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SYMPLECTIC RATIONAL BLOWDOWNS

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Abstract

We prove that the rational blowdown, a surgery on smooth 4-manifolds introduced by Fintushel and Stern, can be performed in the symplectic category. As a consequence, interesting families of smooth 4-manifolds, including the exotic $K3$ surfaces of Gompf and Mrowka, admit symplectic structures.

A basic problem in symplectic topology is to understand what smooth manifolds admit a symplectic structure (a closed non-degenerate 2-form). In this paper we focus on this question in dimension 4. Currently, the primary methods for constructing smooth (irreducible) 4-manifolds in such a way that one can distinguish them by Donaldson or Seiberg-Witten invariants are surgery constructions which use complex manifolds as building blocks. These surgery methods are (smooth) logarithmic transforms, rational blowdowns, and connect sums along surfaces. It is interesting to see when these surgeries can be performed in the symplectic category. In this paper we prove that performing a rational blowdown of a symplectic manifold along symplectic surfaces yields a symplectic manifold. This result establishes that certain exotic 4-manifolds, including the exotic $K3$ surfaces of Gompf and Mrowka [9], are symplectic.

In any even dimension, two symplectic manifolds can be summed along codimension 2 symplectic submanifolds to yield a new symplectic manifold. We refer to this symplectic operation, which was proposed by Gromov [11], as the symplectic sum. Gompf [8] used the symplectic sum to construct a plethora of interesting symplectic manifolds, including the first examples of simply connected symplectic 4-manifolds that are

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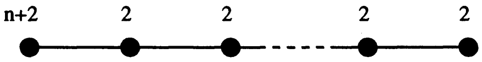

FIGURE 1. The plumbing diagram for C_n, $n \geq 2$

not homotopic to any complex surface and some exotic $K3$ surfaces. More recently, Fintushel and Stern [6] have used the connect sum along smoothly embedded tori to produce a rich class of exotic 4-manifolds, some homeomorphic to a $K3$ surface, many of which cannot admit a symplectic structure.

The logarithmic transform was first studied in the smooth category by Gompf and Mrowka [9] who used it to produce the first examples of irreducible 4-manifolds that are not complex. Subsequently, Fintushel and Stern [5] introduced the rational blowdown and showed that in certain situations a smooth logarithmic transform can be achieved via a sequence of blowups followed by a rational blowdown. The rational blowdown is a surgery in which a neighborhood of a chain of spheres C_n, $n \geq 2$, represented by the plumbing diagram in Figure 1 is replaced by a rational (homology) ball B_n.

Using the rational blowdown, Fintushel and Stern constructed other interesting examples of smooth 4-manifolds. Their examples led them to ask whether the rational blowdown of a symplectic 4-manifold along symplectic spheres is a symplectic operation. Theorem 1.3 of the next section asserts the answer is yes. As a consequence, an infinite family of surfaces not homotopic to a complex surface, constructed by Fintushel and Stern in [5], are symplectic. Furthermore, the complete set of Gompf-Mrowka examples of exotic $K3$ surfaces [9] are also symplectic, extending the results in [8].

The Fintushel-Stern examples that are symplectic as a consequence of Theorem 1.3 are constructed from the simply connected minimal elliptic surfaces $E(n)$, $4 \leq n \in \mathbf{Z}$, that have Euler characteristic $\chi(E(n)) = 12n$. In $E(n)$ one can find two copies of C_{n-2}, embedded so that the spheres are symplectic (consult [7] and [5]). Performing a symplectic rational blowdown along one of these chains of spheres yields a manifold $G(n)$ whose homotopy type is different from any complex manifold. Blowing down both chains of spheres yields a manifold diffeomorphic to a Horikawa surface $H(n)$, a complex manifold of general type (which is therefore Kähler).

The Gompf-Mrowka examples are obtained from the $K3$ surface by performing smooth logarithmic transforms along three pairs of tori (in which a neighborhood of a torus having trivial normal bundle is removed and replaced using a diffeomorphism of the boundary not homotopic to the identity). The resulting manifolds are denoted $K(p_1, q_1; p_2, q_2; p_3, q_3)$, with p_i, q_i relatively prime. Modulo certain relations between the p_i and q_i, these manifolds are mutually non-diffeomorphic but are homeomorphic to either a $K3$ surface or $3\mathbf{C}P^2 \# 19\overline{\mathbf{C}P}^2$. In [8], Gompf showed that $K(p_1, q_1; 1, 1; p_3, q_3)$ are symplectic by presenting them as symplectic sums of simply connected Dolgachev surfaces. The work of Fintushel and Stern [5] shows that all of the $K(p_1, q_1; p_2, q_2; p_3, q_3)$ can be constructed by blowing up points and then performing a rational blowdown. Because the necessary submanifolds can be chosen to be symplectic, Theorem 1.3 implies that all the $K(p_1, q_1; p_2, q_2; p_3, q_3)$ are symplectic.

