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Mixed Problem for Weakly Hyperbolic Equations of Second Order with Degenerate First Order Boundary Condition

Dedicated to Professor Haruo Sunouchi on his 60th birthday

Masaru TANIGUCHI

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Introduction

In this paper, we consider a mixed problem for second order hyperbolic equations permitting them to degenerate on the initial surface, and prove the existence and uniqueness theorem for classical solutions. The point of our proof is to derive the energy estimate. To do so, we reduce our mixed problem to the one with positive boundary condition for symmetric hyperbolic pseudo differential systems of first order. This device was made previously in our works [6], [13] and [14]. Also, we use the theory of the pseudo differential operator with parameter (see [2] and [4]).

A mixed problem with Dirichlet boundary condition for above equations was studied by Baranovskii [3], Kimura [5], Krasnov [7] and Oleinik [12].

In §1, we give the notation and state our main theorem. In §2, we explain the method of reduction mentioned above through examples of mixed problems for wave equation, which will make our arguments in §3 clear. In §3, we consider the mixed problem with zero initial data and obtain the energy estimate. In §4, we give the proof of the main theorem.

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Writing this paper, we have been informed the result of Kubo [8], which is obtained independently of our paper.

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§ 1. Notation and the Main Theorem.

(i) $\|u\|_{m,\omega}\cdots$ the norm of the Sobolev space $H_m(\omega)$.

(ii) $H^\infty(\omega)=\bigcap_{m=0}^{\infty} H_m(\omega)$.

(iii) $u(t)\in \mathscr{E}_t^r(E)\cdots u(t)$ is r-times continuously differentiable in t as an E-valued function.

(iv) $\|u(t)\|^2_{m,\mu,\Omega}=e^{-2\mu t}\|u(t)\|^2_{m,\Omega}$.

(v) $\|u(t)\|^2_{m,\mu}=e^{-2\mu t}\|u(t)\|^2_{m,R^n_+}$.

(vi) $\langle\!\langle u(t)\rangle\!\rangle^2_{m,\mu,S}=e^{-2\mu t}\|u(t)\|^2_{m,S}$.

(vii) $\langle\!\langle u(t)\rangle\!\rangle^2_{m,\mu}=e^{-2\mu t}\|u(t)\|^2_{m,R^{n-1}}$.

(viii) $(\ ,\ )\cdots$the inner product in $L^2(R^n_+)$.

(ix) $(\ ,\ )_\Omega\cdots$the inner product in $L^2(\Omega)$.

(x) $\langle\ ,\ \rangle\cdots$the inner product in $L^2(R^{n-1})$.

(xi) $((\ ,\ ))\cdots$the inner product in C^j.

(xii) $D=\{z\in C\mid |z|<1\}$.

(xiii) $S^m(\mu)=\{p(t, y_1, y', \eta' : \sigma)\in C^\infty([0, T]\times \bar{R}^1_+\times R^{n-1}\times R^{n-1})|$ for any $\theta=(\theta_1, \theta_2, \theta_3, \theta_4)$, there is a constant C_θ such that

$$\left|\left(\frac{\partial}{\partial t}\right)^{\theta_1}\left(\frac{\partial}{\partial y_1}\right)^{\theta_2}\left(\frac{\partial}{\partial y'}\right)^{\theta_3}\left(\frac{\partial}{\partial \eta'}\right)^{\theta_4} p\right|\leqq C_\theta(|\eta'|^2+\mu)^{(m-|\theta_4|)/2}$$

where $\sigma=\sqrt{\mu}$ is a positive constant such that $\mu\geqq 1$.

(xiv) $\mathcal{S}^m(\mu)\cdots$ the set of pseudo differential operators with respect to $y'=(y_2, \cdots, y_n)$ with their symbols $p\in S^m(\mu)$.

