-1}{}^tZ_2$ has real nonnegative eigenvalues $\lambda_1, \cdots, \lambda_r$. It yields that

$$A/B = \prod_i (1 + \lambda_i)^{-1} \geqslant \prod_i (1 - \lambda_i) \geqslant 1 - \sum_i \lambda_i = 1 - |v|^2,$$

thus

$$C/B \leqslant |v|^2, \qquad C = B - A. \tag{1.31}$$

Moreover the function $(C/B)/|v|^2$ tends to 1 as $|v|$ tends to 0. From (1.30) and (1.31), we have the estimations

$$\|\psi_k\| \prec (B/C)^{(k-1)/2}(B/A)^{r/2+[(k-1)/2]+[rq/2]}$$

which yields easily that

$$\|\psi\| \prec (B/C)^{(rq-1)/2}(B/A)^{r/2+rq-1}. \tag{1.32}$$

Here $\prec$ denotes the inequality $\leqslant$ up to a positive constant factor.

1.17. For a complex number s, let $h_s(t)$ be the function given by

$$h_s(t) = -\int_t^\infty x^{-s}(x - r)^{qr/2-1}dx \qquad (\operatorname{Re}(s) > \tfrac{1}{2}qr). \tag{1.33}$$

Obviously $h_s(t)$ satisfies the conditions:

$$\begin{aligned} &\text{(i)} \quad h_s'(t) = t^{-s}(t - r)^{qr/2-1}, \\ &\text{(ii)} \quad h_s(r) = -r^{qr/2-s}\frac{\Gamma(s - qr/2)\Gamma(qr/2)}{\Gamma(s)}. \end{aligned} \tag{1.34}$$

Definition 1.17. Let ω_s be the differential form defined by

$$\omega_s = \frac{-1}{h_s(r)}d\left(h_s\left(r + |v|^2\right)\psi\right).$$

By (i) of (1.34)

$$\omega_s = \frac{-1}{h_s(r)}\left\{2\left(r+|v|^2\right)^{-s}|v|^{rq-1}d|v| \wedge \psi + h_s\left(r+|v|^2\right)d\psi\right\}.$$

Clearly $|v|^{rq-1}\psi_k \wedge d|v|$ $(k \leqslant rq-1)$ are smooth and by (1.22), $|v|^{rq-1}d|v| \wedge \psi$ is also smooth. By Proposition 1.13, $d\psi$ is G-invariant. It follows that ω_s is smooth. By Lemma 1.7 and (1.32),

$$\|\omega_s\| \prec (B/A)^{r/2+rq-(\mathrm{Re}(s)-qr/2)}. \tag{1.35}$$

By Proposition 1.15, the form ω_s is invariant under the identity component G_1^0 of G_1. Let Γ_1 be a tension free discrete subsgroup of G_1^0 such that $\mathrm{vol}(\Gamma_1 \backslash \mathscr{D}_1) < \infty$. The form ω_s for $\mathrm{Re}(s) \gg 0$ can be viewed as a dual form of $\Gamma_1 \backslash \mathscr{D}_1$ in the quotient $\Gamma_1 \backslash \mathscr{D}$. In the sequel, we shall clarify its geometric implication. First, we present a decomposition of the volume form on $\mathscr{D}$ and some integral formulas.

1.18. For $g \in G$ and $Z \in \mathscr{D}$,

$$d(gZ) = \tau(g,Z)dZj(g,Z)^{-1}. \tag{1.36}$$

We know that

$$\det(\tau(g,Z)) = \det(j(g,Z))^{-1}. \tag{1.37}$$

Now let

$$\{dZ\} = \prod_{i=1}^{p}\prod_{j=1}^{q} dZ_{ij}.$$

It follows that (1.36) and (1.37) yield

$$\{dgZ\} = \det(j(g,Z))^{-(p+q)}\{dZ\}.$$

Recall $A(Z) = \det(E - {}^tZZ)$. By Lemma 1.6,

$$A(gZ) = \det(j(g,Z))^{-2}A(Z).$$

Hence the invariant volume element $dv_{\mathscr{D}}$ on $\mathscr{D}$ is given by

$$dv_{\mathscr{D}} = A^{-(p+q)/2}\{dZ\}. \tag{1.38}$$

1.19. For $\binom{Z_1}{0} \in \mathscr{D}_1$, let F_{Z_1} be its fiber

$$F_{Z_1} = \{Z \in \mathscr{D} | Z_1 \text{ fixed}\}.$$

Let $g \in G_1$ be the element

$$g = \begin{pmatrix} (E - Z_1{}^tZ_1)^{-1/2} & 0 & -(E - Z_1{}^tZ_1)^{-1/2}Z_1 \\ 0 & E_r & 0 \\ -(E - {}^tZ_1Z_1)^{-1/2}\,{}^tZ_1 & 0 & (E - {}^tZ_1Z_1)^{-1/2} \end{pmatrix}.$$

Then g carries F_{Z_1} isometrically onto F_0, and

$$g\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ Z_2(E - {}^tZ_1Z_1)^{-1/2} \end{pmatrix}.$$

On F_0, the volume element is $\det(E - {}^tZ_2Z_2)^{-(r+q)/2}\{dZ_2\}$, so F_{Z_1} has the volume element

$$dv_F = A^{-r/2}(B/A)^{q/2}\{dZ_2\}.$$

It follows that

$$dv_{\mathscr{D}} = (B/A)^{(p-r)/2}\, dv_{\mathscr{D}_1}\, dv_F, \tag{1.39}$$

where $dv_{\mathscr{D}_1} = B^{-(p+q-r)/2}\{dZ_1\}$ is the invariant volume element of $\mathscr{D}_1$.

1.20. Lemma. *We have the integration formulas*:

(i)

$$\int_{\mathscr{D}} A^{s/2}\{dZ\} = \frac{\prod_{i=1}^{p}\Gamma((2+1+i)/2)\prod_{j=1}^{q}\Gamma((s+1+j)/2)}{\prod_{\lambda=1}^{p+q}\Gamma((s+1+\lambda)/2)}\pi^{pq/2} \qquad (\operatorname{Re}(s) > -2).$$

(ii)

$$\int_{\Gamma_1\setminus\mathscr{D}} (A/B)^{s/2}\, dv_{\mathscr{D}} = \frac{\prod_{i=1}^{r}\Gamma((s-p-q+1+i)/2)\prod_{j=1}^{q}\Gamma((s-p-q+1+j)/2)}{\prod_{\lambda=1}^{r+q}\Gamma((s-p-q+1+\lambda)/2)}\pi^{rq/2} \cdot \operatorname{vol}(\Gamma_1\setminus\mathscr{D}_1) \qquad (\operatorname{Re}(s) > p+q-2).$$

Proof. (i) Introduce $f(s,p,q) = \int_{\mathscr{D}} A^{s/2}\{dZ\}$. By (1.39) with $r = 1$, one deduces the recursion relation

$$f(s,p,q) = f(s+1,p-1,q)f(s,1,q).$$

Now we have that

$$\begin{aligned} f(s,1,q) &= \int_{\sum_i x_i^2 \leq 1} \left(1 - \left(x_1^2 + \cdots + x_q^2\right)\right)^{s/2} dx_1\, dx_2 \cdots dx_q \\ &= \frac{\pi^{q/2}}{\Gamma(q/2)}\int_0^1 (1-t)^{s/2} t^{q/2-1}\, dt \\ &= \frac{\Gamma(s/2+1)}{\Gamma(s/2+q/2+1)}\pi^{q/2} \qquad (\operatorname{Re}(s) > -2). \end{aligned}$$

Then (i) follows by simple induction.

(ii) By (1.39), the integral has the value

$$\int_{\Gamma_1\setminus\mathscr{D}} (A/B)^{(s+r-p)/2}\, dv_F\, dv_{\mathscr{D}_1}.$$

As A/B, dv_F are G_1^0-invariant, the integral

$$\int_{F_{Z_1}} (A/B)^{(s+r-p)/2}\, dv_F$$

is independent of Z_1. At $Z_1 = 0$, its value is

$$\int_{\mathscr{D}_2} \det(E - {}^tZ_2Z_2)^{(s-p-q)/2}\{dZ_2\},$$

where $\mathscr{D}_2 = \{Z_2 \mid {}^tZ_2Z_2 < E_q\}$. Hence by (i), we obtain the desired formula.

1.21. Now we are ready to present the geometric meaning of the form ω_s. Let ϕ be any smooth $((p-r)q)$-form of $\Gamma_1 \backslash \mathscr{D}$. Assume that

$$\|\phi\| \prec (B/A)^N \tag{1.40}$$

for a certain integer N. By (1.35) and (ii) of Lemma 1.20, the integral $\int_{\Gamma_1\backslash\mathscr{D}} \phi \wedge \omega_s$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$.

Theorem 1.21. *Let ϕ be a smooth closed $((p-r)q)$-form of $\Gamma_1 \backslash \mathscr{D}$ satisfying condition* (1.40). *Then*

$$\int_{\Gamma_1\backslash\mathscr{D}} \omega_s \wedge \phi = \int_{\Gamma_1\backslash\mathscr{D}_1} \phi \qquad (\mathrm{Re}(s) \gg 0).$$

Proof. Let $M = \Gamma_1 \backslash \mathscr{D}_1$. Then M is a complete Riemannian manifold. Let x_0 be a fixed point of M. For $t > 0$, let B_t be the closed ball with center x_0 and radius t. Let ∂B_t be the boundary and $\mathrm{vol}(\partial B_t)$ its volume with respect to the induced metric. Clearly we have that

$$\int_0^\infty \mathrm{vol}(\partial B_t)\, dt = \mathrm{vol}(M) < \infty.$$

This readily implies that

$$\lim_{t\to\infty} \mathrm{vol}(\partial B_t) = 0. \tag{1.41}$$

For t, $\varepsilon > 0$ and $l \gg 0$, let $N(t, \varepsilon, l)$ be the subset of $\Gamma_1 \backslash \mathscr{D}$ consisting of all $Z = \binom{Z_1}{Z_2}$ such that

(i) $\binom{Z_1}{0} \in B_t$,

(ii) $|v(Z)| \geqslant \varepsilon$,

(iii) the distance, $d(Z, \mathscr{D}_1)$ from Z to $\mathscr{D}_1$ is less than or equal to l.

The absolute convergence condition yields that

$$\int_{\Gamma_1\backslash\mathscr{D}} \omega_s \wedge \phi = \lim_{t\to\infty} \lim_{\substack{\varepsilon\to 0\\ l\to\infty}} \int_{N(t,\varepsilon,l)} \omega_s \wedge \phi. \tag{1.42}$$

It is obvious that

$$\partial N(t, \varepsilon, l) = \mathscr{F}_\varepsilon \cup \mathscr{F}_l \cup \mathscr{F}_\partial,$$

where

$$\mathcal{F}_\varepsilon = \left\{ Z \in N(t,\varepsilon,l) \,|\, |v(Z)| = \varepsilon \right\},$$
$$\mathcal{F}_l = \left\{ Z \in N(t,\varepsilon,l) \,|\, d(Z,\mathcal{D}_1) = l \right\},$$
$$\mathcal{F}_\partial = \left\{ Z \in N(t,\varepsilon,l) \,\Big|\, \begin{pmatrix} Z_1 \\ 0 \end{pmatrix} \in \partial B_t \right\}.$$

Now observe that

$$\omega_s \wedge \phi = d\left(\frac{-h_s(r+|v|^2)}{h_s(r)} \psi \wedge \phi \right).$$

By Stoke's theorem

$$\int_{N(t,\varepsilon,l)} \omega_s \wedge \phi = -\int_{\mathcal{F}_l} \frac{h_s(r+|v|^2)}{h_s(r)} \psi \wedge \phi - \int_{\mathcal{E}_\varepsilon} \frac{h_s(r+|v|^2)}{h_s(r)} \psi \wedge \phi$$
$$- \int_{\mathcal{F}_\partial} \frac{h_s(r|v|^2)}{h_s(r)} \psi \wedge \phi.$$

Let I_l, I_ε and I_∂ be the integrals on the right-hand side of the above equality over $\mathcal{F}_l$, $\mathcal{F}_\varepsilon$, $\mathcal{F}_\partial$ with proper orientations respectively. By (1.32), we may assume that

$$\left\| h_s(r+|v|^2) \psi \wedge \phi \right\| \prec (B/C)^{(rq-1)/2} (A/B)^a$$

with $a \gg 0$. By an argument similar to [33, Proposition 1.10], there exists a constant $b > 0$ such that $\mathrm{vol}(\mathcal{F}_l) \prec e^{bl} \mathrm{vol}(B_t)$. By (i) of (1.9), $A/B \prec e^{-2d(Z,\mathcal{D}_1)}$. Hence we obtain that

$$I_l \prec \mathrm{vol}(B_t) e^{(b-2a)l}.$$

As $a \gg 0$, it yields that

(1.43) $$\lim_{l \to \infty} I_l = 0.$$

The same reasoning as in [33, Proposition 2.5] implies

(1.44) $$\lim_{\varepsilon \to 0} I_\varepsilon = -\int_{B_t} \phi.$$

To estimate $|I_\partial|$, we first integrate over the fiber. Let η be the volume form of ∂B_t. By (1.39), $\|\eta \wedge dv_F\| \succ (A/B)^c$ for a certain constant c. Since $\mathrm{Re}(s) \gg 0$, by (1.32) and (i) of Lemma 1.20 over the fiber,

(1.45) $$|I_\partial| \prec \mathrm{vol}(\partial B_t).$$

Now (1.41)–(1.45) easily yield

$$\int_{\Gamma_1\backslash\mathscr{D}} \omega_s \wedge \phi = \lim_{t\to\infty}\int_{B_t}\phi = \int_{\Gamma_1\backslash\mathscr{D}_1}\phi.$$

Remark. Proposition 1.13 is the main result in [7]. If X is compact and m is even, one can construct a dual form of the zero set X_v of the section v from $K^{m/2}$. For our case, X is noncompact and m is not necessarily even. To overcome such difficulties, we replace ψ by $h_s\psi$, where h_s is a certain function on $\mathscr{D}$ parametrized by a complex number s. The differential $\omega_s = d(h_s\psi)$ by Theorem 1.21 clearly exhibits the property of a dual form.

2. Polynomials constructed from dual forms

In this section, we define now polynomials with differential forms on $\mathscr{D}$ as values which will be used in a later section to construct geometric theta functions. For simplicity of formulas, we pull back the construction to a proper vector space with a linear action of G.

2.1. Let $n = p + q$, $V = M_{n1}(\mathbf{R})$ and $W = M_{nq}(\mathbf{R})$. For $X, Y \in W$ or V, define

$$\langle X, Y\rangle = {}^tXQY, \tag{2.1}$$

where Q is the matrix

$$Q = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}.$$

Definition 2.1. Let W_- be the subset of W consisting of X such that $\langle X, X\rangle < 0$.

For $X \in W$, denote

$$X = \begin{pmatrix} X_+ \\ X_- \end{pmatrix} \tag{2.2}$$

with $X_+ \in M_{pq}(\mathbf{R})$ and $X_- \in M_{qq}(\mathbf{R})$. If $X \in W_-$, the condition $\langle X, X\rangle < 0$ implies readily that X_- is invertible. Hence we can define a map π: $W_- \to \mathscr{D}$ given by

$$\pi(X) = X_+X_-^{-1} \qquad (X \in W_-). \tag{2.3}$$

2.2. Let $G = O(p, q)$. Since $G \subset \mathrm{GL}(n, \mathbf{R})$, G acts on W from the left by matrix multiplication. It is easy to see that W_- is G-invariant and the map π is G-equivariant. For $X \in W$, let $X^\perp$ be the set

$$X^\perp = \{\, Y \in W | \langle X, Y\rangle = 0\}.$$

Lemma 2.2. *For $X \in W_-$, $X^\perp$ is a subspace of the tangent space $T_X(W_-)$ at X satisfying the following conditions*:

(i) $g_* X^\perp = (gX)^\perp$, $g \in G$.

(ii) π_* *identifies* $X^\perp$ *with the tangent space* $T_{\pi(X)}(\mathscr{D})$ *of* $\mathscr{D}$ *at* $\pi(X)$.

Proof. Since $\langle\ ,\ \rangle$ is G-invariant, (i) is obvious. Observe that π is G-equivariant and G acts transitively on $\mathscr{D}$. It suffices to verify (ii) with $\pi(X) = 0$. In this case $X_+ = 0$ and as a consequence

$$X^\perp = \left\{ \begin{pmatrix} Y \\ 0 \end{pmatrix} \middle| Y \in M_{pq}(\mathbf{R}) \right\}, \qquad \pi_*\begin{pmatrix} Y \\ 0 \end{pmatrix} = YX_-^{-1}.$$

One concludes easily that $\pi_* | X^\perp$ is an isomorphism onto $T_0(\mathscr{D})$.

2.3. Let r be a positive integer with $r \leqslant p/2$. For $M \in M_{nr}(\mathbf{R})$, denote by $\langle M \rangle$ the subspace of V spanned by the columns of M and by $G_{\langle M \rangle}$ the subgroup of G leaving the subspace $\langle M \rangle$ invariant. Now we assume that $\langle M, M \rangle > 0$.

Here we consider the trivial bundle

$$\mathscr{D} \times M_{q1}(\mathbf{R}) \otimes \langle M \rangle$$

with a group action of $G_{\langle M \rangle}$. For $g \in G_{\langle M \rangle}$, $Y \in M_{q1}(\mathbf{R})$ and $X \in \langle M \rangle$, let

$$g(Z, Y \otimes X) = \left(gZ, {}^t j^{-1}(g, Z) Y \otimes gX\right). \tag{2.4}$$

Let $e_1, \cdots, e_n$ be the standard basis of V and $G_1 = G_{\langle e_{p-r+1}, \cdots, e_p \rangle}$. Identify $M_{q1}(\mathbf{R}) \otimes \langle e_{p-r+1}, \cdots, e_p \rangle$ with $M_{qr}(\mathbf{R})$ in the obvious manner. Then (2.4) coincides with (1.10). More generally for $h \in G$, (2.4) yields a map

$$h\colon \mathscr{D} \times \left(M_{q1}(\mathbf{R}) \otimes \langle M \rangle\right) \to \mathscr{D} \times \left(M_{q1}(\mathbf{R}) \otimes \langle hM \rangle\right).$$

By a simple diagram chasing, we have the following lemma.

Lemma 2.3. *The following diagram is commutative for* $g \in G_{\langle M \rangle}$:

$$\begin{array}{ccc} \mathscr{D} \times \left(M_{q1}(\mathbf{R}) \otimes \langle M \rangle\right) & \xrightarrow{\ h\ } & \mathscr{D} \times \left(M_{q1}(\mathbf{R}) \otimes \langle hM \rangle\right) \\ \downarrow g & & \downarrow hgh^{-1} \\ \mathscr{D} \times \left(M_{q1}(\mathbf{R}) \otimes \langle M \rangle\right) & \xrightarrow{\ h\ } & \mathscr{D} \times \left(M_{q1}(\mathbf{R}) \times \langle hM \rangle\right) \end{array}$$

2.4. As $\langle M, M \rangle > 0$, by Definition 1.3 there is a totally geodesic subdomain $\mathscr{D}_{\langle M \rangle}$ corresponding to the positive space $\langle M \rangle$. Let $\mathscr{D}_1 = \mathscr{D}_{\langle e_{p-r+1}, \cdots, e_p \rangle}$. Choose an element $g \in G$ such that

$$g\left\langle e_{p-r+1}, \cdots, e_p \right\rangle = \langle M \rangle. \tag{2.5}$$

We would like to identify $M_{q1}(\mathbf{R}) \otimes \langle M \rangle$ with $M_{qr}(\mathbf{R})$ and transfer the discussion in §1 for $\mathscr{D}_1$ to the general case. Let $M_1, \cdots, M_r$ be the columns of M and $M_1^*, \cdots, M_r^*$ in $\langle M \rangle$ given by

$$\left\langle M_i, M_j^* \right\rangle = \delta_{ij} \qquad (i, j = 1, \cdots, r).$$

Here we introduce the identification

$$\sum_{i=1}^{r} Y_i \otimes M_i^* = (Y_1 \cdots Y_r). \tag{2.6}$$

Let N be the matrix $N = M\langle M, M\rangle^{-1/2}$. Here $\langle M, M\rangle^{1/2}$ is the unique positive definite matrix whose square is $\langle M, M\rangle$. Clearly N satisfies the condition $\langle N, N\rangle = E_r$. It follows that by using a product of g and some element in $G_{\langle M\rangle}$, we may assume that

$$g^{-1}N = (e_{p-r+1} \cdots e_p).$$

Let $M^* = (M_1^* \cdots M_r^*)$. Then $M^* = N\langle M, M\rangle^{-1/2}$, and

$$\begin{aligned} g^{-1}(Z, Y) &= g^{-1}(Z, Y{}^tM^*) \\ &= \left(g^{-1}Z, {}^tj(g^{-1}, Z)^{-1}Y{}^t(g^{-1}M^*)\right) \\ &= \left(g^{-1}Z, {}^tj(g^{-1}, Z)^{-1}Y\langle M, M\rangle^{-1/2\,t}(g^{-1}N)\right) \\ &= \left(g^{-1}Z, {}^tj(g^{-1}, Z)^{-1}Y\langle M, M\rangle^{-1/2}\right). \end{aligned} \tag{2.7}$$

Recall the fiber metric introduced in (1.11) is given by $\langle Y, Y\rangle_Z = \mathrm{tr}({}^tYH_ZY)$.