In the next section we give a precise definition of the symplectic rational blowdown and state the main theorem. The essence of the proof of Theorem 1.3 is in our choice of model spaces for C_n and a collar neighborhood of the boundary of B_n. Indeed, using a model for $L(n^2, n-1) \times (0, \infty)$ as a guide, we endow B_n with a symplectic structure such that the complement of the spheres in C_n is symplectomorphic to a collar of B_n. The gluing is then clear. To describe the model spaces we use symplectic boundary reduction which is the main step in the procedure of symplectic cutting (defined by Lerman [12]). We define symplectic boundary reduction in Section 2 and construct our model spaces in Section 3. We then prove Theorem 1.3 in the last section.

Remark 0.1. Theorem 1.3 can also be deduced as a straightforward application of the 3-fold sum, a symplectic surgery developed by the author. The 3-fold sum is a sum along positively intersecting symplectic surfaces and is part of a generalization (in dimension 4) of the symplectic sum [17]. We sketch this alternative proof of Theorem 1.3 in Remark 4.1, referring the reader to [18] for details on the 3-fold sum.

We use the notation $[b_1, b_2, \ldots, b_n]$ to denote the *negative* continued fraction expansion $b_1 - 1/(b_2 - 1/(\ldots - 1/b_n)\ldots)$.

1. The symplectic rational blowdown

The symplectic rational blowdown generalizes the blowing down of a -4 sphere (a sphere with self-intersection -4), in which a neighborhood of the sphere is replaced by the complement of a conic in $\mathbf{C}P^2$. As observed by Gompf [8], this can be achieved using the symplectic sum (assuming the -4 sphere is symplectic).

Let C_n, $n \geq 2$, be a tubular neighborhood of a union of spheres $\cup_{i=1}^{n-1} S_i$ such that $S_1 \cdot S_1 = -(n+2)$, $S_i \cdot S_i = -2$ for $i = 2, \ldots, n-1$, $S_i \cdot S_{i+1} = 1$, for $i = 1, \ldots, n-2$, and $S_i \cdot S_j = 0$ otherwise. Thus C_n is a plumbing of disk bundles over spheres represented by the diagram in Figure 1. The boundary of C_n is the lens space $L(n^2, n-1)$ which also bounds a manifold B_n that has the same rational homology as a ball (see [2]). In Section 3 we define symplectic models (C_n, ω_{C_n}) and (B_n, ω_{B_n}) whose symplectic structures depend on the areas of the spheres $\{S_i\}_{i=1}^{n-1}$. The symplectic structure ω_{C_n} is chosen so that the spheres $\{S_i\}_{i=1}^{n-1} \subset (C_n, \omega_{C_n})$ are symplectic and intersect orthogonally with respect to ω_{C_n}. (Note that an orthogonal intersection of symplectic surfaces is necessarily positive.)

Definition 1.1. Suppose there is a symplectic embedding

$$\psi : (C_n, \omega_{C_n}) \to (M, \omega).$$

Let $M^- = M - \psi\left(\cup_{i=1}^{n-1} S_i\right)$ and let B_n be a rational homology ball with no prescribed symplectic structure. A **symplectic rational blowdown** of (M, ω) along the spheres $\psi\left(\cup_{i=1}^{n-1} S_i\right)$ is a closed manifold $\widetilde{M} = M^- \cup_\varphi B_n$, where φ is some diffeomorphism, together with a symplectic structure $\tilde{\omega}$ such that $(M^-, \tilde{\omega})$ and (M^-, ω) are symplectomorphic.

Remark 1.2. If (M, ω) contains a union of symplectic spheres with the same intersection pattern (including self-intersections) as $\cup_{i=1}^{n-1} S_i \subset C_n$, then there is a symplectic embedding $\psi : (C_n, \omega_{C_n}) \hookrightarrow (M, \omega)$. In fact, the spheres in M can be isotoped to make the intersections orthogonal, keeping them symplectic all the while (cf. [14]). Then by a version of the symplectic neighborhood theorem (Proposition 3.5) C_n is symplectomorphic to a neighborhood of the isotoped spheres.