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega=S$. We consider the mixed problem

$$(1.1)\quad \begin{cases} L[u]=\dfrac{\partial^2 u}{\partial t^2}-2t^k\sum_{j=1}^n h_j(t, x)\dfrac{\partial^2 u}{\partial t\partial x_j}-t^{2k}\sum_{i,j=1}^n a_{ij}(t, x)\dfrac{\partial^2 u}{\partial x_i\partial x_j} \\ \qquad +a_0(t, x)\dfrac{\partial u}{\partial t}+t^{k-1}\sum_{j=1}^n a_j(t, x)\dfrac{\partial u}{\partial x_j}+d(t, x)u=f(t, x) \\ u(0, x)=u_0(x)\ ,\quad u_t(0, x)=u_1(x) \\ B[u]\Big|_S=t^k\left(\sqrt{\sum_{i,j=1}^n a_{ij}(t, s)\nu_i\nu_j}\dfrac{\partial u}{\partial \nu}+X(t, s)u\right)-\beta(t, s)\dfrac{\partial u}{\partial t}+\gamma(t, s)u\Big|_S \\ \qquad =g(t, s)\quad (t, x)\in(0, T)\times\Omega \end{cases}$$

where $x=(x_1, \cdots, x_n)$, k is a positive integer, T is a positive constant, the coefficients h_j, a_{ij}, a_0, a_j and $d \in \mathscr{B}([0, T]\times\bar{\Omega})$, the coefficients β and $\gamma \in \mathscr{B}([0, T]\times S)$, $s \in S$, $\nu(s)=(\nu_1(s), \cdots, \nu_n(s))$ is the inner unit normal at s, $(\partial/\partial\nu)=\sum_{j=1}^n \nu_j(s)(\partial/\partial x_j)$, and $X(t, s)$ is a smooth tangent vector field of S over C.

For any $s_0 \in S$, there is a following smooth coordinate transformation $\Psi: V \to W$ such that

(i) $\Psi(s_0)=y_0=(0, y_0')=(0, y_{02}, \cdots, y_{0n})$,

(ii) V and W are neighborhoods of s_0 and y_0 respectively,

(iii) $\Psi: V \to W$ is a bijection,

(iv) $\Psi(V\cap\Omega)=W\cap \boldsymbol{R}_+^n$, $\boldsymbol{R}_+^n=\{y=(y_1, y_2, \cdots, y_n)|y_1>0\}$.

(v) $\Psi(V\cap S)=W\cap \boldsymbol{R}^{n-1}$,

and

(vi) L is transformed into the $\tilde{L}$ where

$$\tilde{L}=\frac{\partial^2}{\partial t^2}-2t^k\sum_{j=1}^n \tilde{h}_j(t, y)\frac{\partial^2}{\partial_i t\partial y_j}-t^{2k}\sum_{i,j=1}^n \tilde{a}_{ij}(t, y)\frac{\partial^2}{\partial y_i\partial y_j}$$
$$+\tilde{a}_0(t, y)\frac{\partial}{\partial t}+t^{k-1}\sum_{j=1}^n \tilde{a}_j(t, y)\frac{\partial}{\partial y_j}+\tilde{d}(t, y) \tag{1.2}$$

for any $y \in W\cap \overline{\boldsymbol{R}_+^n}$.

We assume the following conditions for the problem (1.1):

$$L_0=\frac{\partial^2}{\partial t^2}-2\sum_{j=1}^n h_j(t, x)\frac{\partial^2}{\partial t\partial x_j}-\sum_{i,j=1}^n a_{i,j}(t, x)\frac{\partial^2}{\partial x_i\partial x_j} \tag{A.I}$$

is regularly hyperbolic on $[0, T]\times\bar{\Omega}$ and $\sum_{i,j=1}^n a_{i,j}(t, s)\nu_i\nu_j>0$.

(A.II) (i) For the above coordinate transformation Ψ, B is transformed into $\tilde{B}$ where