The following lemma is immediate from (2.7) and the above metric formula.

Lemma 2.4. *Let $g \in G$ be an element satisfying the condition*

$$g^{-1}M\langle M, M\rangle^{-1/2} = (e_{p-r+1} \cdots e_p).$$

Then the pull back $(g^{-1})^*\langle\ ,\ \rangle_Z$ of $\langle\ ,\ \rangle_Z$ is given by the symmetric matrix

$$(E - {}^tZZ)^{-1} \otimes \langle M, M\rangle^{-1}.$$

2.5. For $X \in W_-$ and $M \in M_{nr}(\mathbf{R})$, let $M_{X^\perp}$ be the component of M which is orthogonal to X with respect to $\langle\ ,\ \rangle$ given in (2.1). In terms of matrix product, we have

$$M = X\langle X, X\rangle^{-1}\langle X, M\rangle + M_{X^\perp}. \tag{2.8}$$

In the sequel, we pull back data on $\mathscr{D}$ to W_- through the projection map π.

Lemma 2.5. $\pi^*(E - {}^tZZ)^{-1} = -X_-\langle X, X\rangle^{-1\,t}X_-$.

Proof. We have that

$$\begin{aligned} E - {}^t\pi(X)\pi(X) &= -{}^t\begin{pmatrix} \pi(X) \\ E \end{pmatrix} Q \begin{pmatrix} \pi(X) \\ E \end{pmatrix} \\ &= -{}^tX_-^{-1\,t}XQXX_-^{-1}; \end{aligned}$$

thus

$$\pi^*(E - {}^tZZ)^{-1} = -X_-\langle X, X\rangle^{-1\,t}X_-.$$

2.6. For $M \in M_{nr}(\mathbf{R})$ with $\langle M, M\rangle > 0$, let $g \in G$ with

$$g^{-1}M\langle M, M\rangle^{-1/2} = (e_{p-r+1} \cdots e_p).$$

Denote by $(B/A)_{\langle M\rangle}$ the function given by

$$(B/A)_{\langle M\rangle} = (g^{-1})^* B/A, \tag{2.9}$$

where B/A is given in Lemma 1.6.

Lemma 2.6. *For $X \in W_-$, we have*

$$\pi^*((B/A)_{\langle M\rangle})(X) = \det\langle M_{X^\perp}, M_{X^\perp}\rangle/\det\langle M, M\rangle.$$

Proof. As in the proof of Lemma 1.7, we have

$$B/A = \det\left\{E_r + Z_2(E - {}^tZZ)^{-1\,t}Z_2\right\}.$$

Let $e' = (e_{p-r+1} \cdots e_p)$. We see that

$$\begin{aligned} &E_r + Z_2(E - {}^tZZ)^{-1\,t}Z_2 \\ &\quad = E_r + \left\langle e', \begin{pmatrix} Z \\ E \end{pmatrix}\right\rangle (E - {}^tZZ)^{-1\,t}\left\langle e', \begin{pmatrix} Z \\ E \end{pmatrix}\right\rangle, \end{aligned}$$

$$\begin{aligned} \left\langle e', g^{-1}\begin{pmatrix} Z \\ E \end{pmatrix}\right\rangle &= \langle ge', X\rangle X_-^{-1} = \left\langle M\langle M, M\rangle^{-1/2}, X\right\rangle X_-^{-1} \\ &= \langle M, M\rangle^{-1/2}\langle M, X\rangle X_-^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} &\pi^*((B/A_{\langle M\rangle}))(X) \\ &\quad = \det\left\{E_r - \langle M, M\rangle^{-1/2}\langle M, X\rangle\langle X, X\rangle^{-1}\langle X, M\rangle\langle M, M\rangle^{-1/2}\right\} \\ &\quad = \det\langle M, M\rangle^{-1}\det\left\{\langle M, M\rangle - \langle M, X\rangle\langle X, X\rangle^{-1}\langle X, M\rangle\right\} \\ &\quad = \det\langle M_{X^\perp}, X_{X^\perp}\rangle/\det\langle M, M\rangle. \end{aligned}$$

2.7. For simplicity of notation, in the following discussion, we shall often omit the notation π^* in the pull back formulas. For $M \in M_{nr}(\mathbf{R})$ with $\langle M, M\rangle > 0$ and $g \in G$ with

$$g^{-1}M\langle M, M\rangle^{-1/2} = e' = (e_{p-r+1} \cdots e_p),$$

$(g^{-1})^*\langle\ ,\ \rangle_Z$ yields the fiber metric

$$H_X = -X_-\langle X, X\rangle^{-1\,t}X_- \otimes \langle M, M\rangle^{-1}. \tag{2.10}$$

Lemma 2.7. *Let v be the smooth section given in §1,8 and $v_M = (g^{-1})^* v$. Then v_M has the expression*

$$v_M(X) = \left(X, {}^tX_-^{-1}\langle X, M\rangle\right)$$

in $W_- \times M_{qr}(\mathbf{R})$.

Proof. For $Z \in \mathscr{D}$, we have that

$$v(g^{-1}Z) = \left(g^{-1}z, \left\langle \begin{pmatrix} g^{-1}Z \\ E \end{pmatrix}, e' \right\rangle \right)$$
$$= \left(g^{-1}Z, \left\langle g^{-1}\begin{pmatrix} Z \\ E \end{pmatrix} j(g^{-1}, Z)^{-1}, e' \right\rangle \right)$$
$$= \left(g^{-1}Z, {}^t j(g^{-1}, Z)^{-1} \left\langle \begin{pmatrix} Z \\ E \end{pmatrix}, M \right\rangle \langle M, M \rangle^{-1/2} \right)$$

and by the identification (2.7)

$$v_M(Z) = \left(Z, \left\langle \begin{pmatrix} Z \\ E \end{pmatrix}, M \right\rangle \right).$$

Pulling back the result to W_-, the desired assertion follows.

2.8. From the fiber metric (2.10), one can introduce a metric connection. Note that $dH_X = H_X\omega + {}^t\omega H_X$, where ω is given by

$$\omega = \{ {}^tX_-^{-1} d\, {}^tX_- - {}^tX_-^{-1} \langle dX, X \rangle \langle X, X \rangle^{-1\,t} X_- \} \otimes I. \tag{2.11}$$

By (1.14), the curvature matrix is given by

$$\Omega = {}^tX_-^{-1} [\langle dX, dX \rangle - \langle dX, X \rangle \langle X, X \rangle^{-1} \langle X, dX \rangle] \langle X, X \rangle^{-1\,t} X_- \otimes I. \tag{2.12}$$

In the following, let us adopt the notation

$$\langle a, b \rangle_\perp = \langle a_{X^\perp}, b_{X^\perp} \rangle. \tag{2.13}$$

Then Ω has the simpler expression

$$\omega = {}^tX_-^{-1} \langle dX, dX \rangle_\perp \langle X, X \rangle^{-1\,t} X_- \otimes I.$$

Now let $e = (e_1, \cdots, e_r)$ be the standard frame fields where

$$e_i = (e_{1i}, \cdots, e_{qi}) \qquad (i = 1, \cdots, r).$$

With respect to the standard frame fields, we have

$$v = \sum_{1 \leqslant i \leqslant r} e_i {}^tX_-^{-1} \langle X, M_i \rangle, \qquad K = -e\Omega H^{-1\,t}e.$$

Here M_i is the ith column of the $n \times r$ matrix M.

Lemma 2.8. *We have the following conditions*:

(i) $dv = \sum_{1 \leqslant i \leqslant r} e_i {}^tX_-^{-1} \langle dX, M_i \rangle_\perp$.

(ii) $K = \sum_{1 \leqslant i, j \leqslant r} e_i {}^tX_-^{-1} \langle M_i, M_j \rangle \langle dX, dX \rangle_\perp X_-^{-1\,t} e_j$.

Proof. The assertions follow by a straightforward computation using our expressions for ω ((2.11)) and Ω ((2.12)).

2.9. Let c, h_X and χ be given by

$$\Gamma(rq/2)c = -(rq-1)!2 \cdot \pi^{rq/2},$$
$$h_X = \det H_X = (\det - \langle X, X \rangle)^{-r} (\det X_-)^{2r} \det \langle M, M \rangle^{-q}, \tag{2.14}$$
$$\chi = h_X^{-1/2} e_{11} \cdots e_{11} \cdots e_{1r} \cdots e_{qr}.$$

Here we consider the expansion of the rq-form

$$\det\langle M, M\rangle^{q/2}(dv)^{qr}/c\chi.$$

From (i) of Lemma 2.8 and (2.14), it follows that

$$\begin{aligned}
&\det\langle M, M\rangle^{q/2}(dv)^{qr}/c\chi \\
&\quad = -\frac{rq}{2}\Gamma\left(\frac{rq}{2}\right)\pi^{-qr/2}s(X_-)^r(\det -\langle X, X\rangle)^{-r/2}\prod_{i=1}^{r}\prod_{t=1}^{q}\langle dX_t, M_i\rangle_\perp \\
(2.15)\quad &\quad = (-1)^{1+qr(r-1)(q-1)/4}\left(\frac{rq}{2}\right)\Gamma\left(\frac{rq}{2}\right)\pi^{-qr/2}s(X_-)^r(\det -\langle X, X\rangle)^{-r/2} \\
&\qquad \cdot\prod_{t=1}^{q}\prod_{i=1}^{r}\langle dX_t, M_i\rangle_\perp,
\end{aligned}$$

where $s(X_-) = \mathrm{sign}(\det(X_-))$.

Definition. Let $f(X, M)$ be the polynomial of M given by

$$\begin{aligned}
f(X, M) = {}&(-1)^{1+qr(r-1)(q-1)/4}\frac{rq}{2}\Gamma\left(\frac{rq}{2}\right)\pi^{-qr/2}(\det -\langle X, X\rangle)^{-r/2} \\
&\cdot s(X_-)^r\prod_{t=1}^{q}\prod_{i=1}^{r}\langle dX_t, M_i\rangle_\perp,
\end{aligned}$$

and denote by $f(Z, M)$ the differential form on $\mathscr{D}$ whose pull back to W_- is $f(X, M)$.

Remark. The polynomial $f(Z, M)$, except $q = 1$, in general is not suitable yet to combine with an exponential function to form a theta function. To obtain the right polynomial, we will notify $f(Z, M)$ by representation theory of $O(p)$.

2.10. We can define $\omega_s(M)$ for $\mathscr{D}_{\langle M\rangle}$ as in §1.17. It is clear from our construction that

$$g^*\omega_s(M) = \omega_s(g^{-1}M).$$

By translation, Theorem 1.21 is also applicable for $\omega_s(M)$ with a proper orientation s_M on $\mathscr{D}_{\langle M\rangle}$. Let

$$Y_0 = \begin{pmatrix} 0 \\ E_r \\ 0 \end{pmatrix}$$

with the $q \times q$ zero matrix at the bottom. The orientations s of $\mathscr{D}$ and s_{Y_0} of $\mathscr{D}_1$ used in Theorem 1.21 are determined by

$$\bigwedge_{i=1}^{p}\bigwedge_{j=1}^{q} dz_{ij}, \quad (-1)^{r(p-r)q^2}\bigwedge_{i=1}^{p-r}\bigwedge_{j=1}^{q} dz_{ij},$$

respectively. Choose $g \in G$ such that

$$gM\langle M, M\rangle^{-1/2} = Y_0, \qquad \det(j(g,z)) > 0 \quad (z \in \mathscr{D}).$$

Since $\omega_s(M\alpha) = \operatorname{sign}(\det \alpha)^q \omega_s(M)$ and $g^* s = \operatorname{sign}(\det(g))^q s$,

$$s_M = \operatorname{sign}(\det(g))^q g^* s_{Y_0}.$$

In the sequel, we use $\mathscr{D}_M$ for $\mathscr{D}_{\langle M\rangle}$ with the orientation s_M.

3. Some invariant theorems

In this section, we discuss some representation and invariant theorems for $O(p)$ needed for our later investigation on geometric theta functions.

3.1. The Casimir operator of $O(p,q)$. Let $G = O(p,q)$ and let $L(G)$ be its Lie algebra. For $X \in M_n(\mathbf{R})$, $n = p + q$, set

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

with $X_{11} \in M_{pp}(\mathbf{R})$, $X_{12} \in M_{pq}(\mathbf{R})$, $X_{21} \in M_{qp}(\mathbf{R})$ and $X_{22} \in M_{qq}(\mathbf{R})$. Then $L(G)$ is given by

$$L(G) = \left\{X \in M_n(\mathbf{R}) \,\middle|\, {}^tX_{11} + X_{11} = 0, {}^tX_{22} + X_{22} = 0, X_{21} = {}^tX_{12}\right\}.$$

Denote by E_{ab} the $n \times n$ matrix with entry 1 at the (a,b)th position and zero elsewhere. One checks easily that $L(G)$ has a basis consisting of the following elements:

$$\begin{aligned} A_{ij} &= E_{ij} + E_{ji} \qquad (i < j), \\ B_{st} &= E_{st} - E_{ts} \qquad (s < t), \\ C_{is} &= E_{is} + E_{si}. \end{aligned}$$

Here we adopt the convention on indexes

$$1 \leqslant i, j \leqslant p, \qquad p < s, t \leqslant n.$$

With respect to the invariant form $B(X,Y) = \operatorname{tr}(X,Y)$, the Casimir operator $C = C(p,q)$ is presented by

$$(3.1) \qquad C = \frac{1}{2}\left\{-\sum_{i<j} A_{ij}A_{ij} - \sum_{s<t} B_{st}B_{st} + \sum_{i,s} C_{is}C_{is}\right\}.$$

3.2. Let $V = \mathbf{R}^n$ and let V^r be identified with $M_{nr}(\mathbf{R})$. Denote by r the representation of G on the smooth functions of V^r given by

$$r(g)f(X) = f(g^{-1}X) \qquad (g \in G,\ X \in V^r).$$

Now introduce the differential operators

$$I_{ab} = \sum_{1 \leqslant \lambda \leqslant r} X_{a\lambda}\frac{\partial}{\partial X_{b\lambda}}, \qquad L_{\lambda\nu} = \sum_{1 \leqslant a \leqslant n} X_{a\lambda}\frac{\partial}{\partial X_{a\nu}}.$$

Then we have

$$r(A_{ij}) = I_{ij} - I_{ji}, \quad r(B_{st}) = I_{st} - I_{ts}, \quad r(C_{is}) = -(I_{is} + I_{si}).$$

It follows that

$$\sum_{i<j} r(A_{ij})r(A_{ij}) = \sum_{\lambda,\nu}\sum_{i,j}\left(X_{i\lambda}X_{i\nu}\frac{\partial^2}{\partial X_{j\lambda}\partial X_{j\nu}} - X_{i\lambda}X_{j\nu}\frac{\partial^2}{\partial X_{j\lambda}\partial X_{i\nu}} \right)$$
$$-(p-1)\sum_{i,\lambda} X_{i\lambda}\frac{\partial}{\partial X_{i\lambda}},$$

$$\sum_{s<t} r(B_{st})r(B_{st}) = \sum_{\lambda,\nu}\sum_{s,t}\left(X_{s\lambda}X_{s\nu}\frac{\partial^2}{\partial X_{t\lambda}\partial X_{t\nu}} - X_{s\lambda}X_{t\nu}\frac{\partial^2}{\partial X_{t\lambda}\partial X_{s\nu}} \right)$$
$$-(q-1)\sum_{s,\lambda} X_{s\lambda}\frac{\partial}{\partial X_{s\lambda}},$$

$$\sum_{i,s} r(C_{is})r(C_{is}) = \sum_{\lambda,\nu}\sum_{i,s}\left(X_{i\lambda}X_{i\nu}\frac{\partial^2}{\partial X_{s\lambda}\partial X_{s\nu}} + X_{s\lambda}X_{s\nu}\frac{\partial^2}{\partial X_{i\lambda}\partial X_{i\nu}} \right)$$
$$+\sum_{\lambda,\nu}\sum_{i,s}\left(X_{i\lambda}X_{s\nu}\frac{\partial^2}{\partial X_{s\lambda}\partial X_{i\nu}} + X_{s\lambda}X_{i\nu}\frac{\partial^2}{X_{i\lambda}\partial X_{s\nu}} \right).$$

A straightforward computation yields the following expression for $r(C(p,q))$.

Lemma 3.2. *The image of $C(p,q)$ under r is given by*

$$r(C(p,q)) = \frac{1}{2}\left\{ \sum_{1\leqslant\lambda,\nu\leqslant r} L_{\lambda\nu}L_{\nu\lambda} + (n-r-1)\sum_{1\leqslant\lambda\leqslant r} L_{\lambda\lambda} \right.$$
$$\left. - \sum_{1\leqslant\lambda,\nu\leqslant r} \langle X_\lambda, X_\nu\rangle \left\langle \frac{\partial}{\partial X_\lambda}, \frac{\partial}{\partial X_\nu} \right\rangle \right\},$$

where X_λ is the λth column of X and

$$\left\langle \frac{\partial}{\partial X_\lambda}, \frac{\partial}{\partial X_\nu} \right\rangle = \sum_{1\leqslant i\leqslant p} \frac{\partial^2}{\partial X_{i\lambda}\partial X_{i\nu}} - \sum_{p<s\leqslant n} \frac{\partial^2}{\partial X_{s\lambda}\partial X_{s\nu}}.$$

3.3. Let $O(n)$ be the real orthogonal group. To discuss the representation theory of $O(n)$, we consider the quadratic form

$$(3.2) \qquad Q(x) = x_1x_n + x_2x_{n-1} + \cdots + x_nx_1.$$

Set $l = [n/2]$ and J the $l \times l$ matrix

$$J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$$

Let A be the matrix

$$(3.3)\qquad A = \begin{cases} \begin{pmatrix} \frac{1}{\sqrt{2}}E & 0 & \frac{i}{\sqrt{2}}J \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}}E & 0 & \frac{-i}{\sqrt{2}}J \end{pmatrix} & (n = 2l+1), \\ \begin{pmatrix} \frac{1}{\sqrt{2}}E & \frac{i}{\sqrt{2}}J \\ \frac{1}{\sqrt{2}}E & \frac{-i}{\sqrt{2}}J \end{pmatrix} & (n = 2l). \end{cases}$$

One sees easily that

$$(3.4)\qquad AO(n)A^{-1} \subset O(Q, \mathbf{C}).$$

The diagonal subgroup T of $AO(n)A^{-1}$ contains a maximal torus consisting of elements of the form

$$(3.5)\qquad a(t) = \begin{cases} \operatorname{diag}(t_1, \cdots, t_l, 1, t_l^{-1}, \cdots, t_1^{-1}), & n = 2l+1, \\ \operatorname{diag}(t_1, \cdots, t_l, t_l^{-1}, \cdots, t_1^{-1}), & n = 2l, \end{cases}$$

where $t_i \in \mathbf{C}$ with $|t_i| = 1$ $(i = 1, \cdots, l)$. It follows that $S = A^{-1}TA$ is a maximal torus of $O(n)$ consisting of elements

$$s(t) = A^{-1}a(t)A.$$

Each character χ of S is determined by an l-tuple of integers $(m_1, \cdots, m_l)$ with

$$\chi(s(t)) = t_1^{m_1} \cdots t_l^{m_l}.$$

Let V be an irreducible complex $SO(n)$-module. It is known that the highest weight of V with respect to S is given by $(m_1, \cdots, m_l)$ satisfying

$$(3.6)\qquad \begin{matrix} m_1 \geqslant \cdots \geqslant m_l \geqslant 0, & n = 2l+1, \\ m_1 \geqslant \cdots \geqslant m_{l-1} \geqslant |m_l|, & n = 2l. \end{matrix}$$

3.4. Now let V be an irredicible *real* $O(n)$-module. We know [39, volume 7] that $V_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}$ is still irreducible as a complex $O(n)$-module. Note that the index of $SO(n)$ in $O(n)$ is 2. It follows from the Clifford's theorem that $V_{\mathbf{C}}|SO(n)$ remains irreducible or breaks up into two irreducible parts. From [39, volume 9], the latter occurs if and only if $n = 2l$ and $V_{\mathbf{C}}$ has a dominant weight $(m_1, \cdots, m_l)$ with $m_l \neq 0$. In this case, the other dominant weight is $(m_1, \cdots, m_{m-l}, -m_l)$. It follows that in all cases, $V_{\mathbf{C}}$ has a *unique* dominant weight $(m_1, \cdots, m_l)$ with $m_l \geqslant 0$.