Theorem 1.3. *Suppose there is a symplectic embedding*

$$\psi : (C_n, \omega_{C_n}) \hookrightarrow (M, \omega).$$

Then there exist a symplectic rational ball (B_n, ω_{B_n}) and a symplectic map φ making $\widetilde{M} = M^- \cup_\varphi (B_n, \omega_{B_n})$ a symplectic rational blowdown of M. The volume of $\widetilde{M}$ is determined by the volume of M and the areas of the spheres $\{S_i\}_{i=1}^{n-1}$.

Note that a smooth rational blowdown is well defined up to diffeomorphism because any diffeomorphism of the boundary of B_n extends over the rational ball [1]. The symplectic blowdown of a -1 sphere is unique because the symplectic structure of any ball that is standard near the boundary is diffeomorphic to the standard structure via a diffeomorphism that is the identity near the boundary [10]. It is an interesting question whether a symplectic rational blowdown is also unique up to symplectomorphism.

2. Symplectic boundary reduction

Let (M, ω) be a symplectic 4-manifold whose boundary is a circle bundle over a surface Σ. Suppose that all vectors tangent to the circle fibers lie in the kernel of $\omega|_{\partial M}$. Then there is a closed symplectic manifold $(\widehat{M}, \hat{\omega})$, unique up to symplectomorphism, that contains a symplectically embedded copy of Σ and is such that $(M - \partial M, \omega)$ and $(\widehat{M} - \Sigma, \hat{\omega})$ are symplectomorphic. We call $(\widehat{M}, \hat{\omega})$ the **symplectic boundary reduction** of (M, ω). It can be realized as the image of a map π which is symplectic when restricted to the interior of M, and which collapses each circle fiber of ∂M to a point. The image $\pi(\partial M)$ is the embedded copy of Σ; it is a symplectic submanifold of $\widehat{M}$. (The above description is true in higher dimensions with Σ being a manifold of dimension 2 less than that of M).

In a neighborhood of a fiber of ∂M, the map π can be described in local coordinates as follows. Any fiber in the boundary of M has a neighborhood symplectomorphic to $(D^2 \times A^2, dx_1 \wedge dy_1 + dp_2 \wedge dq_2)$ where $A^2 = \{0 \leq p_2 < \epsilon\} \subset \mathbf{R} \times S^1$ and q_2 is defined mod 1. With respect to these local coordinates, π is the projection

$$\pi: \ (x_1, y_1, p_2, q_2) \to \left(x_1, y_1, \sqrt{\frac{p_2}{\pi}} \cos(2\pi q_2), \sqrt{\frac{p_2}{\pi}} \sin(2\pi q_2)\right).$$

One can also take the symplectic boundary reduction of a symplectic 4-manifold when its boundary is not smooth, but rather has corners. Specifically, we allow the boundary of (M, ω) to have more than one smooth component, pairs of which meet along Lagrangian tori (tori T of half the dimension of M such that $\omega|_T = 0$). The definition of symplectic boundary reduction in this context is the same as above except that the interior of M is symplectomorphic to the complement of a union of intersecting symplectic surfaces in $\widehat{M}$. Examples 2.1 and 2.2 are local models for boundary reduction near a corner on the boundary of M. Note that we always take the boundary reduction only along the closed part of ∂M.

Here and throughout this paper we use models that are obtained from $T^*T^2 = \mathbf{R}^2 \times T^2$ with the standard symplectic structure $\omega_0 = dp \wedge dq$ where $p = (p_1, p_2)$ are coordinates on $\mathbf{R}^2$ and $q = (q_1, q_2)$ are coordinates on T^2 defined mod 1.

Example 2.1. : $(\mathbf{R}^4, \mathbf{dx} \wedge \mathbf{dy})$. Let Q be the first quadrant of $\mathbf{R}^2$ and $\overline{Q}$ its closure. Consider $Q \times T^2 \subset (T^*T^2, \omega_0)$ and define the map $\pi : Q \times T^2 \to \mathbf{R}^4$ with coordinates (x_1, y_1, x_2, y_2) by the formula

$$(x_i, y_i) = \left(\sqrt{\frac{p_i}{\pi}} \cos(2\pi q_i), \sqrt{\frac{p_i}{\pi}} \sin(2\pi q_i) \right).$$

It is a symplectomorphism between $Q \times T^2$ and the complement of the coordinate planes $x_1 = y_1 = 0$ and $x_2 = y_2 = 0$ in $(\mathbf{R}^4, dx \wedge dy)$. Extending π to $\overline{Q} \times T^2$ we get a projection to $\mathbf{R}^4$ in which the image of the torus $p_1 = p_2 = 0$ is the origin and the image of each circle fiber on the rest of the boundary of $\overline{Q} \times T^2$ is a point on one of the coordinate planes. The image of this projection, which is all of $\mathbf{R}^4$, is the boundary reduction of $\overline{Q} \times T^2$.