$$\tilde{B}=\frac{1}{\sqrt{\tilde{a}_{11}(t, 0, y')}}\left\{t^k\left[\tilde{a}_{11}(t, 0, y')\frac{\partial}{\partial y_1}+\sum_{j=2}^n \tilde{a}_{1j}(t, 0, y')\frac{\partial}{\partial y_j}\right]\right.$$
$$\left.+\tilde{h}_1(t, 0, y')\frac{\partial}{\partial t}\right\}+t^k\sum_{j=2}^n \tilde{\alpha}_j(t, y')\frac{\partial}{\partial y_j}$$
$$-\tilde{\beta}(t, y')\cdot\left(1+\frac{\tilde{h}_1(t, 0, y)^2}{\tilde{a}_{11}(t, 0, y')}\right)^{1/2}\left\{\frac{\partial}{\partial t}-\left(1+\frac{\tilde{h}_1(t, 0, y')^2}{\tilde{a}_{11}(t, 0, y')}\right)^{-1}\right.$$
$$\left.\cdot\, t^k\sum_{j=2}^n\left[\tilde{h}_j(t, 0, y')\frac{\partial}{\partial y_j}-\frac{\tilde{h}_1(t, 0, y')}{\tilde{a}_{11}(t, 0, y')}\tilde{a}_{1j}(t, 0, y')\frac{\partial}{\partial y_j}\right]\right\}$$
$$+\tilde{\gamma}(t, y') \tag{1.3}$$

for any $y=(0, y')=(0, y_2, \cdots, y_n) \in W \cap \boldsymbol{R}^{n-1}$.

(ii) The quadratic equation, for any $(0, y') \in W \cap \boldsymbol{R}^{n-1}$,

$$(c+1)z^2+2bz+(c-1)=0 \tag{1.4}$$

has roots in $D=\{z \in \boldsymbol{C} \mid |z|<1\}$ where

$$\begin{cases} b=\sum_{d=2}^{n} \tilde{\alpha}_j(t, y')\eta_j/d(\eta') , \quad c=\tilde{\beta}(t, y') \\ d(\eta')=\Big[\sum_{i,j=2}^{n} \tilde{a}_{ij}(t, 0, y')\eta_i\eta_j-\dfrac{1}{\tilde{a}_{11}(t, 0, y')}\Big(\sum_{j=2}^{n} \tilde{a}_{1j}(t, 0, y')\eta_j\Big)^2 \\ \qquad +\Big(1+\dfrac{\tilde{h}_1(t, 0, y')^2}{\tilde{a}_{11}(t, 0, y')}\Big)^{-1} \cdot \Big(\sum_{j=2}^{n} \tilde{h}_j(t, 0, y')\eta_j \\ \qquad -\dfrac{\tilde{h}_1(t, 0, y')}{\tilde{a}_{11}(t, 0, y')}\sum_{j=2}^{n} \tilde{a}_{1j}(t, 0, y')\eta_j\Big)^2\Big]^{1/2} . \end{cases} \tag{1.5}$$

REMARK 1. Let M be

$$\begin{aligned} M=\frac{\partial}{\partial t^2}-2\Big(\alpha_1\frac{\partial}{\partial y_1}+\sum_{j=2}^{n} \alpha_j\frac{\partial}{\partial y_j}\Big)\frac{\partial}{\partial t} \\ -\Big(\beta_{11}\frac{\partial^2}{\partial y_1^2}+2\sum_{j=2}^{n} \beta_{1j}\frac{\partial^2}{\partial y_1\partial y_j}+\sum_{i,j=2}^{n} \beta_{ij}\frac{\partial^2}{\partial y_i\partial y_j}\Big) . \end{aligned} \tag{1.6}$$

If M is regularly hyperbolic with respect to t and $\beta_{11}>0$, then we have the symbol

$$\sigma(M)=\tilde{\xi}^2+\tilde{d}(\eta')^2-\tilde{\tau}^2 \tag{1.7}$$

which corresponds to the symbol of wave equation $\xi^2+\eta^2-\tau^2$ where

$$\begin{cases} \tilde{\tau}=\Big(1+\dfrac{\alpha_1^2}{\beta_{11}}\Big)^{1/2}\Big\{\tau-\Big(1+\dfrac{\alpha_1^2}{\beta_{11}}\Big)^{-1}\Big(\sum_{j=2}^{n} \alpha_j\eta_j-\dfrac{\alpha_1}{\beta_{11}}\sum_{j=2}^{n} \beta_{1j}\eta_j\Big)\Big\} \\ \tilde{\xi}=\dfrac{1}{\sqrt{\beta_{11}}}\Big(\beta_{11}\xi+\sum_{j=2}^{n} \beta_{1j}\eta_j+\alpha_1\tau\Big) \\ \tilde{d}(\eta')=\Big[\sum_{i,j=2}^{n} \beta_{ij}\eta_i\eta_j-\dfrac{1}{\beta_{11}}\Big(\sum_{j=2}^{n} \beta_{1j}\eta_j\Big)^2 \\ \qquad +\Big(1+\dfrac{\alpha_1^2}{\beta_{11}}\Big)^{-1}\Big(\sum_{j=2}^{n} \alpha_j\eta_j-\dfrac{\alpha_1}{\beta_{11}}\sum_{j=2}^{n} \beta_{1j}\eta_j\Big)^2\Big]^{1/2} \end{cases} \tag{1.8}$$