Definition 3.4. An *irreducible real $O(n)$-module V* is said to have *signature* $(m_1,\cdots,m_l)$ if $m_1 \geqslant \cdots \geqslant m_l \geqslant 0$ and $(m_1,\cdots,m_l)$ is a dominant weight of $V_{\mathbf{C}}$.

3.5. Let Q be given as in (3.2). Let π be a finite dimensional irreducible complex representation of $SO(Q,\mathbf{C})$. Here we compute the value $\pi(C)$ of the Casimir operator C of $SO(Q,\mathbf{C})$. In the following, we discuss in detail the case $n = 2l+1$. The Lie algebra of $SO(Q,\mathbf{C})$ has a basis given by

$$A_{ij} = \begin{pmatrix} E_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -JE_{ji}J \end{pmatrix} \qquad (1 \leqslant i, j \leqslant l),$$

$$X_i = \begin{pmatrix} 0 & e_i & 0 \\ 0 & 0 & -{}^te_iJ \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_i^- = \begin{pmatrix} 0 & 0 & 0 \\ {}^te_i & 0 & 0 \\ 0 & -Je_i & 0 \end{pmatrix} \qquad (1 \leqslant i \leqslant l),$$

$$Y_{ij} = \begin{pmatrix} 0 & 0 & J(E_{ij}-E_{ji}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_{ij}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (E_{ij}-E_{ji})J & 0 & 0 \end{pmatrix} \qquad (1 \leqslant i < j \leqslant l).$$

With respect to $\operatorname{tr}(XY)$, the Casimir operator C has the form

$$C = \frac{1}{2}\left\{ \sum_{1 \leqslant i,j \leqslant l} A_{ij}A_{ji} + \sum_{1 \leqslant i \leqslant l} \left(X_iX_i^- + X_i^-X_i\right) - \sum_{1 \leqslant i \leqslant j \leqslant l} \left(Y_i^-Y_{ij} + Y_{ij}Y_{ij}^-\right)\right\}.$$

By the Lie algebra structure

$$C = \frac{1}{2}\left\{ \sum_{1 \leqslant i \leqslant l} \left(A_{ii}A_{ii} + A_{ii}\right) + 2\sum_{1 \leqslant i \leqslant l-1} (l-i)A_{ii} \right. \tag{3.7}$$
$$\left. +2\sum_{i<j} A_{ji}A_{ij} + 2\sum_{1 \leqslant i \leqslant l} X_i^-X_i - 2\sum_{1 \leqslant i<j \leqslant l} Y_{ij}^-Y_{ij}\right\}.$$

Lemma 3.5. *If π has dominant weight $(m_1,\cdots,m_l)$, then*

$$2\pi C = \sum_{1 \leqslant i \leqslant l} m_i^2 + 2\sum_{1 \leqslant i \leqslant l-1} (l-i)m_i + \varepsilon \sum_{1 \leqslant i \leqslant l} m_i, \tag{3.8}$$

where $\varepsilon = 1$ if $n = 2l+1$ and $\varepsilon = 0$ if $n = 2l$.

Proof. Immediate from formula (3.7) and a similar expression for $n = 2l$.

3.6. Corollary *Let V be an irreducible real $O(n)$-module with signature $(m_1, \cdots, m_l)$. Then the Casimir operator C_V of C on V is a scalar multiple with value given by* (3.8).

Proof. If $V_{\mathbf{C}}$ is irreducible as an $SO(n)$-module and the dominant weight is $(m_1, \cdots, m_l)$, then C_V has the value given by (3.8). If $V_{\mathbf{C}}$ decomposes into two irreducible $SO(n)$-modules, $V_{\mathbf{C}}$ has dominant weights $(m_1, \cdots, m_l)$ and $(m_1, \cdots, m_{l-1}, -m_l)$. In this case, $n = 2l$ and it is clear that both irreducible spaces yield the same eigenvalue for C determined by (3.8).

3.7. Let $G = SO(n)$ and let $H = SO(n-1)$ be the subgroup of G given by

$$H = \left\{ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \middle| h \in SO(n\text{-}1) \right\}.$$

Let $l = [n/2]$ and $\pi(n; m_1, \cdots, m_l)$ be the irreducible representation of $SO(n)$ with dominant weight $(m_1, \cdots, m_l)$. The spectral decomposition of $\pi|\mathrm{H}$ is given by the following theorem.

Theorem 3.7 [40, *Theorems* 2 *and* 3, pp. 378–379]. (i) *If $n = 2l + 1$, then*

$$\pi(n; m_1, \cdots, m_l)\big|H = \sum \pi(n-1; s_1, \cdots, s_l),$$

where the summation runs over $m_1 \geqslant s_1 \geqslant m_2, \cdots, m_{l-1} \geqslant s_{l-1} \geqslant m_l$ and $m_l \geqslant s_l \geqslant -m_l$.

(ii) *If $n = 2l$, then*

$$\pi(n; m_1, \cdots, m_l)\big|H = \sum \pi(n-1; s_1, \cdots, s_{l-1}),$$

where the summation runs over $m_1 \geqslant s_1 \geqslant m_2, \cdots, m_{l-2} \geqslant s_{l-2} \geqslant m_{l-1}$ and $m_{l-1} \geqslant s_{l-1} \geqslant |m_l|$.

3.8. Let r be a nonnegative integer with $r \leqslant l$ and $\pi_{n,r}$ the irreducible representation of $SO(n)$ with dominant weight

$$\big(\underbrace{a, \cdots, a}_{r}, \underbrace{0, \cdots, 0}_{l-r}\big), \qquad a > 0.$$

Lemma 3.8. *The restriction $\pi_{n,r}|SO(n-r)$ of π to $SO(n-r)$ contains the trivial representation exactly once.*

Proof. For a nontrivial irreducible representation $\pi = \pi(n{:}\ m_1, \cdots, m_l)$, let $h(\pi)$ be the *largest* index i such that $m_i \neq 0$. From Theorem 3.7, it yields that

$$h(\tau) \geqslant h(\pi) - 1 \tag{3.9}$$

for any irreducible representation τ of $SO(n-1)$ with $\tau \leqslant \pi\,|\,SO(n-1)$. Let $m_{n,r}$ be the multiplicity of the trivial representation in $\pi_{n,r}|SO(n-r)$. From (3.9), we have the recursion relation $m_{n,r} = m_{n-1,r-1}$; hence, by a simple induction $m_{n,r} = m_{n-r,0} = 1$.

3.9. Proposition. *Let V be a **real irreducible** $O(n)$-module with **signature***

$$\big(\underbrace{a,\cdots,a}_{r},\underbrace{0,\cdots,0}_{l-r}\big),\qquad a>0,\ l=[n/2].$$

Then the multiplicity of the trivial representation of $O(n-r)$ in $V\,|\,O(n-r)$ is at most 1.

Proof. Let $H = O(n-r)$ and V^H be the subspace of elements of V fixed by H. It is easy to see

$$(V^H)\otimes_R \mathbf{C} = (V\otimes_{\mathbf{R}}\mathbf{C})^H. \tag{3.10}$$

Consider then the space $V_{\mathbf{C}} = V\otimes_{\mathbf{R}}\mathbf{C}$.

Case 1: $V_{\mathbf{C}}|SO(n)$ *is irreducible.* Then it is the representation $\pi_{n,r}$ and our result is immediate from Lemma 3.8.

Case 2: $V_{\mathbf{C}}|SO(n)$ *breaks into two irreducible subspaces.* It follows that $V_{\mathbf{C}}$ has an irreducible $SO(n)$-module W of dominant weight

$$\big(\underbrace{a,\cdots,a}_{r},\underbrace{0,\cdots,0}_{l-r}\big).$$

Choose an element τ in $O(n-r)$ with $\det(\tau) = -1$. Then we have $V_{\mathbf{C}} = W\oplus \tau W$; as a consequence

$$V_{\mathbf{C}}^{SO(n-r)} = W_1\oplus \tau W_1,$$

where $W_1 = W^{SO(n-r)}$. By Lemma 3.8, $\dim_{\mathbf{C}}(W_1) = 1$. Since the matrix of $\tau\,|\,W_1\oplus\tau W_1$ is $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, one concludes readily that the eigensubspace of $\tau\,|\,W_1\oplus \tau W_1$ of eigenvalue 1 is one dimensional. Hence we have that $V_{\mathbf{C}}^H$ is of dimension 1 and by (3.10), $\dim_{\mathbf{R}}(V^H) = 1$.

3.10. Let M be the $n\times r$ matrix of indeterminates $M = (M_{ij})$ and let $W = \mathbf{R}[M]$ be the ring of polynomials. For a nonnegative integer s, let W_s be the subspace of $\mathbf{R}[M]$ given by

$$W_s = \{f(M)\in \mathbf{R}[M]\,|\,f(Mk) = \det(k)^s f(M),\ k\in \mathrm{GL}(r,\mathbf{R})\}. \tag{3.11}$$

The space W_s is trivial if $r>n$. We assume that $r\leqslant n$. Let I be any subset of $I_0 = \{1,\cdots,n\}$ with cardinality r. Let $i_1,\cdots,i_r$ be elements of I in increasing order and denote by ϕ_I the polynomial

$$\phi_I(M) = \det(M_I),$$

where M_I is the $r\times r$ matrix such that the νth row of (M_I) equals the i_νth row of M $(\nu = 1,\cdots,r)$. The fundamental theorem of invariant theory for $\mathrm{GL}(r)$ [10, Theorem 7.2] yields the structure of W_s.

Lemma 3.10. *The space W_s is the linear span of $\phi_{I_1}\cdots\phi_{I_s}$, where $I_1,\cdots,I_s$ run over subsets of $\{1,\cdots,n\}$ of cardinality r.*

3.11. By the preceding lemma, the space W_s is contained in the space of homogeneous polynomials of degree rs. Let $\Delta_{\lambda\nu}$ be the differential operators given by

$$\Delta_{\lambda\nu} = \sum_{i=1}^{n} \frac{\partial^2}{\partial M_{i\lambda}\partial M_{i\nu}} \qquad (1 \leqslant \lambda, \nu \leqslant r).$$

Recall that a function f is *pluriharmonic* if it satisfies

$$\Delta_{\lambda\nu} f = 0 \qquad (1 \leqslant \lambda, \nu \leqslant r). \tag{3.12}$$

By [18, Remark, p. 18], $W = H \otimes \mathbf{R}[{}^t MM]$, where H is the space of pluriharmonic polynomials. By [10, Theorems 6.9 and 6.13], the space $H_s = H \cap W_s$ is irreducible with signature

$$\omega_0 = (\underbrace{s, \cdots, s}_{r}, \underbrace{0, \cdots, 0}_{l-r}), \qquad l = [n/2].$$

Moreover the occurrence of ω_0 in H is 1. Let W^0 (resp. H^0, W_s^0) be the isotypic component of the representation of $O(n)$ with signature ω_0 in W (resp. H, W_s). It follows that

$$W^0 = H^0 \otimes \mathbf{R}[{}^t MM] = H_s \otimes \mathbf{R}[{}^t MM],$$

and so all polynomials in W^0 have degree at least rs. As a consequence, $W_s^0 = H_s$.

Theorem 3.11. *Let $f \in W_s$ and $r \leqslant n/s$. The following statements are equivalent*:

(i) *f is pluriharmonic.*

(ii) *f is harmonic,*

(iii) *$f \in W_s^0$.*

(iv) $Tf = 0$, $T = \sum_{1 \leqslant \lambda, \nu \leqslant r} \langle M_\lambda, M_\nu \rangle \Delta_{\lambda\nu}$.

3.12. Lemma. *Let $f(M)$ be a polynomial such that*

(i) $f(Mh) = \chi(h) f(M)$, $h \in \mathrm{GL}(r)$,

(ii) $f(gM) = \rho(g) f(M)$, $g \in O(n)$.

If $r < n$, there exist a constant c and a nonnegative integer s such that

$$f(M) = c \det({}^t MM)^s.$$

Proof. Immediate from [18, Theorems 6.9 and 6.13].

4. Harmonic forms with highest signature

4.1. Let $\mathscr{D}$ be the symmetric space associated to the group $O(p,q)$. For fixed $g \in G$, the function $\det(j(g, Z))$ has constant sign $s(g)$ over $\mathscr{D}$. Denote by G the subgroup $O^+(p,q)$ of $O(p,q)$, where

$$O^+(p,q) = \{ g \in O(p,q) | s(g) = 1 \}.$$

For $Z \in \mathscr{D}$, let O_Z be the isotropic subgroup of G at Z. It is easy to see that O_Z is isomorphic to $O(p) \times SO(q)$. For convenience, we also realize $\mathscr{D}$ as the space of all maximal negative subspaces of $\mathbf{R}^n$, $n = p + q$. The element Z is then identified with the subspace $\langle Z \rangle$ spanned by the columns of the matrix $\binom{Z}{E_q}$. Let K_Z be the subgroup of G defined by

$$K_Z = \{g \in g \mid gv = v \qquad \text{for} \quad v \in \langle Z \rangle\}.$$

Clearly K_Z is isomorphic to $O(p)$.

4.2. Let $\wedge^{rq}(\mathscr{D})_Z$ be the space of differential forms of degree rq at Z. By choosing a proper orthonormal basis of the cotangent space of $\mathscr{D}$ at Z, $\wedge^{rq}(\mathscr{D})_Z$ is isomorphic to $\wedge^{rq}(V)$ with $V = M_{pq}(\mathbf{R})$; moreover the action of K_Z on $\wedge^{rq}(V)$ becomes the representation of $O(p)$ in the tensored space.

4.3. Let $l = [p/2]$ and $r \leqslant l$. For a quadratic form Q of $\mathbf{R}^p$, we consider the representation of $O(Q, \mathbf{C})$ on $\wedge^{rq}(V) \otimes_{\mathbf{R}} \mathbf{C}$. It is easy to see that $\wedge^{rq}(V)$ has a unique irreducible $O(p)$-submodule with signature

$$\omega_0 = \Big(\underbrace{q, \cdots, q}_{r}, \underbrace{0, \cdots, 0}_{l-r}\Big).$$

Let π_0 be the projection of $\wedge^{rq}(V)$ onto this irreducible subspace with respect to the decomposition of $\wedge^{rq}(V)$ into isotypic subspaces of $O(p)$.

4.4. Let e_{ij}, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$, be the standard basis of V. Let ϕ be the element

$$\phi = \prod_{p-r+1 \leqslant i \leqslant p} \prod_{1 \leqslant j \leqslant q} e_{ij}. \tag{4.1}$$

Here product means exterior product. Now let A and J be the matrices given in §3.3. We have that ((3.4))

$$AO(p)A^{-1} \subset O(Q, \mathbf{C}),$$

with $Q(x) = x_1 x_p + x_2 x_{p-2} + \cdots + x_p x_1$.

Consider the element $A\phi$. Let T be the maximal torus of $AO(p)A^{-1}$ in (3.3). We have the expression

$$A\phi = (-1)^{r(r-1)q/2} \left(\frac{i}{\sqrt{2}} \right)^{rq} \prod_{1 \leqslant i \leqslant r} \prod_{1 \leqslant j \leqslant q} e_{ij} + \text{terms} \tag{4.2}$$

$$\text{(of lower weight with respect to } T\text{)}.$$

Since $AO(p)A^{-1}$ is compact, $\wedge^{rq}(V) \otimes_{\mathbf{R}} \mathbf{C}$ has an invariant inner product $\langle\ ,\ \rangle$. Let ψ be the element

$$\psi = \prod_{1 \leqslant i \leqslant r} \prod_{1 \leqslant j \leqslant q} e_{ij}. \tag{4.3}$$

From (4.2), $\langle A\phi, \psi\rangle \neq 0$. Note that $\mathbf{C}\psi$ is the weight space of $\bigwedge^{rq}(V) \otimes_{\mathbf{R}} \mathbf{C}$ of weight ω_0. As a consequence $\pi_0(\phi) \neq 0$.

Proposition 4.4. *Let W^0 be the irreducible $O(p)$-module of $\bigwedge^{rq}(V)$ of signature ω_0. Then the multiplicity of the trivial representation of $O(p-r)$ in W^0 is* 1; *moreover every $O(p-r)$ invariant element of W^0 is a multiple of $\pi_0(\phi)$.*

Proof. We know that ϕ is $O(p-r)$-invariant. By the above discussion, $\pi_0(\phi)$ is a nonzero $O(p-r)$-invariant element of W^0. By Proposition 3.9, the multiplicity of the trivial representation in $W \mid O(p-r)$ is at most 1, thus our assertion follows.

4.5. Let v_0 be a nonzero $O(p-r)$-invariant element of W^0. Since

$$\langle \phi, \pi_0(\phi)\rangle = \langle \pi_0(\phi), \pi_0(\phi)\rangle \neq 0,$$

it yields that the coefficient of ϕ in v_0 with respect to the basis e_{ij} is nonzero. Let

$$* : \textstyle\bigwedge^{rq}(V) \to \bigwedge^{(p-r)q}(V)$$

be the ordinary $*$ operation. Let ξ be the element

$$\xi = \prod_{1 \leqslant i \leqslant p-r} \prod_{1 \leqslant j \leqslant q} e_{ij}.$$

Lemma 4.5. *We have the following conditions*:

(i) *The coefficient of ϕ in v_0 is nonzero.*

(ii) *The coefficient of ξ in $* v_0$ is nonzero.*

Proof. (i) is established above.

(ii) follows by a variant argument of (i).

4.6. Let $M \in M_{nr}(\mathbf{R})$. Recall that in §2.9 we have defined a polynomial $f(Z, M)$ such that its pull back to W_- has the expression

$$(4.4)\quad \begin{aligned} f(X, M) &= a_r s(X_-)^r (\det - \langle X, X\rangle)^{-r/2} \prod_{t=1}^{q} \prod_{i=1}^{r} \langle dX_t, M_i\rangle_\perp, \\ a_r &= (-1)^{1+qr(r-1)(q-1)/4} \left(\frac{rq}{2}\right) \Gamma\left(\frac{rq}{2}\right) \pi^{-qr/2}. \end{aligned}$$

By §4.3, we know that $\bigwedge^{rq}(\mathscr{D})_Z$ has a unique K_Z-irreducible submodule $\bigwedge_0^{rq}(\mathscr{D})_Z$ of signature

$$\omega_0 = \Big(\underbrace{q, \cdots, q}_{r}, \underbrace{0, \cdots, 0}_{l-r}\Big), \qquad l = [p/2].$$

Definition 4.6. Let $F(Z, M)$ be the projection of $f(Z, M)$ in $\bigwedge^{rq}(\mathscr{D})_Z$ with respect to the decomposition of $\bigwedge^{rq}(\mathscr{D})_Z$ into isotypic subspaces of K_Z.