Example 2.2. : $(\mathbf{R}^4, \mathbf{dx} \wedge \mathbf{dy})$ again. Now consider any closed positive cone C in $\mathbf{R}^2$ defined by integral vectors u and v such that the matrix $B = [u\, v]$ is in $GL(2, \mathbf{Z})$. The boundary reduction of

$$C \times T^2 \subset (T^*T^2, \omega_0)$$

is $\mathbf{R}^4$ with the standard symplectic structure. This follows because $(C \times T^2, \omega_0)$ is symplectomorphic to $(\overline{Q} \times T^2, \omega_0)$ via the map $\varphi(p, q) = (Bp + r, B^{-T} q)$ where r is the vertex of the cone $C \subset \mathbf{R}^2$.

Definition 2.3. If Σ is a surface in the image of ∂M under symplectic boundary reduction of (M, ω), then we call the preimage $\pi^{-1}(\Sigma)$ the **boundary along** Σ.

3. Model neighborhoods

By construction, all of our model spaces admit Hamiltonian 2-torus actions, and all of the figures we draw in $\mathbf{R}^2$ are in fact images of moment maps for the torus actions, though we do not appeal to that language in this paper. However, the reader should note that this means that in our figures, each interior point represents a torus, each edge point represents a circle and each vertex represents a single point.

We begin by describing a symplectic structure $\omega_{n,m}$ on $V_{n,m} = L(n,m) \times (0,\infty)$, for $n \geq m \geq 1$ relatively prime, that is induced from the standard structure $\omega_0 = dp \wedge dq$ on T^*T^2 via boundary reduction. Recall that a lens space $L(n,m)$ can be presented as the union of two solid tori glued together via a map ϕ of their boundaries such that $\phi_*\mu_2 = -m\mu_1 + n\lambda_1$ where μ_i, λ_i are meridinal and longitudinal cycles on the boundaries.

Example 3.1. : $(V_{n,m}, \omega_{n,m})$. Let $(V_{n,m}, \omega_{n,m})$ be the boundary reduction of $U_{n,m} \times T^2 \subset (T^*T^2, \omega_0)$ where $U_{n,m} = \{p_1 \geq 0\} \cap \{p_2 \geq \frac{m}{n}p_1\} \cap \{p_2 > 0\}$. The 3-dimensional submanifold of $V_{n,m}$ that is the image (under boundary reduction) of $(\{p_2 = 1\} \cap U_{n,m}) \times T^2$ is a union of two solid tori with shared boundary $\{p_1 = c,\ p_2 = 1\} \times T^2$ for some $0 < c < \frac{m}{n}$. The boundary of the solid torus that is the image of $(\{p_1 \leq c,\ p_2 = 1\} \cap U_{n,m}) \times T^2$ has a meridian whose tangent vectors are $\frac{\partial}{\partial q_1}$, while the other one has a meridian whose tangent vectors are $-m\frac{\partial}{\partial q_1} + n\frac{\partial}{\partial q_2}$. Since these two meridians are identified, the 3-manifold is $L(n,m)$. Hence, $(V_{n,m}, \omega_{n,m})$ is a symplectic model for $L(n,m) \times (0,\infty)$.

For the following examples, let L_0, L_1 be the lines $\{p_1 = 0\}$, $\{p_2 = 0\}$ in $\mathbf{R}^2$.

Example 3.2. : (N_b, ω_{N_b}). Let L_2 be the line $\{p_2 = \frac{1}{b}(p_1 - a)\}$, $a > 0$. Consider the closed domain in $\mathbf{R}^2$ lying between L_0 and L_2 and bounded below by L_1. Let U_b be a neighborhood of L_1 in this domain. Then the boundary reduction (N_b, ω_{N_b}) of $U_b \times T^2$ (taken along the closed edges) is a symplectic neighborhood of a sphere of self-intersection $-b$ and area a. Indeed, the image of $(L_1 \cap U_b) \times T^2$ is a sphere of area a since it is the boundary reduction of a cylinder $\{(p_1, q_1) | 0 \leq p_1 \leq a\}$ with symplectic form $dp_1 \wedge dq_1$. As per Examples 2.1 and 2.2, N_b is the union of two polydisks, and the boundary of N_b is $L(b,1)$, verifying that the sphere has self-intersection $-b$.