for $(\xi, \eta')=(\xi, \eta_2, \cdots, \eta_n) \in \boldsymbol{R}^n$ (see [10]).

The assumption (A.II)-(i) is associated with the above representation. Also, this representation is a base to transform the operators L and B in (1.1) into $\tilde{L}$ in (1.2) and $\tilde{B}$ in (1.3), and is used to obtain the energy estimate in §3.

Let us define $u_{r+2}(x)$ recursively by the formulae

$$u_{r+2}(x)=-\sum_{j=0}^{r}{}_rC_j\{A_1^{(j)}(0, x: D_x)u_{r+1-j}(x) + A_2^{(j)}(0, x: D_x)u_{r-j}(x)\}+f^{(r)}(0, x) \quad (r=0, 1, 2, \cdots) \tag{1.9}$$

where

$$\begin{cases} L=D_t^2+A_1(t, x: D_x)D_t+A_2(t, x: D_x) \\ A_i^{(j)}(t, x: \xi)=(D_t^jA_i)(t, x: \xi) \\ f^{(r)}(t, x)=(D_t^rf)(t, x) \\ D_t=\dfrac{\partial}{\partial t}, \text{ etc. } . \end{cases} \tag{1.10}$$

DEFINITION. We say that the data $\{u_0(x), u_1(x), f(t, x), g(t, s)\}$ satisfy the compatibility condition of infinite order provided that

$$\left(\frac{\partial}{\partial t}\right)^r (B[u]|_S)\Big|_{t=0}=\left(\frac{\partial}{\partial t}\right)^r g\Big|_{t=0} \quad (r=0, 1, 2, \cdots)$$

i.e.

$$\sum_{j=0}^{r}{}_rC_j\{B_1^{(j)}(0, s)u_{r+1-j}(x)+B_2^{(j)}(0, s: D_x)u_{r-j}(x)\}\Big|_S=\left(\frac{\partial}{\partial t}\right)^r g\Big|_{t=0} \quad (r=0, 1, 2, \cdots) \tag{1.11}$$

where

$$\begin{cases} B=B_1(t, s)D_t+B_2(t, s: D_x) \\ B_i^{(j)}=\left(\dfrac{\partial}{\partial t}\right)^j B_i \quad (i=1, 2) . \end{cases} \tag{1.12}$$

We now state our result:

MAIN THEOREM. *For any data $u_0(x), u_1(x) \in H^\infty(\Omega)$, $f(t, x) \in \mathscr{E}_t^\infty(H^\infty(\Omega))$ and $g(t, s) \in \mathscr{E}_t^\infty(H^\infty(S))$ which satisfy the compatibility condition of infinite order, there is a unique solution $u(t, x)$ of the problem* (1.1) *which belongs to $\mathscr{E}_t^\infty(H^\infty(\Omega))$.*

§2. Some examples for reduction to symmetric hyperbolic system.

In this section, we consider mixed problems for wave equation with Dirichlet, Neumann and the oblique derivative boundary conditions, and discuss how to reduce the mixed problems for wave equation to those for symmetric hyperbolic systems of first order with non-negative boundary condition. By this reduction, we can easily obtain the energy inequality.