4.7. Lemma *The form $F(Z, M)$ satisfies the following conditions*:

(i) *For $g \in G$, $(g^{-1})^* F(Z, M) = F(Z, gM)$.*

(ii) *Let $M_{Z^\perp}$ be the component of M orthogonal to Z (Z as a subspace of $\mathbf{R}^n$). Then $F(Z, M)$ is just a polynomial in $M_{Z^\perp}$.*

(iii) *For* $h \in \mathrm{GL}(r)$,

$$F(Z, Mh) = \det(h)^q F(Z, M). \tag{4.5}$$

(iv) $F(Z, M) \neq 0$.

Proof. (i) is immediate from formula (4.4). By our construction,

$$\langle dX_t, M_i \rangle_\perp = \langle dX_t, (M_i)_{Z^\perp} \rangle,$$

hence $F(Z, M)$ is a polynomial in $M_{Z^\perp}$. Let $f_t(X, M)$ be the form

$$f_t(X, M) = \prod_{1 \leqslant i \leqslant r} \langle dX_t, M_i \rangle_\perp .$$

One sees easily that

$$f_t(X, Mh) = \det(h) f_t(X, M)$$

and as a consequence, (iii) follows by formula (4.4). Let

$$M = \begin{pmatrix} 0 \\ E_r \\ 0 \end{pmatrix}$$

with upper and lower zero matrices of the size $(p - r) \times q$ and $q \times q$ respectively. At $Z = 0$, $f(Z, M)$ is a nonzero multiple of ϕ given in (4.1). Then by Proposition 4.4, $F(Z, M) \neq 0$.

4.8. In the sequel, we consider forms of the type

$$\tau(a) = \det\langle M_{Z^\perp}, M_{Z^\perp} \rangle^a F(Z, M) \qquad a = -(p + 2q - r - 1)/2. \tag{4.6}$$

Let $C(K_Z)$ be the Casimir operator of K_Z on polynomials of $M_{Z^\perp}$. Since $\det\langle M_{Z^\perp}, M_{Z^\perp} \rangle$ is K_Z-invariant and by our definition $F(Z, M_{Z^\perp})$ belongs to the isotypic component of $\mathbf{R}[M_{Z^\perp}]$ of signature

$$\Big(\underbrace{q, \cdots, q}_{r}, \underbrace{0, \cdots, 0}_{l - r}\Big), \qquad l = [p/2],$$

thus

$$C(K_z)\tau(a) = \tfrac{1}{2} qr(p + q - r - 1)\tau(a) \tag{4.7}$$

by a simple computation using Corollary 3.6. Let $C(G)$ be the Casimir operator of G on $\mathbf{R}[M]$. Note that $\tau(a)$ depends only on $M_{Z^\perp}$. By a straightforward computation, $\tau(a)(M_{Z^\perp})$ is again pluriharmonic. By Lemma 3.2, it is easy to see that

$$C(G)\tau(a) = C(K_Z)\tau(a) + \frac{q}{2} \sum_{1 \leqslant i \leqslant r} L_{ii}\tau(a), \tag{4.8}$$

where

$$L_{ii} = \sum_{1 \leqslant \lambda \leqslant n} M_{\lambda i} \frac{\partial}{\partial M_{\lambda i}}.$$

Clearly

$$L_{ii}\tau(a) = (2a+q)\tau(a) \qquad (1 \leqslant i \leqslant r);$$

hence we obtain the relation

$$C(G)\tau(a) = \frac{qr}{2}\{(p+q-r-1)+(2a+q)\}\tau(a) = 0. \tag{4.9}$$

4.9. We say a differential form $\omega(z)$ of $\mathscr{D}$ of degree rq is of *highest signature* if

$$\omega(Z) \in \Lambda_0^{rq}(\mathscr{D})_Z, \tag{4.10}$$

i.e., at every Z, $\omega(Z)$ belongs to the isotypic subspace of signature

$$\Big(\underbrace{q,\cdots,q}_{r},\underbrace{0,\cdots,0}_{l-r}\Big), \qquad l = [p/2].$$

By a harmonic form $\omega(Z)$ of $\mathscr{D}$, following Hodge we mean a differential form $\omega(Z)$ of $\mathscr{D}$ satisfying $d\omega = 0$ and $d(*\,\omega) = 0$. The condition is stronger than $\Delta\omega = 0$ in general. We have the following theorem on harmonic forms of highest signature.

Theorem 4.9. *We have the following conditions*:

(i) *The form* $(\det\langle M_{Z^\perp}, M_{Z^\perp}\rangle)^{-(p+2q-r-1)/2}F(Z,M)$ *is a harmonic form of highest signature invariant under* G_M.

(ii) *Conversely every* G_M*-invariant harmonic form of degree* rq *of highest signature is a constant multiple of the form in* (i).

Proof. Let τ be the differential form given by

$$\tau(Z,M) = \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-(p+2q-r-1)/2}F(Z,M). \tag{4.11}$$

We prove the theorem in several steps.

Step 1. By (i) of Lemma 4.7, it yields that

$$(g^{-1})^*\tau(Z,M) = \tau(Z,gM). \tag{4.12}$$

Let C and $C(G)$ be the Casimir operators of G on $\Lambda^{rq}(\mathscr{D})$ and $\mathbf{R}(M)$ respectively. From (4.12), $C\tau = C(G)\tau$. Then by formula (4.9), we have the condition $C\tau = 0$.

Step 2. Consider the product $\tau \wedge (*\,\tau)$. We have

$$\tau \wedge (*\,\tau) = \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-(p+2q-r-1)}F(M,Z) \wedge (*\,F(Z,M)). \tag{4.13}$$

Set

$$F(Z,M) \wedge *\,F(Z,M) = h(Z,M)\,dv_{\mathscr{D}},$$

where $h(Z,M)$ is a polynomial in $M_{Z^\perp}$ and $dv_{\mathscr{D}}$ is the invariant volume element of $\mathscr{D}$. Since $dv_{\mathscr{D}}$ is G-invariant, by Lemma 4.7,

$$\begin{aligned} h(Z,M\alpha) &= \det(\alpha)^{2q}h(Z,M), \qquad \alpha \in \mathrm{GL}(r),\\ h(Z,gM) &= h(Z,M), \qquad g \in K_Z. \end{aligned} \tag{4.14}$$

Appealing to Lemma 3.12, we have

$$h(Z, M) = \left(\det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^q\right) a(Z).$$

As a consequence of the condition

$$(g^{-1})^* F(Z, M) = F(Z, gM),$$

$a(Z)$ must be G-invariant. It follows that $a(Z)$ is a constant c such that

$$\tau \wedge (*\tau) = c \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-(p+q-r-1)} dv_{\mathscr{D}}. \tag{4.15}$$

Step 3. Assume that $\langle M, M\rangle > 0$. Let Γ_M be a uniform discrete subgroup of G_M where

$$G_M = \{g \in G \mid g \text{ leaves } M \text{ pointedwisely fixed}\}.$$

Then by formula (1.39) for $dv_{\mathscr{D}}$ and Lemmas 1.20, 2.6

$$\int_{\Gamma_M\setminus\mathscr{D}} \tau \wedge *\tau = b \operatorname{vol}(\Gamma_M \setminus G_M), \tag{4.16}$$

where b is given by

$$b = c\frac{\prod_{i=1}^r \Gamma(p+q-2r-1+i)/2 \prod_{j=1}^q \Gamma(p+q-2r-1+j)/2}{\prod_{\lambda=1}^{r+q} \Gamma(p+q-2r-1+\lambda)/2}$$
$$\cdot \pi^{rq/2} \det\langle M, M\rangle^{-(p+q-r-1)}.$$

Step 4. Let $L^2_{rq,M}$ be the space of G_M-invariant forms ϕ of G of degree rq such that

$$\int_{G_M\setminus G} \langle \phi, \phi\rangle \, d\bar{g} < \infty,$$

where $d\bar{g}$ is a fixed Haar measure on $G_M \setminus G$. By (4.16), the pull back of τ lies in $L^2_{rq,M}$. Hence Kuga's lemma on harmonic forms [3, p. 49, Theorem 2.5] is applicable. It follows that τ is a harmonic form for $\langle M, M\rangle > 0$. Since $\tau(Z, M)$ depends on M and Z as a rational function, it is a harmonic form wherever it is defined. This proves (i).

Now let $\omega(Z)$ be a differential form satisfying condition (ii). Note that $G_M \cap K_Z$ is isomorphic to $O(m)$ with $m \geqslant p - r$. If $p - r < m$, by Theorem 3.7, $\omega(Z)$ has to be trivial. Hence we may assume that $m = p - r$. By Proposition 4.4, there exists a function $b(Z)$ with

$$\omega(Z) = \tau(Z, M) b(Z).$$

Then the conditions $d\omega = 0$ and $d(*\omega) = 0$ yield

$$db(Z) \wedge \tau = 0, \qquad db(Z) \wedge (*\tau) = 0. \tag{4.17}$$

With the aid of Lemma 4.5, condition (4.17) implies that $db(Z) = 0$, i.e., $b(Z)$ is a constant. Thus we have established assertion (ii).

5. Some integral formulas

In this section, we consider various realizations and parametrizations of $\mathscr{D}$ and present various expressions of the invariant volume element of $\mathscr{D}$. The purpose is to evaluate some integrals which will be encountered in our study of the Fourier coefficients of the lifting map.

5.1. Let $p \geqslant q > 0$, $n = p + q$ and let Q_1 be the matrix

$$Q_1 = \begin{pmatrix} 0 & 0 & E_q \\ 0 & E_{p-q} & 0 \\ E_q & 0 & 0 \end{pmatrix}.$$

Denote by G the orthogonal group leaving Q_1 invariant,

$$G = \left\{ g \in \mathrm{GL}_n(\mathbf{R}) \middle| {}^t g Q_1 g = Q_1 \right\}.$$

Given an $n \times n$ matrix X, we use the black form which is suitable for Q_1,

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix},$$

where X_{11}, $X_{33} \in M_{qq}(\mathbf{R})$ and $X_{22} \in M_{p-q,p-q}(\mathbf{R})$. The Lie algebra $L(G)$ consists of those X satisfying the conditions:

$$(5.1) \qquad \begin{aligned} X_{11} + {}^t X_{33} &= 0, & X_{22} + {}^t X_{22} &= 0, \\ X_{12} + {}^t X_{23} &= 0, & X_{21} + {}^t X_{32} &= 0, \\ X_{13} + {}^t X_{13} &= 0, & X_{31} + {}^t X_{31} &= 0. \end{aligned}$$

For $t = (t_1, \cdots, t_q)$ with $t_i > 0$ $(i = 1, \cdots, q)$, let us set

$$(5.2) \qquad d(t) = \mathrm{diag}\left(t_1, \cdots, t_q, 1, \cdots, 1, t_1^{-1}, \cdots, t_q^{-1}\right)$$

and denote by T the group consisting of all these $d(t)$. We know that T is a maximal connected $\mathbf{R}$-diagonalizable subgroup of G. Let α_i be the character of T given by

$$d(t)^{\alpha_i} = t_i \qquad (i = 1, \cdots, q).$$

Let $\mathscr{N}$ be the subalgebra of $L(G)$ consisting of X such that $X_{21} = 0$, $X_{31} = 0$, $X_{32} = 0$, $X_{22} = 0$ and X_{11} is upper triangular. Then $\mathscr{N}$ is a maximal T-invariant nilpotent subalgebra of $L(G)$ and it defines an order of the roots of G with respect to T. From (5.1) one easily sees that the set Σ_+ of positive roots consists of

$$(5.3)\qquad \begin{aligned} &\alpha_i\alpha_j^{-1}(1), \\ &\alpha_i\alpha_j(1), \qquad 1 \leqslant i < j \leqslant q, \\ &\alpha_i(p-q), \qquad 1 \leqslant i \leqslant q, \end{aligned}$$

where the number in the parenthesis indicates the multiplicity of the root.

5.2. Let S be the matrix given by

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}}E_q & 0 & \frac{1}{\sqrt{2}}E_q \\ 0 & E_{p-q} & 0 \\ \frac{1}{\sqrt{2}}E_q & 0 & \frac{-1}{\sqrt{2}}E_q \end{pmatrix}.$$

Clearly S satisfies the conditions

$$S^2 = E_n, \quad {}^tS = S, \quad SQ_1S = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}$$

and consequently $O(p,q) = SGS$. Let $a(t)$ be the element of $O(p,q)$ given by $a(t) = Sd(t)S$. A straightforward computation shows that $a(t)$ is of the form

$$a(t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

$$A = \begin{pmatrix} D & 0 \\ 0 & E_{p-q} \end{pmatrix}, \quad B = \begin{pmatrix} H \\ 0 \end{pmatrix}, \quad C = (H \quad 0),$$

where

$$D = \operatorname{diag}\left(\frac{t_1 + t_1^{-1}}{2}, \dots, \frac{t_q + t_q^{-1}}{2} \right),$$

$$H = \operatorname{diag}\left(\frac{t_1 - t_1^{-1}}{2}, \dots, \frac{t_q - t_q^{-1}}{2} \right).$$

Consider the group action of $O(p,q)$ on $\mathscr{D}$. We have that

$$(5.4)\quad a(t)O = \begin{pmatrix} M(t) \\ 0 \end{pmatrix}, \qquad M(t) = \operatorname{diag}\left(\frac{t_1 - t_1^{-1}}{t_1 + t_1^{-1}}, \dots, \frac{t_q - t_q^{-1}}{t_q + t_q^{-1}} \right).$$

5.3. Let $K = O(p) \times O(q)$ and $A = STS$. By a theorem of Cartán,

$$(5.5)\qquad O(p,q) = KAK.$$

By [12, Proposition 1.17, p. 381] the invariant measure of $O(p,q)$ with respect to the Cartan decomposition (5.5) has a form

$$(5.6)\qquad dg = c \prod_{\alpha \in \Sigma_+} \sinh a^\alpha dk_1 da\, dk_2,$$

where c is a nonzero constant. From the data of positive roots (5.3), it yields

$$\begin{aligned} dg &= c\prod_{i=1}^{q}\left(\frac{t_i - t_i^{-1}}{2}\right)^{p-q} \\ &\quad\cdot \prod_{1\leqslant i<j\leqslant q}\left(\frac{t_i^2 + t_i^{-2} - t_j^2 - t_j^{-2}}{4}\right) dk_1 da(t)\, dk_2. \end{aligned} \tag{5.7}$$

Now set

$$\lambda_i = \frac{t_i - t_i^{-1}}{t_i + t_i^{-1}} \qquad (i = 1,\cdots,q).$$

In terms of λ_i, (5.7) becomes

$$\begin{aligned} dg &= c\prod_{i=1}^{q}\left(1 - \lambda_i^2\right)^{-(p+q)/2}\lambda_i^{(p-q)} \\ &\quad\cdot \prod_{1\leqslant i<j\leqslant q}\left(\lambda_i^2 - \lambda_j^2\right) dk_1 d\lambda_1 \cdots d\lambda_q dk_2. \end{aligned} \tag{5.8}$$

5.4. Recall that $\mathscr{D} = O(p,q)/K$ and the identification is given by $g \mapsto gO$. For any real $p \times q$ matrix $Z \in \mathscr{D}$, there exist $\tau \in O(p)$ and $\sigma \in O(p)$ such that

$$Z = \tau\begin{pmatrix} D(\lambda) \\ 0 \end{pmatrix}\sigma, \tag{5.9}$$

where

$$D(\lambda) = \operatorname{diag}(\lambda_1,\cdots,\lambda_q), \qquad 0 \leqslant \lambda_1 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_q < 1.$$

From (5.8), the invariant measure $dv_{\mathscr{D}}$ on $\mathscr{D}$ in terms of the decomposition (5.9) can be presented by

$$\begin{aligned} dv_{\mathscr{D}} &= c_1\prod_{i=1}^{q}\left(1 - \lambda_i^2\right)^{-(p+q)/2}\lambda_i^{p-q} \\ &\quad\cdot \prod_{1\leqslant i<j\leqslant q}\left(\lambda_i^2 - \lambda_j^2\right) d\lambda_1 \cdots d\lambda_q d\tau d\sigma, \end{aligned} \tag{5.10}$$

where c_1 is a nonzero constant.

5.5. For $Z \in \mathscr{D}$, let $\{dZ\}$ be the Euclidean measure. The invariant measure $dv_{\mathscr{D}}$ is also given by (1.38)

$$dv_{\mathscr{D}} = \det(E - {}^tZZ)^{-(p+q)/2}\{dZ\}.$$

Comparing it with (5.10), we see easily that

$$\{dZ\} = c_1\prod_{i=1}^{q}\lambda_i^{p-q}\prod_{1\leqslant i<j\leqslant q}\left(\lambda_i^2 - \lambda_j^2\right) d\lambda_1 \cdots d\lambda_q d\tau d\sigma. \tag{5.11}$$

5.6. Let α: $\mathscr{D} \to M_{pq}(\mathbf{R})$ be the map

$$(5.12) \qquad \alpha(Z) = Z(E - {}^tZZ)^{-1/2} \qquad (Z \in \mathscr{D}).$$

For any $W \in M_{pq}(\mathbf{R})$, $W(E + {}^tWW)^{-1/2}$ lies in $\mathscr{D}$. Thus we have a map β: $M_{pq}(\mathbf{R}) \to \mathscr{D}$ given by

$$(5.13) \qquad \beta(W) = W(E + {}^tWW)^{-1/2} \qquad (W \in M_{pq}(\mathbf{R})).$$

One checks readily that α, β are inverse to each other.

For $Z \in \mathscr{D}$, the decomposition (5.9) yields

$$Z = \tau \begin{pmatrix} D(\lambda) \\ 0 \end{pmatrix} \sigma.$$

By a direct and simple computation,

$$W = \alpha(Z) = \tau \begin{pmatrix} D(\nu) \\ 0 \end{pmatrix} \sigma,$$

where $\nu_i = \lambda_i / \sqrt{1 - \lambda_i^2}$ $(i = 1, \cdots, q)$. Then it follows that (5.11) implies

$$(5.14) \qquad \alpha^*\left(\det(E + {}^tWW)^{-1/2}\{dW\}\right) = dv_{\mathscr{D}}.$$

Theorem 5.6. *Let $dv_{\mathscr{D}}$ be the invariant volume element*

$$dv_{\mathscr{D}} = \det(E - {}^tZZ)^{-(p+q)/2}\{dZ\}$$

of $\mathscr{D}$ and $W = Z(E - {}^tZZ)^{-1/2}$. Then we have the integral formula

$$\int_{\mathscr{D}} f(Z)\, dv_{\mathscr{D}}(Z) = \int_{M_{pq}(\mathbf{R})} f\left(W(E + {}^tWW)^{-1/2}\right)\det(E + {}^tWW)^{-1/2}\{dW\}$$

for integrable functions $f(Z)$ of $\mathscr{D}$.