Example 3.3. : (C_n, ω_{C_n}) and $(C_n^-, \omega_{C_n^-})$. Suppose the spheres $\{S_i\}_{i=1}^{n-1}$ have areas $\{a_i\}_{i=1}^{n-1}$. Let n_i, m_i be the relatively prime integers such that $\frac{n_i}{m_i} = [n+2, 2, \ldots, 2]$ for a continued fraction expression of length i, and define vectors $r_i = \begin{pmatrix} n_{i-1} \\ m_{i-1} \end{pmatrix}$ with $n_0 = 1$ and $m_0 = 0$.

For $i = 2, \dots, n$, let $t_i = \sum_{j=1}^{i-1} a_j r_j$ and define lines L_i, $i = 2, \dots, n$ by the parametric equations $tr_i + t_i$, $t \in (-\infty, \infty)$. Consider the closed domain bounded below by the union of lines L_i, $i = 1, \dots, n-1$ and lying between the lines L_0, L_n. Let U_{C_n} be a neighborhood in this domain of the lines L_i, $i = 1, \dots, n-1$. The boundary reduction of $U_{C_n} \times T^2$ is a symplectic model for (C_n, ω_{C_n}).

To see this notice that $U_{C_n} \times T^2$ is the union of $U_{n+2} \times T^2$ (as defined in Example 3.2) and the images of $n-2$ copies of $U_2 \times T^2$ under symplectic maps $(p,q) \mapsto (T_i p + t_i, T_i^{-T} q)$, $i = 2, \dots, n-1$ where $T_i = R_{n+2} R_2^{i-2}$ with

$$R_k = \begin{pmatrix} k & -1 \\ 1 & 0 \end{pmatrix}.$$

These maps induce symplectomorphisms that plumb the disk bundles N_{n+2}, $N_2, \dots N_2$ to form (C_n, ω_{C_n}).

We let $(C_n^-, \omega_{C_n^-})$ be the boundary reduction of $U_{C_n^-} \times T^2$ where $U_{C_n^-} = U_{C_n} - \{L_i\}_{i=1}^{n-1}$. This is symplectomorphic to the complement of the surfaces $\{S_i\}_{i=1}^{n-1} \subset (C_n, \omega_n)$ and hence is a symplectic model for a collar neighborhood of the boundary of M^-.

We now define a symplectic rational ball $(B'_n, \omega_{B'_n})$ that is a complement of two spheres in a rational ruled surface. In the proof of Theorem 1.3 we will see how to modify the symplectic structure on this rational ball to obtain the ball (B_n, ω_{B_n}) which is required for the gluing. Let F_{n-1} be a rational ruled surface that contains symplectic sections Σ_{n+1} and Σ_{-n+1} with self-intersections $n+1, -n+1$. (For instance F_{n-1} can be a projectivized plane bundle with holomorphic sections Σ_+ and Σ_- of self-intersections $n-1, -n+1$ respectively. In this case $[\Sigma_{n+1}] = [\Sigma_+] + [f]$ and $[\Sigma_{-n+1}] = [\Sigma_-]$ where f is a fiber.) Since $\Sigma_{n+1} \cdot \Sigma_{-n+1} = 1$ and the spheres span the rational homology of F_{n-1}, the complement $F_{n-1} - (\Sigma_{n+1} \cup \Sigma_{-n+1})$ is a rational ball with boundary $L(n^2, n-1)$. Let $(B'_n, \omega_{B'_n})$ be this rational ball with the symplectic structure inherited from F_{n-1}. (Note that the ruled surface F_{n-1} has a symplectic structure, well-defined up to symplectomorphism by the areas $\alpha_{n+1}, \alpha_{-n+1}$ of the two sections [13].)