For convenience, formula (5.14) can also be written as

$$(5.15) \qquad \{dW\} = \det(E - {}^tZZ)^{-(p+q+1)/2}\{dZ\}.$$

5.7. In the sequel, we discuss various realizations of $\mathscr{D}$ with the intention of easier comprehension of the boundary of $\mathscr{D}$. Let p, q, s be integers such that $p \geqslant q > 0$, $q \geqslant s \geqslant 0$. Denote by Q_2 the symmetric matrix

$$(5.16) \qquad Q_2 = \begin{pmatrix} 0 & 0 & E_s \\ 0 & E_{p-s,q-s} & 0 \\ E_s & 0 & 0 \end{pmatrix}$$

and by G the orthogonal group leaving Q_2 invariant. The symmetric space $\mathscr{D}$ associated to G is realized as the space of all real $p \times q$ matrices Z such that

$$(5.17) \qquad {}^tN(Z)Q_2N(Z) < 0,$$

where $N(Z)$ stands for the matrix $N(Z) = \binom{Z}{E_q}$. For $g \in G$, we use the block matrix form $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in M_{pp}(\mathbf{R})$ and $D \in M_{qq}(\mathbf{R})$. The action of g on $\mathscr{D}$ is the usual fractional transformation

$$gZ = (AZ + B)(CZ + D)^{-1}.$$

5.8. For $Z \in \mathscr{D}$, we decompose it into the block form

$$Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}, \tag{5.18}$$

with $Z_1 \in M_{s,q-s}$, $Z_2 \in M_{ss}$, $Z_3 \in M_{p-s,q-s}$ and $Z_4 \in M_{p-s,s}$. Then the inequality (5.17) reads

$$ {}^t(Z_3\ Z_4)\begin{pmatrix} Z_3 \\ Z_4 \end{pmatrix} < \begin{pmatrix} E_{q-s} & -{}^tZ_1 \\ -Z_1 & -{}^tZ_2 - Z_2 \end{pmatrix}. \tag{5.19}$$

Note also that (5.19) is equivalent to

$$\begin{aligned} &{}^tZ_3Z_3 < E_{q-s}, \\ &{}^tZ_4Z_4 < -({}^tZ_2 + Z_2) - (Z_1 + {}^tZ_4Z_3)(E - {}^tZ_3Z_3)^{-1}({}^tZ_1 + {}^tZ_3Z_4). \end{aligned} \tag{5.20}$$

Let $A(Z)$ be the function given by

$$A(Z) = \det(-{}^tN(Z)Q_2N(Z)). \tag{5.21}$$

In terms of Z_1, Z_3, Z_2, Z_4, we have the expression

$$A = \det(E - {}^tZ_3Z_3)\det(M),$$

$$M = -\{{}^tZ_4Z_4 + {}^tZ_2 + Z_2 + (Z_1 + {}^tZ_4Z_3)(E - {}^tZ_3Z_3)^{-1}({}^tZ_1 + {}^tZ_3Z_4)\}.$$

The invariant volume element of $\mathscr{D}$ is given by

$$dv_{\mathscr{D}} = A^{-(p+q)/2}\{dZ\}, \tag{5.22}$$

where $\{dZ\}$ as usual stands for $\prod_{i=1}^p \prod_{j=1}^q dZ_{ij}$.

5.9. Let F_0 be the $n \times s$ matrix given by

$$F_0 = \begin{pmatrix} E_s \\ 0 \end{pmatrix}$$

and G_0 the subgroup

$$G_0 = \{g \in G \mid gF_0 = F_0\}.$$

For any $g \in G_0$, write

$$g = \begin{pmatrix} E_s & X & C \\ 0 & D & Y \\ 0 & 0 & E_s \end{pmatrix}.$$

The condition $g \in O(Q_2)$ yields equivalently the conditions

(5.23) $\quad D \in O(p-s, q-s), \quad X = -{}^tYQ_3D, \quad C + {}^tC = -{}^tYQ_3Y,$

where $Q_3 = E_{p-s,q-s}$. For any $Y \in M_{n-2s,s}(\mathbf{R})$ and *skew symmetric* $s \times s$ matrix S, denote by $n(Y, S)$ the element of G_0 given by

$$n(Y,S) = \begin{pmatrix} E_s & -{}^tYQ_3 & C(Y,S) \\ 0 & E_{n-2s} & Y \\ 0 & 0 & E_s \end{pmatrix}, \tag{5.24}$$

where $C(Y, S) = -\frac{1}{2}{}^tYQ_3Y + S$. Let N be the set consisting of all $n(Y, S)$. One sees readily that G_0 is the semi-direct product, i.e., that

$$G_0 = O(p-s, q-s) \ltimes N. \tag{5.25}$$

5.10. Given any $(n-2s) \times s$ matrix Y, let

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \tag{5.26}$$

with $Y_1 \in M_{p-s,s}$ and $Y_2 \in M_{q-s,s}$. A simple computation yields the transformation formula of $n(Y, S)$ on $\mathscr{D}$

$$n(Y,S)Z = \begin{pmatrix} Z_1 - {}^tY_1Z_3 + {}^tY_2 & Z_2 - Z_1Y_2 - {}^tY_1Z_4 + {}^tY_1Z_3Y_2 + \Delta \\ Z_3 & Z_4 + Y_1 - Z_3Y_2 \end{pmatrix},$$

$$\Delta = -\tfrac{1}{2}{}^tYY + S. \tag{5.27}$$

For $m \in O(p-s, q-s)$, let

$$m = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

with $A_1 \in M_{p-s,p-s}$ and $D_1 \in M_{q-s,q-s}$. Identify m with the element of G_0 defined by

$$\begin{pmatrix} E_s & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & E_s \end{pmatrix}.$$

Then the transformation formula of m on $\mathscr{D}$ is given by

$$mZ = \begin{pmatrix} Z_1(C_1Z_3 + D_1)^{-1} & Z_2 - Z_1(C_1Z_3 + D_1)^{-1}C_1Z_4 \\ mZ_3 & (A_1 - (mZ_3)C_1)Z_4 \end{pmatrix}. \tag{5.28}$$

5.11. Now let X be the $n \times r$ matrix $(r \geqslant s)$ given by $X = \binom{E_r}{0}$, and G_X the subgroup of G leaving X fixed. Clearly $G_X \subset G_0$. Denote by N_1 the subgroup of N consisting of all $n(Y, S)$ such that Y_1 in (5.26) is of the form

$$Y_1 = \begin{pmatrix} 0 \\ Y_1' \end{pmatrix},$$

where Y_1' is a $(p-r)\times s$ matrix. Then one readily shows that

$$G_X = O(p-r, q-s) \ltimes N_1. \tag{5.29}$$

5.12. For Z_3 and Z_4, we decompose further

$$Z_3 = \begin{pmatrix} Z_3'' \\ Z_3' \end{pmatrix}, \qquad Z_4 = \begin{pmatrix} Z_4'' \\ Z_4' \end{pmatrix} \tag{5.30}$$

with $Z_3' \in M_{p-r,q-s}$ and $Z_4' \in M_{p-r,s}$.

By condition (5.20), we have that ${}^tZ_3'Z_3' < E_{q-s}$. As a consequence, there exists $m \in O(p-r,q-s)$ such that $mZ_3' = 0$. Then choosing Y, S properly with $n(Y,S) \in N_1$, one can bring Z into $W = n(Y,S)mZ$ satisfying the conditions:

$$W_1 = 0, \quad {}^tW_2 = W_2, \quad W_3' = 0, \quad W_4' = 0. \tag{5.31}$$

Now denote by F the subset of $\mathscr{D}$ consisting of all Z satisfying (5.31). By the above discussion, we have that

$$\mathscr{D} = G_X F. \tag{5.32}$$

From (5.27) and (5.28), one sees easily that an element $g \in G_X$ satisfies $gF \cap F \neq \varnothing$ if and only if $g \in O(p-r)\times O(q-s)$; moreover $O(p-r)\times O(q-s)$ leaves F invariant. Set

$$Z_0 = \begin{pmatrix} 0 & -E_s \\ 0 & 0 \end{pmatrix},$$

$$\mathscr{D}_1 = G_X Z_0. \tag{5.33}$$

Here we view $\mathscr{D}$ as a fibered space over $\mathscr{D}_1$. For each $Z \in \mathscr{D}_1$, there exists $g \in G_X$ with $gZ_0 = Z$. Then the fiber F_Z over Z is determined by $F_Z = gF$.

The above discussion indicates that it is independent of our choice of g. Now let

$$B = \det\left(E - {}^tZ_3'Z_3'\right) \tag{5.34}$$

and α be the form

$$\alpha = B^{-(p+q+s-r)/2}\{dZ_3'\}\{dZ_4'\}\{dZ_1\}\left\{d\left(\frac{Z_2 - {}^tZ_2}{2}\right)\right\}. \tag{5.35}$$

Examining the transformation formulas (5.27) and (5.28), one concludes easily that α is G_X-invariant. Thus we can view α as a G_X-invariant measure $dv_{\mathscr{D}_1}$ on $\mathscr{D}_1$. For $Z \in \mathscr{D}_1$, let dv_F be the form over the fiber F_Z given by

$$dv_F = B^{-(r-s)/2}\{dZ_3''\}\{dZ_4''\}\left\{d\left(\frac{Z_2 + {}^tZ_2}{2}\right)\right\}. \tag{5.36}$$

By our construction, it is also G_X-invariant. From (5.22), (5.35) and (5.36), we have the following decomposition of $dv_{\mathscr{D}}$, with respect to the fibered structure mentioned above,

$$dv_{\mathscr{D}} = (B/A)^{(p+q)/2}\, dv_F\, dv_{\mathscr{D}_1}. \tag{5.37}$$

5.13. Let $\mathscr{S}_l^+$ be the space of all positive definite real $l \times l$ matrices. The group $\mathrm{GL}(l, \mathbf{R})$ acts on $\mathscr{S}_l^+$ by

$$T[g] = {}^t gTg, \qquad T \in \mathscr{S}_l^+,\ g \in \mathrm{GL}(l, \mathbf{R}).$$

Let $d\mu(T)$ be the measure on $\mathscr{S}_l^+$ defined by

$$d\mu(T) = \det(T)^{-(l+1)/2}\{dT\}, \tag{5.38}$$

where $\{dT\}$ is the Euclidean measure on $\mathscr{S}_l^+$. It is easy to see that $d\mu(T)$ is $\mathrm{GL}(l, \mathbf{R})$-invariant and moreover it satisfies the condition

$$d\mu(T) = d\mu(T^{-1}). \tag{5.39}$$

For $T \in \mathscr{S}_l^+$, we can express

$$T = D[B], \tag{5.40}$$

where

$$D = \mathrm{diag}(\lambda_1, \cdots, \lambda_l) \qquad (\lambda_i > 0),$$

$$B = \begin{pmatrix} 1 & b_{12} & \cdots & b_{1l} \\ & 1 & \cdots & b_{2l} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \qquad (b_{ij} \in \mathbf{R}).$$

With respect to this parametrization,

$$\{dT\} = \lambda_1^{l-1} \cdots \lambda_{l-1}\, d\lambda_1 \cdots d\lambda_l \prod_{i<j\leqslant l} db_{ij}. \tag{5.41}$$

Lemma 5.13. *For $\eta \in \mathscr{S}_l^+$, we have the integral*

$$\int_{\mathscr{S}_l^+} e^{-\mathrm{tr}(\eta T)} \det(T)^a \{dT\} = \Gamma_l(a) \det(\eta)^{-(l+1+2a)/2},$$

where

$$\Gamma_l(a) = \pi^{l(l-1)/4} \prod_{i=1}^{l} \Gamma\left(\frac{2a+1+i}{2}\right).$$

Proof. Changing the variable $W = \eta^{1/2} T \eta^{1/2}$ in the integral, we obtain that the integral has the value

$$\det(\eta)^{-(1+l+2a)/2} \int_{\mathscr{S}_l^+} e^{-\mathrm{tr}(W)} \det(W)^a \{dW\}$$

$$= \det(\eta)^{-1(1+l+2a)/2} \pi^{l(l-1)/4} \prod_{i=1}^{l} \Gamma\left(\frac{2a+1+i}{2}\right).$$

5.14. For $X = \binom{E_r}{0} \in M_{nr}(\mathbf{R})$, clearly we have

$$(5.42)\qquad \begin{aligned} \langle X, X\rangle &= \begin{pmatrix} 0 & 0 \\ 0 & E_{r-s} \end{pmatrix}, \\ \langle X, N(Z)\rangle &= \begin{pmatrix} 0 & E_s \\ Z_3'' & Z_4'' \end{pmatrix}. \end{aligned}$$

For $Z \in F$, recall that Z satisfies condition (5.31)

$$\begin{aligned} &Z_1 = 0, \\ &Z_2 = S, \qquad {}^tS = S, \\ &Z_3' = 0, \qquad Z_4' = 0. \end{aligned}$$

Let $Z_3'' = W$ and $Z_4'' = V$. A straightforward computation shows that

$$(5.43)\qquad \begin{aligned} -\langle N(Z), N(Z)\rangle &= \begin{pmatrix} E - {}^tWW & -{}^tWV \\ -{}^tVW & -2S - {}^tVV \end{pmatrix} \\ &= \begin{pmatrix} E - {}^tWW & 0 \\ 0 & T \end{pmatrix} \left[\begin{pmatrix} E & -(E - {}^tWW)^{-1}\,{}^tWV \\ 0 & E \end{pmatrix} \right], \end{aligned}$$

where

$$T = -\left\{ 2S + {}^tVV + {}^tVW(E - {}^tWW)^{-1}\,{}^tWV \right\}.$$

Thus (5.21) and (5.43) yield

$$(5.44)\qquad A = \det(E - {}^tWW)\det(T).$$

5.15. Set

$$(5.45)\qquad \begin{aligned} \langle X_Z, X_Z\rangle &= \langle X, N(Z)\rangle \langle N(Z), N(Z)\rangle^{-1} \langle N(Z), X\rangle, \\ \langle X_{Z^\perp}, X_{Z^\perp}\rangle &= \langle X, X\rangle - \langle X_Z, X_Z\rangle. \end{aligned}$$

From (5.42) and (5.43),

$$(5.46)\qquad -\langle X_Z, X_Z\rangle = \begin{pmatrix} (E - {}^tWW)^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} \left[\begin{pmatrix} 0 & {}^tW \\ E & {}^tV(E - W{}^tW)^{-1} \end{pmatrix} \right].$$

Now make the change of variable

$$(5.47)\qquad U = (E - W{}^tW)^{-1}V.$$

Note that $B = 1$ on F. By the formula (5.36),

$$(5.48)\qquad \begin{aligned} &A^{-(p+q)/2}dv_F \\ &\quad = A^{-(p+q)/2}\det(E - W{}^tW)^s 2^{-s(s+1)/2}\{dW\}\{dU\}\{dT\} \end{aligned}$$

on F.

In terms of W, U, T,

$$-\langle X_Z, X_Z\rangle = \begin{pmatrix} T^{-1} & T^{-1}{}^tU \\ UT^{-1} & UT^{-1}{}^tU + W(E - {}^tWW)^{-1}{}^tW \end{pmatrix};$$

$$(5.49) \qquad \langle X_{Z^\perp}, X_{Z^\perp}\rangle = \begin{pmatrix} T^{-1} & T^{-1}{}^tU \\ UT^{-1} & UT^{-1}{}^tU + (E - W{}^tW)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} T^{-1} & 0 \\ 0 & (E - W{}^tW)^{-1} \end{pmatrix} \left[\begin{pmatrix} E & {}^tU \\ 0 & E \end{pmatrix}\right]$$

by (5.42). This implies in particular,

$$(5.50) \qquad \det(X_{Z^\perp}, X_{Z^\perp}) = \det T^{-1} \det(E - W{}^tW)^{-1} = A^{-1}.$$

Finally we set $L = W(E - {}^tWW)^{-1/2}$. From Theorem 5.6,

$$\{dL\} = \det(E - {}^tWW)^{-(r+q-2s+1)/2}\{dW\}$$

and consequently by (5.48) we have the expression

$$(5.51) \qquad \begin{aligned} &A^{-(p+q)/2} dv_F \\ &\quad = 2^{-s(s+1)/2} A^{-(p+q)/2} \det(E - {}^tWW)^{(r+q+1)/2}\{dL\}\{dU\}\{dT\} \end{aligned}$$

on F.

5.15. Now we are ready to consider integrals of the form

$$(5.52) \quad I(\eta, a) = \int_{\Gamma_X \backslash \mathscr{D}} \det\langle X_{Z^\perp}, X_{Z^\perp}\rangle^{a-(p-r-1)/2} e[-2i \operatorname{tr} \eta\langle X_Z, X_Z\rangle]\, dv_{\mathscr{D}},$$

where $\eta \in \mathscr{S}_r^+$ and Γ_X is a lattice of G_X. To evaluate the integral, we appeal to the decomposition of $dv_{\mathscr{D}}$ ((5.37)). Clearly the function involved in the integration is G_X-invariant. It follows that

$$I(\eta, a) = J(\eta, a)\operatorname{vol}(\Gamma_X \backslash \mathscr{D}_1),$$

$$(5.53) \quad \begin{aligned} J(\eta, a) = \int_F &\det\langle X_{Z^\perp}, X_{Z^\perp}\rangle^{a-(p-r-1)/2} e[-2i \operatorname{tr} \eta\langle X_Z, X_Z\rangle] \\ &\cdot A^{-(p+q)/2} dv_F. \end{aligned}$$

By (5.50) and (5.51),

$$(5.54) \quad \begin{aligned} J(\eta, a) = 2^{-s(s+1)/2} \int_F &(\det T)^{-a-(q+r+q)/2} \det(E + {}^tLL)^a \\ &\cdot e[-2i \operatorname{tr} \eta\langle X_Z, X_Z\rangle] \cdot \{dL\}\{dU\}\{dT\}, \end{aligned}$$

with $T \in \mathscr{S}_s^+$, $U \in M_{r-s,s}(\mathbf{R})$ and $L \in M_{r-s,q-s}(\mathbf{R})$. Decompose η into the block form

$$\eta = \begin{pmatrix} \eta_{11} & \eta_{12} \\ {}^t\eta_{12} & \eta_{22} \end{pmatrix},$$

where $\eta_{11} \in M_{ss}(\mathbf{R})$ and $\eta_{22} \in M_{r-s,r-s}(\mathbf{R})$. Then we have that

$$\begin{aligned}(5.55)\qquad -\operatorname{tr}(\eta\langle X_Z, X_Z\rangle) &= \operatorname{tr}\left(\eta_{11}T^{-1} + \eta_{12}UT^{-1}\right)\\ &\quad + \operatorname{tr}\left({}^t\eta_{12}T^{-1}\,{}^tU + \eta_{22}L{}^tL + \eta_{22}UT^{-1}\,{}^tU\right).\end{aligned}$$

Set $Y = \eta_{22}^{1/2}UT^{-1/2}$. We have

$$\begin{aligned}(5.56)\qquad &\operatorname{tr}\left(\eta_{12}UT^{-1}\right) + \operatorname{tr}\left({}^t\eta_{12}T^{-1}{}^tU + \eta_{22}UT^{-1}{}^tU\right)\\ &\quad = \operatorname{tr}\left(Y + \eta_{22}^{-1/2}{}^t\eta_{12}T^{-1/2}\right)\left({}^tY + T^{-1/2}\eta_{12}\eta_{22}^{-1/2}\right)\\ &\quad - \operatorname{tr}\left(\eta_{12}\eta_{22}^{-1}{}^t\eta_{12}T^{-1}\right).\end{aligned}$$

We integrate (5.54) first over U. By (5.55) and (5.56),

$$J(\eta, a) = 2^{-(r-s)s-s(s+1)/2}\det(\eta_{22})^{-s/2}J_1 \cdot J_2,$$

where

$$\begin{aligned}(5.57)\qquad J_1 &= \int_{\mathscr{S}_s^+} (\det T)^{-a-(q+s+1)/2} e\left[2i\operatorname{tr}\left(\left(\eta_{11} - \eta_{22}\eta_{22}^{-1}{}^t\eta_{12}\right)T^{-1}\right)\right]\{dT\},\\ J_2 &= \int_{M_{r-s,q-s}(\mathbf{R})} \det(E + {}^tLL)^a e\left[2i\operatorname{tr}\left(\eta_{22}L{}^tL\right)\right]\{dL\}.\end{aligned}$$

Assume that $a = 0$. By Lemma 5.13,

$$\begin{aligned}J_1 &= \int_{\mathscr{S}_s^+} e\left[2i\operatorname{tr}\left(\left(\eta_{11} - \eta_{11}\eta_{22}^{-1}{}^t\eta_{12}\right)T^{-1}\right)\right](\det T)^{-q/2}(\det T)^{-(s+1)/2}\{dT\}\\ (5.58)\qquad &= \det\left(\eta_{11} - \eta_{12}\eta_{22}^{-1}{}^t\eta_{12}\right)^{-q/2}\int_{\mathscr{S}_s^+} e[2i\operatorname{tr} T](\det T)^{(q-s-1)/2}\{dT\}\\ &= (4\pi)^{-qs/2}\det\left(\eta_{11} - \eta_{12}\eta_{22}^{-1}{}^t\eta_{12}\right)^{-q/2}\prod_{i=1}^{s}\Gamma\left(\frac{q-s+i}{2}\right)\pi^{s(s-1)/4}.\end{aligned}$$