Example 3.4. : $(A'_n, \omega_{A'_n})$. We can assume the two sections $\Sigma_{n+1}, \Sigma_{-n+1}$ intersect orthogonally with respect to the symplectic structure on F_{n-1}, isotoping one of them if necessary (cf. [14]). Define L_2, L_3

parametrically in t by $tr_2 + \alpha_{n+1} r_1$ and $tr_3 + \alpha_{-n+1} r_2 + \alpha_{n+1} r_1$ where

$$r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} -n-1 \\ -1 \end{pmatrix} \quad \text{and} \quad r_3 = \begin{pmatrix} -n^2 \\ -n+1 \end{pmatrix}.$$

Assuming $\alpha_{n+1} > (n+1)\alpha_{-n+1}$, consider the closed domain in $\mathbf{R}^2$ that lies between L_0, L_3 and below L_1, L_2. Let $U_{A'_n}$ be a neighborhood of $L_1 \cup L_2$ in this domain, minus the two lines. The boundary reduction $(A'_n, \omega_{A'_n})$ of $U_{A'_n} \times T^2$ is a model for a collar neighborhood of the boundary of $(B'_n, \omega_{B'_n})$.

For our model neighborhoods to be useful we need the following simple modification of the symplectic neighborhood theorem.

Proposition 3.5. *Consider two embeddings of a configuration of spheres $j : C \hookrightarrow (M, \omega)$ and $j' : C \hookrightarrow (M', \omega')$, such that the areas and self-intersections of the images of a given sphere are the same and all intersections are orthogonal (and hence positive) with respect to the ambient symplectic form. Then $j(C)$ and $j'(C)$ have symplectomorphic neighborhoods.*

The proof is a standard application of Moser's method to turn a diffeomorphism into a symplectomorphism except that one must be careful in the vicinity of the intersection points. Details of how to handle such situations were provided in [15].

Corollary 3.6. *Suppose (M, ω) and (M', ω') are symplectic manifolds with boundary such that the kernel of the symplectic form (restricted to the boundary) defines a union of smooth components fibered by circles and intersecting along Lagrangian tori. Let $\pi(M), \pi'(M')$ be their symplectic boundary reductions. If $\pi(\partial M), \pi'(\partial M')$ define configurations of symplectomorphic submanifolds with the same intersection patterns in $\pi(M), \pi'(M')$, then $\partial M, \partial M'$ have symplectomorphic collar neighborhoods.*

Proof. This follows from the fact that there is a unique way to extend to the boundaries of M, M' the symplectic map $(\pi')^{-1} \circ \phi \circ \pi$ where ϕ is a symplectomorphism of neighborhoods of $\pi(\partial M)$ and $\pi'(\partial M')$. q.e.d.

4. Proof of Theorem 1.3

Proof. To blow down a symplectic -4-sphere in (M, ω) one simply takes the symplectic sum of M with $\mathbf{C}P^2$ along the -4-sphere in M

and a conic Q in $\mathbf{C}P^2$, as shown in [8]. This is the rational blowdown for $n = 2$. For notational convenience we restrict our attention to the cases $n \geq 3$.

Recall the domains $U_{n,m}$, $U_{C_n^-}$ and $U_{A_n'}$ defined in Examples 3.1, 3.3, and 3.4. The boundary reduction of the product of each of these with T^2, viewed as a subset of (T^*T^2, ω_0), is a symplectic model for $V_{n,m}$, C_n^- and A_n' respectively. Corollary 3.6 implies that there are symplectic embeddings

$$\psi_1 : (C_n^-, \omega_{C_n^-}) \hookrightarrow M - \psi(\cup_{i=1}^{n-1} S_i)$$

and

$$\psi_2 : (A_n', \omega_{A_n'}) \hookrightarrow (B_n', \omega_{B_n'}) = F_{n-1} - \{\Sigma_{-n+1}, \Sigma_{n+1}\}.$$

Notice that there is a translation of the domain $U_{C_n^-}$ that is a subset of $U_{n^2,n-1}$ such that its closed edges are subsets of the two edges of $U_{n^2,n-1}$. Since the translation in the p-coordinates is a symplectomorphism of T^*T^2, this implies that $(C_n^-, \omega_{C_n^-})$ is symplectomorphic to a submanifold of $(V_{n^2,n-1}, \omega_{n^2,n-1})$. Call this symplectic embedding ϕ_1.

Now choose a rational surface F_{n-1} that has sections $\Sigma_{-n+1}, \Sigma_{n+1}$ with areas $\alpha_{n+1} > (n+1)\alpha_{-n+1} > 0$ such that

$$(n-1)\alpha_{n+1} + \alpha_{-n+1} < \sum_{i=1}^{n-1} \left((n-1)n_{i-1} - n^2 m_{i-1}\right) a_i,$$

where the n_i, m_i are defined in Example 3.3, and the a_i are the areas of the spheres $S_i \subset C_n$. Since $\alpha_{n+1} > (n+1)\alpha_{-n+1} > 0$, there is a symplectic embedding $\phi_2 : A_n' \hookrightarrow V_{n,m}$; like ϕ_1 it is a translation in the p-coordinates. Because of our choice of areas, the image of $U_{A_n'}$ under ϕ_2 lies below the image of $U_{C_n^-}$ under ϕ_1. Hence, the union $\phi_1(C_n^-) \cup \phi_2(A_n')$ is a collar neighborhood of a submanifold A_n of $V_{n^2,n-1}$, that can be simultaneously glued onto $(M - \psi(\cup_{i=1}^{n-1} S_i), \omega)$ and $(B_n', \omega_{B_n'})$ via symplectomorphisms $\psi_1 \circ \phi_1^{-1}$ and $\psi_2 \circ \phi_2^{-1}$. By letting

$$B_n = B_n' \cup_{\psi_2 \circ \phi_2^{-1}} A_n$$

with induced symplectic structure ω_{B_n}, the manifold

$$\widetilde{M} = \left(M - \psi(\cup_{i=1}^{n-1} S_i)\right) \cup_{\psi_1 \circ \phi_1^{-1}} B_n$$

with the induced symplectic structure $\tilde{\omega}$ is a symplectic rational blowdown of (M, ω) along the spheres $\psi(\cup_{i=1}^{n-1} S_i)$.

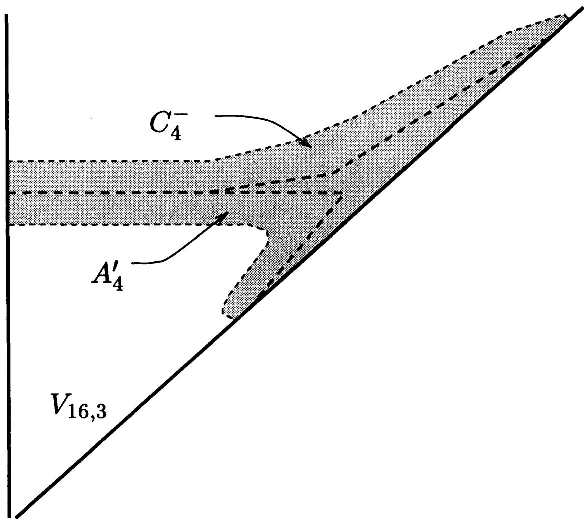

FIGURE 2. Images of C_4^- and A_4' in $V_{16,3} = L(16,3) \times (0,\infty)$

Figure 4 illustrates symplectic embeddings ϕ_1, ϕ_2 of C_4^- and A_4' into $V_{16,3} = L(16,3) \times (0,\infty)$ by showing the images of $U_{C_4^-}$ and $U_{A_4'}$ in $U_{16,3}$. Here we have chosen $3\alpha_5 + \alpha_{-3} = \sum_{i=1}^{3}(3n_i - 16m_i)a_i$. The shaded domain represents A_4, a collar neighborhood of the boundary of the ball B_4 we need for gluing.

The volume of $\widetilde{M}$ is independent of any choice of rational ball that fits. Indeed, suppose $B_{n,1}, B_{n,2}$ are two choices of rational balls for which a symplectic rational blowdown exists. By shrinking the gluing loci, we can assume that the boundaries of the $B_{n,i}$ both have collar neighborhoods that are symplectomorphic to a common C_n^-. But C_n^- is also a collar neighborhood of the boundary of a neighborhood Z of spheres $\Sigma'_{n+1} \cup \Sigma'_{-n+1}$ of self intersections $n+1, -n+1$. (See Figure 3.) Therefore for some symplectic map φ, each $B_{n,i} \cup_\varphi Z$ is a closed symplectic manifold containing a sphere of positive self-intersection, and hence is a rational ruled surface [10]. The cohomology class of the symplectic form on each of these ruled surfaces is set by the areas of the spheres $\Sigma'_{n+1}, \Sigma'_{-n+1}$, thus the volume is the same in both cases. But this implies that the volume of $B_{n,i}$ is independent of i. q.e.d.