The value of J_2 in this case is easily seen to be

$$(5.59)\qquad J_2 = \det(\eta_{22})^{-(q-s)/2}2^{-(r-s)(q-s)}.$$

Observe that

$$\det\eta = \det\eta_{22}\det\left(\eta_{11} - \eta_{12}\eta_{22}^{-1}{}^t\eta_{12}\right),$$

and by (5.57)–(5.59)

$$J(\eta, 0) = 2^{-rq-s(s+1)/2}\pi^{s(s-2q-1)/4}\det(\eta)^{-q/2}\prod_{i=1}^{s}\Gamma\left(\frac{q-s+i}{2}\right).$$

Theorem 5.15. *We have the integrals*

$$\begin{aligned}&\int_{\Gamma_X\backslash I\mathscr{D}} \det\langle X_{Z^\perp}, X_{Z^\perp}\rangle^{-(p-r-1)/2} e\left[-2i\operatorname{tr}\eta\langle X_Z, X_Z\rangle\right] dv_{\mathscr{D}}\\ &\quad = 2^{-rq-s(s+1)/2}\pi^{s(s-2q-1)/4}\prod_{i=1}^{s}\Gamma\left(\frac{q-s+i}{2}\right)\det(\eta)^{-q/2}\operatorname{vol}(\Gamma_X\backslash\mathscr{D}_1).\end{aligned}$$

5.16. Let $M \in M_{nr}(\mathbf{R})$ such that $\dim\langle M\rangle = r$ and $\langle M, M\rangle \geqslant 0$. Let s be the dimension of maximal isotropic subspaces of $\langle M\rangle$. With respect to proper basis, we may assume that $Q = Q_2$ and there exists $g \in G$ with $g\langle M\rangle = \langle X\rangle$, where $X = \binom{E_r}{0}$. Let $I(\eta, M)$ be the integral

(5.60)
$$I(\eta, M) = \int_{\Gamma_M\backslash\mathscr{D}} \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-(p-r-1)/2} e\left[-2 \operatorname{tr} \eta i\langle M_Z, M_Z\rangle\right] dv_{\mathscr{D}}.$$

As $g\langle M\rangle = \langle X\rangle$, there exists $\beta \in \mathrm{GL}(r, \mathbf{R})$ satisfying the condition

(5.61)
$$gM = X\beta.$$

By the property of invariance of $dv_{\mathscr{D}}$ and functions in the integral $I(\eta, M)$, it follows that

(5.62)
$$\begin{aligned} I(\eta, M) &= \int_{g\Gamma_M g^{-1}\backslash\mathscr{D}} \det\langle gM_{Z^\perp}, gM_{Z^\perp}\rangle^{-(p-r-1)/2} \\ &\qquad \cdot e\left[-2 \operatorname{tr} \eta i\langle gM_Z, gM_Z\rangle\right] dv_{\mathscr{D}} \\ &= |\det\beta|^{-(p-r-1)} I({}^t\beta\eta\beta, X) \quad \text{(by substitution (5.61))}. \end{aligned}$$

By Theorem 5.15 (on $I({}^t\beta\eta\beta, X)$), (5.62) yields that

(5.63)
$$I(\eta, M) = c|\det\beta|^{-(p+q-r-1)}(\det\eta)^{-q/2}\operatorname{vol}(\Gamma_M\backslash\mathscr{D}_M),$$

where c is given by

$$c = 2^{-rq-s(s+1)/2}\pi^{s(s-2q-1)/4}\prod_{i=1}^{s}\Gamma\left(\frac{q-s+i}{2}\right).$$

6. Theta functions and the geometric lifting

In this section, we recall some of the basic results on Weil representations and theta functions for the reductive pair $O(p,q) \times \mathrm{Sp}(2r, \mathbf{R})$ in a form convenient for our presentation of the geometric lifting.

First we briefly discuss spherical polynomials with respect to a quadratic form. The result will be used to derive the desired transformation formulas for our theta functions.

6.1. Let A be a nondegenerate symmetric $n \times n$ matrix and let Q be the quadratic form $Q(x) = {}^t xAx$. Denote by Δ_Q the Laplacian

(6.1)
$$\Delta_Q = \sum_{1\leqslant i,j\leqslant n} b_{ij}\frac{\partial^2}{\partial x_i \partial x_j},$$

where $A^{-1} = (b_{ij})$.

Definition 6.1. A polynomial $f(x)$ is called a spherical polynomial with respect to Q if $\Delta_Q f = 0$.

6.2. Since A is nondegenerate symmetric, there is a matrix B with $A = {}^tBB$. By changing variables $y = Bx$, one can translate the assertions for spherical polynomials with respect to ${}^tyy$ to the general case.

Theorem 6.2 [28, *Theorem* 18]. *Let $f(x)$ be a homogeneous polynomial of degree l in $x_1, \cdots, x_n$ with complex coefficients. Then the following statements are equivalent*:

(i) *$f(x)$ is a spherical polynomial with respect to ${}^txAx$.*

(ii) *f is a linear sum of functions of the form $({}^t\zeta Ax)^l$ with ${}^t\zeta A\zeta = 0$.*

6.3. Let $f(x)$ be a polynomial in $x_1, \cdots, x_n$ and $A \in \mathrm{GL}(n, \mathbf{C})$. Set

$$(6.2) \qquad (Af)(x) = f(A^{-1}x), \qquad f\left(\frac{\partial}{\partial x}\right) = f\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right).$$

Lemma 6.3. *Let $f(x)$ be a homogeneous spherical polynomial of degree l with respect to ${}^txAx$. Then we have the condition*

$$(Af)\left(\frac{\partial}{\partial x}\right)e^{\lambda {}^txAx} = (2\lambda)^l f(x) e^{\lambda {}^txAx} \qquad (\lambda \in \mathbf{C}).$$

Proof. By (ii) of Theorem 6.2, we may assume that f is a linear sum of polynomials of the form $({}^t\zeta Ax)^l$ with ${}^t\zeta A\zeta = 0$, $\zeta \in \mathbf{C}^n$. The assertion is easily verified in this case by straightforward differentiation.

6.4. Let N be the $p \times r$ matrix of indeterminates and $f(N)$ a polynomial satisfying

$$(6.3) \qquad f(Nh) = \det(h)^q f(N) \qquad (h \in \mathrm{GL}(r, \mathbf{C})).$$

Proposition 6.4. *Let $f(N)$ be a spherical polynomial with respect to* $\mathrm{tr}({}^tNN)$, *and ${}^tA = A \in M_{rr}(\mathbf{C})$. Then we have the condition*

$$f\left(\frac{\partial}{\partial N}\right)e^{\lambda \mathrm{tr}({}^tNNA)} = (2\lambda)^{rq} f(N)(\det A)^q e^{\lambda \mathrm{tr}({}^tNNA)}.$$

Proof. From (6.3), f is homogeneous of degree rq. By continuity, we may assume that A is nondegenerate. Choose B such that $A = B{}^tB$. Let $M = NB$. Then $f(M)$ is a spherical polynomial with respect to $\mathrm{tr}\,{}^tMM$. It follows that

$$(6.4) \qquad f\left(\frac{\partial}{\partial M}\right)e^{\lambda \mathrm{tr}({}^tMM)} = (2\lambda)^{rq} f(M) e^{\lambda \mathrm{tr}({}^tMM)}.$$

However we have that

$$(6.5) \qquad \begin{aligned} f(M) &= f(NB) = (\det B)^q f(N), \\ \frac{\partial}{\partial M} &= \left(\frac{\partial}{\partial N}\right){}^tB^{-1}, \\ f\left(\frac{\partial}{\partial M}\right) &= f\left(\frac{\partial}{\partial N}\right)(\det B)^{-q}. \end{aligned}$$

By (6.4) and (6.5),

$$f\left(\frac{\partial}{\partial N}\right)e^{\lambda \operatorname{tr}({}^tNNA)} = (2\lambda)^{rq} f(N)(\det A)^q e^{\lambda \operatorname{tr}({}^tNNA)}.$$

Now we return to the discussion on Weil representations. For convenience, we explain briefly the pertinent results in [23] and notations used here.

6.5. Let R be a finite dimensional commutative algebra over $\mathbf{R}$. Assume that R is isomorphic to $\mathbf{R} \oplus \cdots \oplus \mathbf{R}$ (m copies). Let V be a finite R-module and $\langle\ ,\ \rangle$: $V \times V \to R$ a nondegenerate bilinear form. Denote by $O(V)$ the orthogonal group

$$O(V) = \{g \in \mathrm{GL}_R(V) \mid g \text{ preserves } \langle\ ,\ \rangle\}.$$

Let e_i be the irreducible idempotents of R and let $V^{(i)} = e_i V$ $(i = 1,\cdots,m)$. Clearly we have the conditions:

$$V = \bigoplus_{i=1}^{m} V^{(i)}, \qquad O(V) = \prod_{i=1}^{m} O(V^{(i)}).$$

Now let

$$V^r = V \oplus \cdots \oplus V \quad (r \text{ copies}).$$

For $X = (X_1,\cdots,X_r)$ and $Y = (Y_1,\cdots,Y_r)$, define an $r \times r$ matrix with entries in R by

$$\langle X, Y\rangle = (\langle X_i, Y_j\rangle) \in M_{rr}(R). \tag{6.6}$$

We have a bilinear form σ: $V^r \times V^r \to \mathbf{R}$,

$$\sigma(X, Y) = \operatorname{tr}_{R/\mathbf{R}}(\operatorname{tr}\langle X, Y\rangle), \tag{6.7}$$

and by which V^r is identified with its dual. We introduce an alternating form A on $V^r \times V^r$ by

$$A(Z, Z') = \sigma(X, Y') - \sigma(Y, X'), \tag{6.8}$$

where $Z = (X, Y)$ and $Z' = (X', Y') \in V^r \times V^r$.

Let J be the skew symmetric matrix

$$J = \begin{pmatrix} 0 & E_r \\ -E_r & 0 \end{pmatrix}$$

and $\mathrm{Sp}(2r, R)$ the symplectic group

$$\mathrm{Sp}(2r, R) = \{g \in \mathrm{GL}(2r, R) \mid {}^t g J g = J\}.$$

The symmetric space associated to $\mathrm{Sp}(2r, R)$ is realized as the Siegel upper half space $\mathscr{H}_r(R)$ of genus r given by

$$\mathscr{H}_r(H) = \{{}^t\tau = \tau \in M_r(R \otimes_{\mathbf{R}} \mathbf{C}) \mid (\tau - {}^t\tau)/i > 0\}. \tag{6.9}$$

Here an element $v \in M_r(R)$ is positive if $e_j v > 0$ ($j = 1, \cdots, m$). Clearly

$$\mathscr{H}_r(R) \simeq \mathscr{H}_r(\mathbf{R})^m.$$

For $g \in \mathrm{Sp}(2r, R)$, set $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in M_r(R)$.

On $\mathscr{H}_r(R)$, $\mathrm{Sp}(2r, R)$ acts by fractional linear transformation

$$g\tau = (a\tau + b)(c\tau + d)^{-1}. \tag{6.10}$$

We have an automorphic factor

$$j(g, \tau) = \det(c\tau + d) \in R. \tag{6.11}$$

For $x \in R$,

$$x = \sum_{i=1}^{m} x_i e_i$$

with $x_i \in \mathbf{R}$ ($i = 1, \cdots, m$).

Given an invertible $x \in R$ and m-tuple $t = (t_1, \cdots, t_m)$ of integers (or half integers), let

$$x^t = x_1^{t_1} \cdots x_m^{t_m}. \tag{6.12}$$

Then we write

$$j(g, \tau)^t = (\det(c\tau + d))^t \tag{6.13}$$

by the above convention.

6.6. We have a homomorphism ρ of $O(V) \times \mathrm{Sp}(2r, R)$ into $\mathrm{Sp}(V^r \times V^r, A)$ defined by

$$\begin{aligned} &(X, Y)\rho(g) = (g^{-1}X, g^{-1}Y), \ g \in O(V), \\ &(X, Y)\rho(g) = (Xa + Yc, Xb + Ya), \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2r, R). \end{aligned} \tag{6.14}$$

Then the Weil representation of $\mathrm{Sp}(V^r \times V^r, A)$ on $L^2(V^r)$ gives rise to a representation (projective unitary representation) of $O(V) \times \mathrm{Sp}(2r, R)$ via ρ. This representation r will be normalized as in [30].

Now introduce subsets Ω, Ω^0 and Ω' of $\mathrm{Sp}(2r, R)$ defined by ($g = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$):

$$\begin{aligned} &\Omega = \{ g \in \mathrm{Sp}(r, R) | c \text{ is nonsingular} \}, \\ &\Omega^0 = \{ g \in \mathrm{Sp}(r, R) | c = 0 \}, \\ &\Omega' = \{ g \in \mathrm{Sp}(r, R) | c \neq 0, \ c \text{ is singular and } d \text{ is nonsingular} \}. \end{aligned} \tag{6.15}$$

Let (p_i, q_i) be the signature of $\langle\ ,\ \rangle | V^{(i)}$ and $\mathscr{D}$ the space of maximal negative definite subspaces of V. For $Z \in \mathscr{D}$, $Z^\perp$ is the orthogonal complement of Z in V. Now we define the majorant $\langle\ ,\ \rangle_Z$ of $\langle\ ,\ \rangle$ on V^r by

$$\langle X, X \rangle_Z = \begin{cases} \langle X, X \rangle & \text{if } X_i \in Z^\perp \\ -\langle X, X \rangle & \text{if } X_i \in \langle Z \rangle \end{cases} \qquad (1 \leqslant i \leqslant r). \tag{6.16}$$

For $\tau \in \mathscr{H}_r(R)$, write

$$\tau = u + iv \tag{6.17}$$

with ${}^t u = u$, ${}^t v = v \in M_r(R)$ and $v > 0$. Define a Schwartz function

$$e_{\tau,Z}(X) = e\left[\tfrac{1}{2}\operatorname{tr}_{R/\mathbf{R}}\left(\operatorname{tr}(u\langle X, X\rangle + iv\langle X, X\rangle_Z)\right)\right]. \tag{6.18}$$

Here $e[x] = \exp(2\pi\sqrt{-1}\,x)$ ($x \in \mathbf{R}$).

Let p and q be the tuples

$$p = (p_1, \cdots, p_m), \qquad q = (q_1, \cdots, q_m). \tag{6.19}$$

6.7. Lemma [23, *Lemma* 8.1]. *For* $g = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \operatorname{Sp}(2r, R)$,

$$r(g)e_{\tau,Z}(X) = \varepsilon(g)\,j(g,\tau)^{(q-p)/2}\left|j(g,\tau)^{-q}\right|e_{g\tau,Z}(X),$$

where

$$\varepsilon(g) = \begin{cases} \left(i^k \operatorname{sgn}\det(c)\right)^{(p-q)/2}, & \textit{if } g \in \Omega, \\ \left(\operatorname{sgn}\det(a)\right)^{(p-q)/2}, & \textit{if } g \in \Omega^0, \\ \varepsilon(g\omega^{-1})\varepsilon(\omega)\,j(g,\tau)^{(p-q)/2}\,j(g\omega^{-1},\omega\tau)^{(q-p)/2}\,j(\omega,\tau)^{(q-p)/2}, & \\ \qquad \textit{if } g \in \Omega'. & \end{cases}$$

Here

$$\omega = \begin{pmatrix} 0 & -E_r \\ E_r & 0 \end{pmatrix}$$

and we use the convention (6.12) and (6.13) for the exponential notations.

6.8. Let k be a totally real number field with $[k:\mathbf{Q}] = m$. Let W be an n-dimensional vector space over k and $\langle\ ,\ \rangle$: $W \times W \to k$ a nondegenerate bilinear form. Let $R = k \otimes_{\mathbf{Q}} \mathbf{R}$ and $V = W \otimes_{\mathbf{Q}} \mathbf{R}$. Clearly $R \simeq \mathbf{R} \oplus \cdots \oplus \mathbf{R}$ (m copies) and $\langle\ ,\ \rangle$ extends to a nondegenerate bilinear form $V \times V \to R$. Let $\mathcal{O}$ be the ring of integers of k and L_0 an $\mathcal{O}$-lattice of W. Assume that $\langle L_0, L_0\rangle \subset \mathcal{O}$. Then L_0 is contained in its dual lattices

$$L_0^* = \left\{w \in W \middle| \operatorname{tr}_{k/\mathbf{Q}}\langle w, L_0\rangle \subset \mathbf{Z}\right\}.$$

Now set

$$\begin{aligned} L &= L_0^r \subset V^r, \\ L^* &= \left\{v \in V^r \middle| \operatorname{tr}_{R/\mathbf{R}}\operatorname{tr}\langle v, L\rangle \subset \mathbf{Z}\right\}. \end{aligned} \tag{6.19}$$

Let $f_\tau(X)$, $\tau \in \mathscr{H}_r(R)$, be Schwartz functions satisfying the condition

$$r(g)f_\tau = \varepsilon(g)\,j(g,\tau)^{-l}\left|j(g,\tau)^{-t}\right|f_{g\tau}, \tag{6.20}$$

where $\varepsilon(g)$ is given in Lemma 6.7, and

$$l = (l_1, \cdots, l_m) \in \left(\tfrac{1}{2}\mathbf{Z}\right)^m, \qquad t = (t_1, \cdots, t_m) \in (\mathbf{Z})^m.$$

Define $f_\tau^* = (\det v)^{t/2} f_\tau$, where v is the imaginary part of τ ((6.17)). Then f_τ^* satisfies the condition

$$r(g) f_\tau^* = \varepsilon(g) j(g, \tau)^{-l} f_{g\tau}^*. \tag{6.21}$$

For $h \in L^*/L$, define a theta function

$$\theta(\tau, f, h) = \sum_{X \equiv h(L)} f_\tau^*(X). \tag{6.22}$$

Let N be a positive integer such that

$$\begin{aligned} &N L_0^* \subset L_0, \\ &N \operatorname{tr}_{k/\mathbf{Q}}(\langle X, Y \rangle) \equiv 0 \pmod 2 \quad \text{for } X, Y \in L_0^*. \end{aligned} \tag{6.23}$$

Denote by $\Gamma_0(N)$ the subgroup of $\mathrm{Sp}(2r, \mathcal{O})$ consisting of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that

$$\begin{aligned} &c \equiv 0\ (N), \\ &\sigma(\operatorname{tr}\langle Xa, Xb \rangle) \equiv 0\ (2) \\ &\sigma(\operatorname{tr}\langle Xc, Xd \rangle) \equiv 0\ (2) \qquad (X \in L). \end{aligned} \tag{6.24}$$

Here $\sigma(x) = \operatorname{tr}_{R/\mathbf{R}}(\operatorname{tr}(x))$.

Now let $\tilde{\Gamma}(N)$ be the subgroup of $\Gamma_0(N)$ consisting of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a \equiv E_r$, $b \equiv 0\ (N)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \tilde{\Gamma}(N)$, let

$$\begin{aligned} \chi(\gamma) = {}& \varepsilon(\gamma)^{-1} c(\gamma\omega^{-1}, \omega)\left(N_{R/\mathbf{R}}(\det d)\right)^{-1/2} \\ & \cdot \sum_{l \in L/L'd} e\left[\tfrac{g}{2}(\langle l, lbd^{-1} \rangle)\right]. \end{aligned} \tag{6.25}$$

Then we have the following transformation formula [23, Proposition 8.4].

Proposition 6.8. *Let $\gamma \in \tilde{\Gamma}(N)$, $f_\tau \in \mathcal{S}(V^r)$ satisfying* (6.20) *and $\theta(\tau, f, h)$ defined as in* (6.22). *Then we have*

$$\theta(\gamma\tau, f, h) = \chi(\gamma) j(\gamma, \tau)^l \theta(\tau, f, h).$$

6.9. Now we assume that

$$\begin{aligned} &\operatorname{sgn}\left(\langle\,,\,\rangle \middle| V^{(1)}\right) = (p, q) \qquad (p + q = n), \\ &\operatorname{sgn}\left(\langle\,,\,\rangle \middle| V^{(j)}\right) = (n, 0) \qquad (j = 2, \cdots, m). \end{aligned} \tag{6.26}$$

Let $G = O_+(V^{(1)}) \times \prod_{j=2}^m O(V^{(j)})$, where $O_+(V^{(1)})$ is the subgroup of $O(V^{(1)})$ consisting of g satisfying

$$\det(j(g, Z)) > 0 \qquad (Z \in \mathcal{D}).$$

Here $j(g, Z)$ is the automorphic factor defined in (1.3).

In this section, up to now, we have used X for a variable in V^r. Now we switch back in notation from X to M. For any $M \in V^r$, let $M^{(1)} = e_1 M$. By choosing an orthonormal basis in $V^{(1)}$, we may assume that $O(V^{(1)}) = O(p,q)$ and our discussions in previous sections are applicable to $O(V^{(1)})$. Now let $M^{(1)} \mapsto F(Z, M^{(1)})$ be the polynomial defined in §4.6. Here we introduce

$$f_{\tau,Z}(M) = F(Z, M^{(1)})e_{\tau,Z}(M), \tag{6.27}$$

where $e_{\tau,Z}(M)$ is the exponential function given by (6.18). In the following, we assume that $r \leqslant [p/2]$.

Lemma 6.9. *For $g \in \Omega^0 \cup \Omega' \cup \Omega \subset \mathrm{Sp}(2r, R)$,*

$$r(g)f_{\tau,Z}(M) = \varepsilon(g)j(g,\tau)^{-P}\left|j(g,\tau)^{-Q}\right|f_{g\tau,Z}(M),$$

where $P = (n/2, n/2, \cdots, n/2)$, and $Q = (q, 0, \cdots, 0)$.

Proof. By our construction, $F(Z, M^{(1)})$ is a differential form of degree rq of highest signature

$$\omega_0 = \Big(\underbrace{q,\cdots,q}_{r},\underbrace{0,\cdots,0}_{[p/2]-r}s\Big),$$

and $F(Z, M^{(1)})$ is actually a polynomial in $M^{(1)}_Z$. For $g \in O_+(V^{(1)})$, by (i) of Lemma 4.7,

$$g^*F(Z, M^{(1)}) = F\big(Z, g^{-1}M^{(1)}\big).$$

This implies that $F(Z, M^{(1)})$ as a polynomial in $M^{(1)}_Z$ also lies in the isotypic component of signature ω_0.

It yields by Theorem 3.11, that $F(Z, M^{(1)}_Z)$ is a spherical polynomial in $M^{(1)}_Z$ with respect to the bilinear form $\mathrm{tr}(\langle\ ,\ \rangle)\,|\,(Z^{\perp})^r$. Moreover by (iii) of Lemma 4.7, $F(Z, M^{(1)})$ satisfies the condition

$$F(Z, M^{(1)}h) = \det(h)^q F(Z, M^{(1)}),$$

$h \in \mathrm{GL}(r)$. Now let x stand for $M^{(1)}_Z$. Consider coordinates of x with respect to an orthonormal basis. By Proposition 6.4,

$$\begin{aligned} &F(Z, \partial/\partial x)\left\{e\left[\tfrac{1}{2}\sigma(\mathrm{tr}\langle M+x, M+x\rangle S)\right]e_{\tau,Z}(M+x)\right\}_{x=0} \\ &\quad = (2\pi i)^{rq}F\big(Z, M^{(1)}_Z\big)\det(S+\tau)^Q e\left[\tfrac{1}{2}\sigma(\mathrm{tr}\langle M, M\rangle S)\right]e_{\tau,Z}(M) \\ &\qquad\qquad \big({}^tS = S \in M_r(R)\big). \end{aligned} \tag{6.28}$$

Clearly we also have the condition

$$\begin{aligned} &F(Z, \partial/\partial x)e\left[\sigma\,\mathrm{tr}(\langle M, X\rangle T)\right] \\ &\quad = (2\pi i)^{rq}F\big(Z, M^{(1)}_Z\big)(\det T)^Q e\left[\sigma\,\mathrm{tr}(\langle M, X\rangle T)\right], T \in M_r(R). \end{aligned} \tag{6.29}$$

Now as a consequence of Lemma 6.7, (6.28), (6.29) and a standard argument as in [20, Proposition 4.2], our assertion follows.

6.10. For $h \in L^*/L$, we define a theta function

$$\theta(\tau, h, Z) = \det(v)^{(q/2,0,\cdots,0)} \sum_{M \equiv h(L)} f_{\tau,Z}(M). \tag{6.30}$$

Proposition 6.10. *The theta function satisfies the conditions*:

(i) $\theta(\gamma\tau, h, Z) = \chi(\gamma) j(\gamma, \tau)^P \theta(\tau, h, Z)$, $\gamma \in \tilde{\Gamma}(N)$.

(ii) *If* $g \in G$ *satisfies the conditions* (a) $gL = L$, *and* (b) g *acts trivially on* L^*/L, *then*

$$g^*\theta(\tau, h, Z) = \theta(\tau, h, Z).$$

Proof. Immediate from Proposition 6.8 and Lemma 6.9.

7. The geometric interpretation of the lifting map

7.1. Recall that $G = O_+(V^{(1)}) \times \prod_{j=2}^m O(V^{(j)})$. Let Γ be the subgroup of G given by

$$\Gamma = \left\{\gamma \in G \middle| \gamma L = L \text{ and } \gamma \text{ acts trivially on } L^*/L\right\}.$$

Replacing Γ by a subgroup of finite index if necessary, we assume that Γ is *neat*; in particular Γ is torsion free. For $M \in V^r$, let

$$\Gamma_M = \{\gamma \in \Gamma | \gamma M = M\}.$$

By our assumption, $V^{(i)}$ $(i = 2, \cdots, m)$ are positive definite, $\prod_{j=2}^m O(V^{(j)})$ is compact; hence we identify Γ with its image in $O_+(V^{(1)})$. By [2, Corollary 13.2],

$$\operatorname{vol}(\Gamma \backslash \mathscr{D}) < \infty.$$

Let $H_0^{rq}(\Gamma \backslash \mathscr{D})$ be the space of *harmonic* differential forms ϕ of $\Gamma \backslash \mathscr{D}$ of degree rq satisfying the conditions:

$$\begin{aligned} &\text{(i)} \quad \phi \text{ is of highest signature (defined in §4.9)},\\ &\text{(ii)} \quad \int_{\Gamma\backslash\mathscr{D}} \phi \wedge (*\phi) < \infty,\\ &\text{(iii)} \quad \phi \text{ is a cusp form.}\end{aligned} \tag{7.1}$$

We define a lifting map $\mathscr{L}^*$ of $H_0^{rq}(\Gamma \backslash \mathscr{D})$ by

$$\mathscr{L}^*(\phi) = \int_{\Gamma\backslash\mathscr{D}} \langle \phi, \theta \rangle = \int_{\Gamma\backslash\mathscr{D}} \phi \wedge (*\theta), \tag{7.2}$$

where θ is given in (6.30). By Proposition (6.10),

$$\mathscr{L}^*(\phi)(\gamma\tau) = \chi(\gamma) j(\gamma, \tau)^{n/2} \mathscr{L}^*(\phi)(\tau), \qquad \gamma \in \tilde{\Gamma}(N). \tag{7.3}$$

Here $n/2$ stands for $(n/2, \cdots, n/2)$. The Fourier expansion of the lifting and its geometric interpretation are the main task of this section.

7.2. Let $G = O_+(p,q)$ be as defined in §4.1, $V = \mathbf{R}^n$, $(n = p + q)$ and $M \in M_{nr}(\mathbf{R}) = V^r$. In §4.6, we have constructed a differential form $F(Z, M)$.

For differential forms α and β of $\mathscr{D}$ of the same degree, let

$$\langle \alpha, \beta \rangle = \alpha \wedge (*\beta). \tag{7.4}$$

The following lemma of $F(Z, M)$ has been established in Step 2 of the proof of Theorem 4.9.

Lemma 7.2. *There exists a nonzero constant c_1 such that*

$$\begin{aligned} \langle F(Z,M), F(Z,M) \rangle &= F(Z,M) \wedge *F(Z,M) \\ &= c_1 \det\langle M_{Z^\perp}, M_{Z^\perp} \rangle^q \, dv_{\mathscr{D}}, \end{aligned}$$

where $dv_{\mathscr{D}}$ is the invariant measure on $\mathscr{D}$.

7.3. Let $h_s(t)$ be the function defined in (1.33)

$$h_s(t) = -\int_t^\infty x^{-s}(x - r)^{qr/2 - 1}\, dx \qquad (\mathrm{Re}(s) > qr/2).$$

Recall that ω_s (Definition 1.17) is the differential form

$$\omega_s = \frac{-1}{h_s(r)} d\left(h_s\left(r + |v|^2\right)\psi\right),$$

where ψ is given in (1.23). For $M \in M_{nr}(\mathbf{R})$ with $\langle M, M \rangle > 0$, one can define ω_s as in §2 for $\mathscr{D}_{\langle M \rangle}$.

Lemma 7.3. *Let $(\omega_s)_0$ be the component of ω_s in the K_Z-irreducible space of the highest signature. Then*

$$(\omega_s)_0 = \frac{-2\det\langle M, M\rangle^{-q/2}\left(r + |v|^2\right)^{-s}}{rqh_s(r)} F(Z, M).$$

Proof. By Lemma 2.8, K yields contribution of K_Z-invariant form. From Proposition 1.13, it follows that

$$\omega_s = \frac{-2}{h_s(r)}\left(r + |v|^2\right)^{-s} |v|^{rq-1} d|v| \wedge \psi_{rq} + \text{terms involving } K.$$

The condition $u_{rq} = 0$ ((1.22)) and the definition of u_k readily imply

$$|v|^{rq-1} d|v| \wedge \psi_{rq} = \frac{1}{rq}\det\langle M, M\rangle^{-q/2} f(Z, M);$$

as a consequence, we have the desired form

$$(\omega_s)_0 = \frac{-2}{rqh_s(r)}\left(r + |v|^2\right)^{-s}\det\langle M, M\rangle^{-q/2} F(Z, M).$$

7.4. Let $M \in M_{nr}(\mathbf{R})$ with $\langle M, M \rangle > 0$. Here we consider the integral

$$(7.5)\qquad I = \int_{\Gamma_M \backslash \mathscr{D}} \left\langle \omega_s, \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-(p+2q-r-1)/2} F(Z, M)\right\rangle.$$

Since by (1.35) and Lemma 2.6, we have

$$\left\| \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-(p+2q-r-1)/2} F(Z, M)\right\| \prec (B/A)^t_{\langle M \rangle}$$

for a certain positive integer t ($t = p + q - r - 1$), the integral is absolutely convergent for $\mathrm{Re}(s) \gg 0$. Let $b = -\frac{1}{2}(p + 2q - r - 1)$. By Lemmas 7.2 and 7.3,

$$\begin{aligned} I &= \int_{\Gamma_M \backslash \mathscr{D}} \left\langle \omega_s, \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^b F(Z, M)\right\rangle \\ (7.6)\qquad &= \int_{\Gamma_M \backslash \mathscr{D}} \left\langle (\omega_s)_0, \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^b F(Z, M)\right\rangle \\ &= C_2 \det\langle M, M\rangle^{-q/2} \int_{\Gamma_M \backslash \mathscr{D}} \left(r + |v|^2\right)^{-s} \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{b+q} dv_{\mathscr{D}}, \end{aligned}$$

where

$$C_2 = \frac{-2}{rq h_s(r)} c_1.$$

Let $Y = M\langle M, M\rangle^{-1/2}$. We have that by a straightforward computation

$$(7.7)\qquad \mathrm{tr}\langle Y_{Z^\perp}, Y_{Z^\perp}\rangle = r + |v|^2.$$

Hence I is given by

$$(7.8)\quad I = C_2 \det\langle M, M\rangle^{b+q/2} \int_{\Gamma_M \backslash \mathscr{D}} \mathrm{tr}(Y_{Z^\perp}, Y_{Z^\perp})^{-s} \det\langle Y_{Z^\perp}, Y_{Z^\perp}\rangle^{b+q} dv_{\mathscr{D}}.$$

7.5. Choose an element $g \in G$ such that

$$gY = Y_0 = \begin{pmatrix} 0 \\ E_r \\ 0 \end{pmatrix},$$

where the zero matrix at the bottom is of size $q \times r$. By translation, replacing Γ_M by $\Gamma_1 = g\Gamma_M g^{-1}$, we may assume that $Y = Y_0$ and $\Gamma_M = \Gamma_1$. In this case,

$$\begin{aligned} I_1 &= \int_{\Gamma_M \backslash \mathscr{D}} \mathrm{tr}(Y_{Z^\perp}, Y_{Z^\perp})^{-s} \det\langle Y_{Z^\perp}, Y_{Z^\perp}\rangle^{b+q} dv_{\mathscr{D}} \\ (7.9)\qquad &= \int_{\Gamma_M \backslash \mathscr{D}} \frac{(B/A)^{b+q}}{\left[\mathrm{tr}\left(E_r + Z_2(E - {}^tZZ)^{-1}\,{}^tZ_2\right)\right]^s} dv_{\mathscr{D}}. \end{aligned}$$

In the following, we adopt the decomposition $Z = \binom{Z_1}{Z_2}$ with $Z_1 \in M_{p-r,q}(\mathbf{R})$ and $Z_2 \in M_{r,q}(\mathbf{R})$.

By formula (1.39), we have

$$dv_{\mathscr{D}} = (B/A)^{(p-r)/2}\, dv_{\mathscr{D}_1}\, dv_F,$$

where $dv_{\mathscr{D}_1}$ and dv_F are the volume element of $\mathscr{D}_1$ and fiber over Z_1, respectively. As everything involved in (7.9) is G_1-invariant, it follows that

$$I_1 = \mathrm{vol}(\Gamma_1/\mathscr{D}_1) \int_F \frac{(B/A)^{1/2}}{\left[\mathrm{tr}\left(E + Z_2(E - {}^tZZ)^{-1}\,{}^tZ_2\right)\right]^s}\, dv_F. \tag{7.10}$$

The integral in (7.10) is

$$\begin{aligned}
I_2 &= \int_F \frac{\det\left(E - Z_2{}^tZ_2\right)^{-1/2}}{\left[\mathrm{tr}\left(E - Z_2{}^tZ_2\right)^{-1}\right]^s}\, dv_F \\
&= \frac{1}{\Gamma(s)} \int_F \int_0^\infty \det\left(E - Z_2{}^tZ_2\right)^{-1/2} e^{-t\,\mathrm{tr}(E_r + Z_2(E - {}^tZ_2Z_2)^{-1}\,{}^tZ_2)} t^{s-1}\, dt\, dv_F \\
&= \frac{1}{\Gamma(s)} \int_0^\infty \int_{M_{rq}(\mathbf{R})} e^{-t\,\mathrm{tr}\, W\,{}^tW} \{dW\}\, e^{-\mathrm{tr}} t^{s-1}\, dt,
\end{aligned}$$

by Theorem 5.6 and consequently

$$\begin{aligned}
I_2 &= \frac{1}{\Gamma(s)} \int_{M_{rq}(\mathbf{R})} e^{-\mathrm{tr}(W\,{}^tW)} \{dW\} \int_0^\infty e^{-\mathrm{tr}} t^{s-rq/2-1}\, dt \\
&= \frac{r^{-s+rq/2}\pi^{rq/2}}{\Gamma(s)} \Gamma\left(s - \frac{rq}{2}\right).
\end{aligned} \tag{7.11}$$

Lemma 7.5. *We have the integral value*

$$\begin{aligned}
&\int_{\Gamma_M \backslash \mathscr{D}} \left\langle \omega_s, \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-(p+2q-r-1)/2} F(Z, M)\right\rangle \\
&\qquad = \frac{2\pi^{rq/2}}{rq\Gamma(rq/2)} c_1\, \mathrm{vol}(\Gamma_M \backslash \mathscr{D}_M) \det\langle M, M\rangle^{-(p+q-r-1)/2}
\end{aligned}$$

for $\langle M, M\rangle > 0$ *and* $\mathrm{Re}(s) \gg 0$.

Proof. By substitutions (7.8) to (7.11), the assertion is immediate.

7.6. In the sequel, we assume that $\mathrm{vol}(\Gamma_M \backslash G_M) < \infty$. Let f be a continuous function of $\Gamma_M \backslash \mathscr{D}$ such that

$$\int_{\Gamma_M \backslash \mathscr{D}} f(Z)\, dv_{\mathscr{D}}(Z) < \infty$$

is absolutely convergent.

Lemma 7.6. *Let* $I_M(f)(Z) = \int_{\Gamma_M \backslash G_M} f(gZ)\,dg$. *Then*

$$\int_{\Gamma_M \backslash \mathscr{D}} f(Z)\,dv_{\mathscr{D}}(Z) = \frac{1}{\operatorname{vol}(\Gamma_M \backslash G_M)} \int_{\Gamma_M \backslash \mathscr{D}} I_M(f)(Z)\,dv_{\mathscr{D}}(Z).$$

Proof. Consider the projection map $\pi\colon G \to G/K = \mathscr{D}$ and the pull back of f on G. The assertion is immediate from the corresponding obvious assertion in G.

In the sequel, we give the Haar measure on G_M such that

$$\operatorname{vol}(\Gamma_M \backslash G_M) = \operatorname{vol}(\Gamma_M \backslash \mathscr{D}_M). \tag{7.12}$$

7.7. Now let $\phi \in H_0^{rq}(\Gamma \backslash \mathscr{D})$. We consider the orbit integral

$$I_M(\phi) = \frac{1}{\operatorname{vol}(G_M/\Gamma_M)} \int_{\Gamma_M \backslash G_M} g^*\phi\,dg. \tag{7.13}$$

It is G_M-invariant. Since ϕ satisfies $d\phi = 0$ and $d(*\phi) = 0$, so does $I_M(\phi)$. By Theorem 4.9, there is a constant c_ϕ such that

$$I_M(\phi)(Z) = c_\phi \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-(p+2q-r-1)/2} F(Z, M). \tag{7.14}$$

In the following, we determine c_ϕ first in the cases $\langle M, M\rangle > 0$ or $\langle M, M\rangle \not> 0$. For the latter case, $\det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-(p+2q-r-1)/2} F(Z, M)$ has singularity, hence $c_\phi = 0$. Then we discuss the case $\langle M, M\rangle > 0$. We have that by Lemma 7.6,

$$\begin{aligned} I &= \int_{\Gamma_M \backslash \mathscr{D}} \langle \omega_s, \phi\rangle = \int_{\Gamma_M \backslash \mathscr{D}} \langle \omega_s, I_M(\phi)\rangle \\ &= c_\phi c_1 \left(\frac{2}{rq}\right) \frac{\pi^{rq/2}}{\Gamma(rq/2)} \det\langle M, M\rangle^{-(p+q-r-1)/2} \operatorname{vol}(\Gamma_M \backslash \mathscr{D}_M) \quad \text{(Lemma 7.5)}. \end{aligned}$$

While by Theorem 1.21,

$$I = \int_{\Gamma_M \backslash \mathscr{D}_M} *\phi.$$

Therefore

$$\begin{aligned} c_\phi &= c_1^{-1}\left(\frac{rq}{2}\right)\Gamma\left(\frac{rq}{2}\right)\pi^{-rq/2} \det\langle M, M\rangle^{(p+q-r-1)/2} \\ &\quad \cdot \operatorname{vol}(\Gamma_M \backslash \mathscr{D}_M)^{-1} \int_{\Gamma_M \backslash \mathscr{D}_M} *\phi. \end{aligned} \tag{7.15}$$

Here the orientation of $\mathscr{D}_M$ is given in §2.10.

7.8. Let τ be an element in the Siegel upper half space of genus r. Set $\tau = u + iv$, with ${}^t u = u$ and ${}^t v = v \in M_r(\mathbf{R})$.