Note that in contrast to the case of blowing down a -4-sphere, collar neighborhoods $(C_n^-, \omega_{C_n^-})$ and $(A_n', \omega_{A_n'})$ cannot be symplectomorphic for $n \geq 3$ so long as we choose $(B_n', \omega_{B_n'})$ to be the complement of a pair

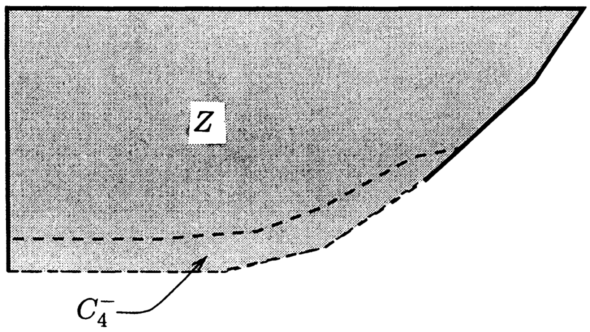

FIGURE 3. Completion Z of rational balls $B_{4,i}$, $i = 1, 2$ to F_3

of symplectic submanifolds of a rational surface. Indeed, for $n \geq 3$ we always have vol $\widetilde{M} >$ vol $M +$ vol B'_n.

Remark 4.1. This theorem can also be proved by using the 3-fold sum, an adaptation of the symplectic sum for positively intersecting surfaces. For details on the 3-fold sum, the reader should consult Symington [18]. Appealing to the 3-fold sum, the proof can be encapsulated in a figure. A blowdown is the sum of M, one copy of $\mathbf{C}P^2$, and $n-2$ ruled surfaces $F_2, \dots, F_{n-1}$. Figure 4 shows how these manifolds should be glued together. A 3-fold sum is performed at each intersection point. Each line segment in the diagram represents a surface along which we are gluing; the numbers labeling the edges are the self-intersection numbers of the corresponding surfaces. Note that the sum of the self-intersection numbers of each pair of corresponding surfaces equals the negative of the number of 3-fold sums that involve the two surfaces, as it must be to perform the sum.

Remark 4.2. Following the same method as in Example 3.3, the neighborhood of any union of spheres defined by a linear plumbing and having negative definite intersection form has a symplectic model with a symplectic structure inherited from T^*T^2 via boundary reduction. Furthermore, after removing the spheres, this model embeds into some $(V_{n,m}, \omega_{n,m})$. Indeed, if the spheres $\{S_i\}_{i=1}^s$ are indexed so that $S_i \cdot S_j = 1$ if $j = i+1$ and $S_i \cdot S_j = 0$ for other $j \neq i$, and if they have self-intersection numbers $-b_1, \dots - b_s$, then the model embeds symplectically in $(V_{n,m}, \omega_{n,m})$ where n, m are the relatively prime positive integers such that $\frac{n}{m} = [b_1, \dots, b_s]$.

It is worth observing that this embedding shows that inside any

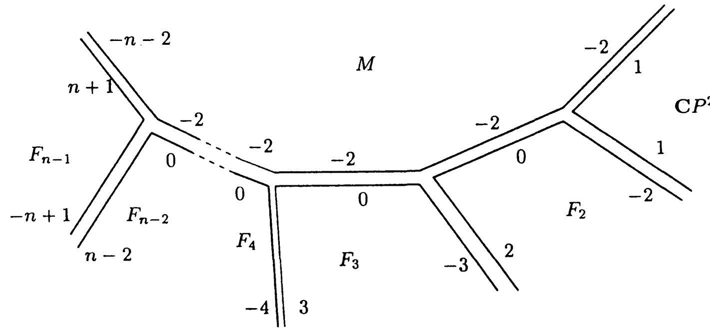

FIGURE 4. Rational blowdown using the 3-fold sum

neighborhood of such a chain of spheres one can find a neighborhood with ω-convex boundary. Note that the boundary of a symplectic manifold (M, ω) is **ω-convex** if there is an outward-pointing vector field X, defined in a neighborhood of the boundary, such that $\mathcal{L}_X \omega = \omega$. Such an X is called an **expanding vector field**. (See [3] and [4] for discussions of symplectic convexity.) Now, the vector field $X = p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2}$ on T^*T^2 induces an expanding vector field $X_{n,m}$ on $V_{n,m}$, and hence on the complement of the spheres $\{S_i\}_{i=1}^s$ in their model neighborhood. Thus, any neighborhood whose boundary is transverse to this expanding vector field is in fact ω-convex.

Question 4.3. Park [16] has studied, in the smooth category, a generalized rational blowdown in which rational balls with boundary $L(n^2, nk - 1)$, $(n, k) = 1$, replace a neighborhood of a chain of spheres with the same boundary. It would be interesting to know whether or not it is possible to perform this generalized blowdown in the symplectic category.

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