Recall that $f_{\tau, Z}(M)$ is given by

$$f_{\tau, Z}(M) = F(Z, M)\, e\left[\tfrac{1}{2}\operatorname{tr}(u\langle M, M\rangle + iv\langle M, M\rangle_Z)\right]. \tag{7.16}$$

Here we study the integral

$$I(\phi, M) = \int_{\Gamma_M \backslash \mathscr{D}} \langle \phi, f_{\tau,Z}(M) \rangle .$$

Since $f_{\tau,Z}(M)$ is a G_M-invariant differential form, by Lemma 7.6,

$$\begin{aligned}\int_{\Gamma_M \backslash \mathscr{D}} \langle \phi, f_{\tau,Z}(M) \rangle &= \int_{\Gamma_M \backslash \mathscr{D}} \langle I_M(\phi), f_{\tau,M}(M) \rangle \\ &= c_\phi \int_{\Gamma_M \backslash \mathscr{D}} \left\langle \det\langle M_{Z^\perp}, M_{Z^\perp} \rangle^{-(p+2q-r-1)/2} F(Z, M), f_{\tau,Z}(M) \right\rangle \\ &\qquad\qquad \text{(from (7.14))} \\ &= c_\phi c_1 \int_{\Gamma_M \backslash \mathscr{D}} \det\langle M_{Z^\perp}, M_{Z^\perp} \rangle^{-(p-r-1)/2} e\left[-\operatorname{tr} iv \langle M_Z, M_Z \rangle\right] dv_{\mathscr{D}} \\ &\qquad \cdot e\left[\tfrac{1}{2} \operatorname{tr}(\tau \langle M, M \rangle)\right].\end{aligned} \tag{7.17}$$

Now let $\dim_{\mathbf{R}}\langle M \rangle = d$ be the dimension of the column space of M. If $d < r$, $I_M(\phi)$ is invariant under $K_{Z,M} \simeq O(p-d)$. By Theorem 3.7, the irreducible representation π of $O(p)$ with signature

$$\Big(\underbrace{a, \cdots, a}_{r}, \underbrace{0, \cdots, 0}_{[p/2]-r}\Big)$$

satisfies the condition $I \not\leq \pi \,|\, O(p-d)$. Hence

$$c_\phi = 0 \quad \text{if } \dim_{\mathbf{R}}\langle M \rangle < r. \tag{7.18}$$

In 7.6, we know also

$$c_\phi = 0 \quad \text{if } \langle M, M \rangle \not\geq 0. \tag{7.19}$$

Thus to compute $I(\phi, M)$, it suffices to study the case $\langle M, M \rangle \geq 0$ and $\dim_{\mathbf{R}}\langle M \rangle = r$.

Lemma 7.8. *Let $I(\phi, M)$ be the integral*

$$I(\phi, M) = \int_{\Gamma_M \backslash \mathscr{D}} \langle \phi, f_{\tau,Z}(M) \rangle .$$

(i) *If* $\dim_{\mathbf{R}}(\langle M \rangle) < r$ *or* $\langle M, M \rangle \not\geq 0$, *then* $I(\phi, M) = 0$.

(ii) *If* $\langle M, M \rangle > 0$, *then*

$$I(\phi, M) = \frac{rq}{2} \Gamma\left(\frac{rq}{2}\right) (2\pi)^{-rq/2} (\det v)^{-q/2} e\left[\tfrac{1}{2} \operatorname{tr}(\tau \langle M, M \rangle)\right] \int_{\Gamma_M \backslash \mathscr{D}_M} * \phi .$$

(iii) *In general, there exists a constant $c(\phi, M)$ such that*

$$I(\phi, M) = c(\phi, M) (\det v)^{-q/2} e\left[\tfrac{1}{2} \operatorname{tr}(\tau \langle M, M \rangle)\right],$$

$$c(\phi, M\alpha) = c(\phi, M) \operatorname{sign}(\det \alpha)^q .$$

Proof. (i) follows from (7.18) and (7.19).

(ii) follows from (7.17) and (5.63). Note that in this case $s = 0$ and $|\det\beta| = \det\langle M, M\rangle^{1/2}$.

(iii) We have that by (7.14)

$$I_M(\phi) = c_{\phi,M}\det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-b/2}F(Z, M),$$
$$I_{M\alpha}(\phi) = c_{\phi,M\alpha}|\det\alpha|^{-b}(\det\alpha)^q\det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-b/2}F(Z, M),$$

with $b = p + 2q - r - 1$.

Since $I_m(\phi) = I_{M\alpha}(\phi)$, it follows that

$$c_{\phi,M\alpha} = |\det\alpha|^{p+q-r-1}\operatorname{sign}(\det(\alpha))^q c_{\phi,M}. \tag{7.20}$$

Then by (7.17),

$$I(\phi, M) = c_1 c_{\phi,M} e\left[\tfrac{1}{2}\operatorname{tr}(\tau\langle M, M\rangle)\right] J(M),$$

where $J(M)$ is the integral

$$J(M) = \int_{\Gamma_M\backslash\mathscr{D}} \det\langle M_{Z^\perp}, M_{Z^\perp}\rangle^{-(p-r-1)/2} e\left[-\operatorname{tr}(iv\langle M_Z, M_Z\rangle)\right] dv_{\mathscr{D}}.$$

By (5.63),

$$J(M\alpha) = |\det\alpha|^{-(p+q-r-1)} J(M). \tag{7.21}$$

Clearly if we set

$$c(\phi, M) = c_1 c_{\phi,M} J(M)(\det v)^{q/2}, \tag{7.22}$$

$I(\phi, M)$ has the desired form and (7.20), (7.21) yield the condition for $c(\phi, M)$.

7.9. Now we return to the general case that $G = O_+(V^{(1)}) \times \prod_{j=2}^m O(V^{(j)})$ as in §6. By our assumption (6.26), if $m > 1$, then

$$\langle X^{(1)}, X^{(1)}\rangle \neq 0 \tag{7.23}$$

for $0 \neq X \in L_0$.

Consider the case $k = \mathbf{Q}$ ($m = 1$). For $M \in V^r$ with the conditions

$$\dim_{\mathbf{R}}\langle M\rangle = r, \quad \langle M, M\rangle \geqslant 0, \quad \langle M, M\rangle \not> 0, \tag{7.24}$$

the subspace $\langle M\rangle^\perp \cap \langle M\rangle \neq 0$. Choose a basis $M_1, \cdots, M_r$ of $\langle M\rangle$ such that

$$M_1 \in \langle M\rangle^\perp \cap \langle M\rangle \cap L_0, \qquad M_\lambda \in L_0, \quad \lambda = 2, \cdots, r. \tag{7.25}$$

Denote by $M(j)$ the sequence of elements in L such that the λth component $M(j)_\lambda$ of $M(j)$ satisfies

$$M(j)_1 = jM_1, \qquad M(j)_\lambda = M_\lambda, \quad \lambda = 2, \cdots, r. \tag{7.26}$$

It follows that

$$\langle M(j), M(j)\rangle = \langle M(1), M(1)\rangle$$

for all j and the $M(j)$'s are not Γ-related.

Lemma 7.9. *Let $M \in L^*$ satisfying condition* (7.24). *Then $I_M(\phi) = 0$ for $\phi \in H_0^{rq}(\Gamma \backslash \mathscr{D})$.*

Proof. We consider the theta function

$$\theta = \det(v)^{q/2} \sum_{M \equiv 0(L)} f_{\tau,Z}(M), \tag{7.27}$$

and the lifting $\mathscr{L}^*$ defined by (7.2). Since ϕ is a cusp form, by the unfolding argument

$$\mathscr{L}^*(\phi) = \det(v)^{q/2} \sum_{\Gamma\backslash L} \int_{\Gamma_M\backslash\mathscr{D}} \langle \phi, f_{\tau,Z}(M)\rangle,$$

where the summation runs over Γ-orbits in L; moreover the summation is absolutely convergent. The sequence $M(j)$ constructed in (7.26) is in L and

$$\Gamma M(j) \neq \Gamma M(i) \tag{7.28}$$

for $i \neq j$. It follows that

$$\sum_{j=1}^{\infty} \left| \int_{\Gamma_M\backslash\mathscr{D}} \langle \phi, f_{\tau,Z}(M(j))\rangle \right| < \infty. \tag{7.29}$$

By (iii) of Lemma 7.8, all the absolute values coincide with one another. Hence (7.28) implies that

$$I(\phi, M(j)) = 0 \quad \text{for all } j.$$

By (iii) of Lemma 7.8, it follows that $I(\phi, M) = 0$; as a consequence $c(\phi, M) = 0$. Since by (7.22), $c(\phi, M)$ is a product of $c_{\phi,M}$ and a positive number, thus $c_{\phi,M} = 0$. We know that

$$I_M(\phi) = c_{\phi,M} \det(M_{Z^\perp}, M_{Z^\perp})^{-(p+2q-r-1)/2} F(Z, M).$$

Thus $I_M(\phi) = 0$.

7.10. Let $S_r(N\mathcal{O})$ be the set

$$S_r(N\mathcal{O}) = \left\{X \in M_r(k) \,\middle|\, {}^tX = X,\ X \equiv 0(N\mathcal{O})\right\}$$

and $S_r^*(N\mathcal{O})$ the set given by

$$S_r^*(N\mathcal{O}) = \left\{X \in M_r(k) \,\middle|\, {}^tX = X,\ \operatorname{tr}_{k/\mathbf{Q}}\operatorname{tr}(XS_r(N\mathcal{O})) \subset \mathbf{Z}\right\}.$$

For $h \in L^*$ and $\eta \in S_r^*(N\mathcal{O})$, let

$$L_{\eta,h} = \left\{M \in L^* \,\middle|\, M \equiv h(L),\ \langle M, M\rangle = 2\eta\right\}. \tag{7.30}$$

It is known that for $\eta > 0$, $L_{\eta,h}$ has only finitely many Γ-orbits

$$(7.31) \qquad L_{\eta,h} = \bigcup_{i=1}^{l(\eta)} \Gamma X_i.$$

Theorem 7.10. *Let $\mathscr{L}^*$ be the lifting map given in* (7.2). *Assume that $k \neq \mathbf{Q}$ or $(p,q) \neq (2.1)$. For $\phi \in H_0^{rq}(\Gamma \setminus \mathscr{D})$, we have the conditions*

(i) *$\mathscr{L}^*(\phi)$ is a cusp form of $\tilde{\Gamma}(N)$ satisfying the condition*

$$\mathscr{L}^*(\phi)(\gamma\tau) = \chi(\gamma) j(\gamma,\tau)^{(n/2,\cdots,n/2)} \mathscr{L}^*(\phi)(\tau), \qquad \gamma \in \tilde{\Gamma}(N).$$

(ii) *$\mathscr{L}^*(\phi)$ has the Fourier expansion*

$$\mathscr{L}^*(\phi)(\tau) = \sum_{\substack{\eta \in S_r^*(N\mathscr{O}) \\ \eta > 0}} a_\eta e[\sigma(\mathrm{tr}(\eta\tau))],$$

where $\sigma = \mathrm{tr}_{k \otimes_Q \mathbf{R}/\mathbf{R}}$, and

$$a_\eta = \frac{rq}{2}\Gamma\left(\frac{rq}{2}\right)(2\pi)^{-rq/2} \sum_{i=1}^{l(\eta)} \int_{\Gamma_{X_i} \setminus \mathscr{D}_{X_i}} *\phi.$$

Proof. The theta function $\theta(\tau, h, Z)$ is given by

$$\theta(\tau, H, Z) = \det(v)^{(q/2,0,\cdots,0)} \sum_{M \equiv h(L)} f_{\tau,Z}(M)$$

and $\mathscr{L}^*(\phi)$ is the integral

$$\mathscr{O}^*(\phi) = \int_{\Gamma \setminus \mathscr{D}} \langle \phi, \theta \rangle.$$

Since ϕ is a cusp form, by the unfolding argument,

$$(7.32) \qquad \mathscr{L}^*(\phi) = \det(v)^{(q/2,0,\cdots,0)} \sum_{\Gamma \setminus (L+h)} \int_{\Gamma_M \setminus \mathscr{D}} \langle \phi, f_{\tau,Z}(M) \rangle.$$

By Lemmas 7.8 and 7.9, we sum over only those orbits ΓM with $\langle M, M \rangle > 0$. The integral

$$\begin{aligned} &\int_{\Gamma_M \setminus \mathscr{D}} \langle \phi, f_{\tau,Z}(M) \rangle \\ &= \prod_{j=2}^{m} e\left[\tfrac{1}{2}\mathrm{tr}\,\tau^{(j)} \langle M^{(j)}, M^{(j)} \rangle\right] \int_{\Gamma_M \setminus \mathscr{D}} \langle \phi, f_{\tau^{(1)},Z}(M^{(1)}) \rangle \\ &= a \prod_{j=1}^{m} e\left[\tfrac{1}{2}\mathrm{tr}\,\tau^{(j)} \langle M^{(j)}, M^{(j)} \rangle\right] (\det v)^{(-q/2,0,\cdots,0)} \int_{\Gamma_M \setminus \mathscr{D}_M} *\phi \end{aligned}$$

((ii) of (Lemma 7.8)) with

$$a = \frac{rq}{2}\Gamma\left(\frac{rq}{2}\right)(2\pi)^{-rq/2}.$$

It follows that

$$(7.33)\qquad \mathscr{L}^*(\phi) = a \sum_{\substack{\Gamma\backslash L+h \\ \langle M, M\rangle > 0}} \left(\int_{\Gamma_M\backslash\mathscr{D}_M} *\phi \right) e\left[\tfrac{1}{2}\sigma(\operatorname{tr}(\langle M, M\rangle\tau))\right].$$

By (i) of Proposition 6.10, we have that

$$(7.34)\quad \mathscr{L}^*(\phi)(\gamma\tau) = \chi(\gamma) j(\gamma,\tau)^{(n/2,\cdots,n/2)} \mathscr{L}^*(\phi)(\tau) \qquad (\gamma \in \tilde{\Gamma}(N)).$$

Thus by (7.34), in (7.33) we sum only over those M with

$$\langle M, M\rangle \in 2S_r^*(N\mathscr{O}).$$

Hence

$$\mathscr{L}^*(\phi) = a \sum_{\substack{\zeta\in S_r^*(N\mathscr{O}) \\ \zeta>0}} \sum_{\Gamma\backslash L_{\zeta,h}} \left(\int_{\Gamma_M\backslash\mathscr{D}_M} *\phi \right) e\left[\sigma(\operatorname{tr}(\zeta\tau))\right].$$

By a usual argument [27, p. 114] one shows that the constant terms of $\mathscr{L}^*(\phi)$ at other cusps are zero. Then (i) and (ii) have been established.

7.11. Let $\mathscr{S}_{n/2}(\tilde{\Gamma}(N),\chi)$ be the space of cusp form $\phi(\tau)$ satisfying

$$\phi(\gamma\tau) = \chi(\gamma) j(\gamma,\tau)^{(n/2,\cdots,n/2)}\phi(\tau) \qquad (\gamma \in \tilde{\Gamma}(N)).$$

Theorem 7.10 shows that the lifting map $\mathscr{L}^*$ is a map

$$(7.35)\qquad \mathscr{L}^*\colon H_0^{rq}(\Gamma\backslash\mathscr{D}) \to \mathscr{S}_{n/2}(\tilde{\Gamma}(N),\chi).$$

Let $\mathscr{L}\colon \mathscr{S}_{n/2}(\tilde{\Gamma}(N),\chi) \to H_0^{rq}(\Gamma\backslash\mathscr{D})$ be the adjoint map of $\mathscr{L}^*$. Now assume that $n = p + q > 4r$. In this case, one can present a concrete description of the map $\mathscr{L}$. Let $\tilde{\Gamma}_\infty$ be the subgroup of $\tilde{\Gamma}(N)$ given by

$$\tilde{\Gamma}_\infty = \tilde{\Gamma}(N) \cap \left\{ \begin{pmatrix} \pm E_r & \beta \\ & \pm E_r \end{pmatrix} \middle| \beta \in S_r(k) \right\}.$$

For $\beta \in S_r^*(N\mathscr{O})$, let

$$(7.36)\qquad \phi_\beta(\tau) = c_{r,N}^{-1} \sum \chi(\gamma)^{-1}(\gamma,\tau)^{-n/2} e\left[\sigma(\operatorname{tr}(\beta\cdot\gamma\tau))\right],$$

where the summation runs over $\tilde{\Gamma}_\infty \backslash \tilde{\Gamma}(N)$ and

$$c_{r,N}^{-1} = \operatorname{vol}(S_r(R)/S_r(N\mathscr{O})).$$

The function $\phi_\beta(\tau)$ is the βth Poincaré series. In the range $n > 4r$, it is absolutely convergent and the function ϕ_β ($\beta \in S_r^*(N\mathscr{O})$) span $\mathscr{S}_{n/2}(\tilde{\Gamma}(N),\chi)$. Let $\langle\ ,\ \rangle$ be the Petersson inner product of $\mathscr{S}_{n/2}$. For

$$\phi = \sum_{\substack{\beta\in S_r^*(N\mathscr{O}) \\ \beta>0}} a(\beta) e\left[\sigma(\operatorname{tr}(\beta\tau))\right],$$

by [23, (10.2)] we then have

$$\langle \phi, \phi_\beta \rangle = b(\det \beta)^{-(t,\dots,t)} a(\beta), \tag{7.37}$$

where $t = \frac{1}{2}(p + q - r - 1)$, and

$$b = 2^{-rm(p+q-r-1)/2}\left(\Gamma(t)\Gamma(t - \tfrac{1}{2}) \cdots \Gamma\left(t - \frac{r-1}{2}\right) \cdot (2\pi)^{-\mathrm{tr}} \pi^{r(r-1)/4}\right)^m.$$

Observe that

$$\left\langle \mathscr{L}^*(\phi), \phi_\beta \right\rangle = ab \det \beta^{-(t,\dots,t)} \sum_{\Gamma \backslash L_{\beta,h}} \int_{\Gamma_M \backslash \mathscr{D}_M} *\phi. \tag{7.38}$$

For $\beta \in S_r^*(N\mathscr{O})$, let

$$C_\beta = \sum_{\substack{M \equiv h\ (L) \\ \langle M, M \rangle = 2\beta \\ \bmod \Gamma}} C_M, \tag{7.39}$$

where C_M is the cycle of the image $\mathscr{D}_M$ in $\Gamma \backslash \mathscr{D}$. Let $\tilde{\phi}_\beta \in H_0^{rq}(\Gamma \backslash \mathscr{D})$ such that

$$\langle \phi, \tilde{\phi}_\beta \rangle = \sum_{\Gamma \backslash L_{\beta,h}} \int_{\Gamma_M \backslash \mathscr{D}_M} *\phi. \tag{7.40}$$

Note that $\tilde{\phi}_\beta$ is the component of the highest signature of the finite part of the dual form of C_β.

Theorem. *Let* $\mathscr{L}$: $\mathscr{S}_{n/2}(\tilde{\Gamma}(N), \chi) \to H_0^{rq}(\Gamma \backslash \mathscr{D})$ *be the adjoint map of* $\mathscr{L}^*$ *and* ϕ_β, $\tilde{\phi}_\beta$ *given by* (7.36) *and* (7.40) *respectively. Then*

$$\mathscr{L}(\phi_\beta) = ab(\det \beta)^{-(t,\dots,t)} \tilde{\phi}_\beta,$$

where $t = \frac{1}{2}(p + q - r - 1)$.

7.12 Remarks. (i) The lifting map $\mathscr{L}^*$ ((7.2)) is meaningful for any cusp harmonic differential forms ϕ of $\Gamma \backslash \mathscr{D}$ of degree rq. However K_Z-invariant bilinear forms between nonisomorphic irreducible modules are trivial. It follows that $\mathscr{L}^*$ always factors through $H_0^{rq}(\Gamma \backslash \mathscr{D})$.

(ii) For $q = 1$ and $\Gamma \backslash \mathscr{D}$ compact, geometric lifting has been studied by Kudla and Millson [23]. For our presentation, in this case $K = 0$ which is immediate from (1.27), and condition (i) of Theorem 3.15 is easily checked to be true for $f(Z, M)$. Hence $F(Z, M) = f(Z, M)$. Our result then coincides with that given in [23] except a constant factor and our additional information that $\mathscr{L}^*$ factors through $H_0^{rq}(\Gamma \backslash \mathscr{D})$ of the space of cusp harmonic differential forms of $\Gamma \backslash \mathscr{D}$ of degree rq of *highest signature*.

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