We set

$$U=\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}=\begin{pmatrix} u_t-u_x+\alpha u_y \\ \alpha(u_t+u_x)+u_y \end{pmatrix} \tag{2.1}$$

where $u_{tt}-u_{xx}-u_{yy}=0$ and α is a complex constant. Then, U satisfies the following equation

$$U_t=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}U_x+\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}U_y\ . \tag{2.2}$$

EXAMPLE 1. We consider the mixed problem with Dirichlet boundary condition

$$\begin{cases} M[u]=\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}+\gamma_1\dfrac{\partial u}{\partial t}+\gamma_2\dfrac{\partial u}{\partial x}+\gamma_3\dfrac{\partial u}{\partial y}+\gamma_4 u=f(t, x, y) \\ u(0, x, y)=u_0(x, y)\ , \quad u_t(0, x, y)=u_1(x, y) \\ B[u]|_{x=0}=u|_{x=0}=g(t, y) \\ (t, x, y)\in \boldsymbol{R}^1_+\times\boldsymbol{R}^1_+\times\boldsymbol{R}^1_+ \end{cases} \tag{2.3}$$

where γ_j is a complex constant ($j=1, \cdots, 4$).

Differentiating $u|_{x=0}=g(t, y)$ with respect to t and y, we have $u_t|_{x=0}=g_t(t, y)$ and $u_y|_{x=0}=g_y(t, y)$. We set

$$U=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix}=\begin{pmatrix} u_t-u_x+\alpha_1 u_y \\ \alpha_1(u_t+u_x)+u_y \\ u_t-u_x+\alpha_2 u_y \\ \alpha_2(u_t+u_x)+u_y \\ u \end{pmatrix}. \tag{2.4}$$

If $\alpha_1\neq\alpha_2$, we have

$$\operatorname{rank}\begin{pmatrix} 1 & -1 & \alpha_1 \\ \alpha_1 & \alpha_1 & 1 \\ 1 & -1 & \alpha_2 \\ \alpha_2 & \alpha_2 & 1 \end{pmatrix}=3\ . \tag{2.5}$$

Therefore, the problem (2.3) is transformed into the system:

$$
(2.6)\quad \begin{cases}
U_t=\begin{pmatrix} -1 & & & & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} U_x+\left(\begin{array}{cc|cc|c} 0 & 1 & & & \\ 1 & 0 & & & \\ \hline & & 0 & 1 & \\ & & 1 & 0 & \\ \hline & & & & 0 \end{array}\right) U_y+D_1U+F_1 \\
\quad =A_1U_x+B_1U_y+D_1U+F_1 \\
U(0,\, x,\, y)=U_0(x,\, y) \\
P_1U|_{x=0}=G_1 \\
(t,\, x,\, y)\in \boldsymbol{R}_+^1\times \boldsymbol{R}_+^1\times \boldsymbol{R}_+^1
\end{cases}
$$

where D_1 is a 5×5 constant matrix,

$$
P_1=\begin{pmatrix} \alpha_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1 & 0 \end{pmatrix},\quad F_1={}^t(f,\, \alpha_1 f,\, f,\, \alpha_2 f,\, 0)
$$

$$
G_1={}^t(2\alpha_1 g_t+(\alpha_1^2+1)g_y,\, 2\alpha_2 g_t+(\alpha_2^2+1)g_y)
$$

α_1 and α_2 are different complex constants, and $|\alpha_j|>1$ $(j=1,\, 2)$.

By simple calculations, we obtain

$$
(2.7)\qquad ((A_1U,\, U))\geqq C((U,\, U))\qquad \text{for any}\quad U\in \operatorname{Ker} P_1
$$

where C is a positive constant.

EXAMPLE 2. We study the mixed problem with the oblique derivative boundary condition

$$
(2.8)\quad \begin{cases}
M[u]=\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}+\gamma_1\dfrac{\partial u}{\partial t}+\gamma_2\dfrac{\partial u}{\partial x}+\gamma_3\dfrac{\partial u}{\partial y}+\gamma_4 u=f(t,\, x,\, y) \\
u(0,\, x,\, y)=u_0(x,\, y),\ u_t(0,\, x,\, y)=u_1(x,\, y) \\
B[u]\Big|_{x=0}=\Big(\dfrac{\partial u}{\partial x}+b\dfrac{\partial u}{\partial y}-c\dfrac{\partial u}{\partial t}+\gamma_5 u\Big)\Big|_{x=0}=g(t,\, y) \\
(t,\, x,\, y)\in \boldsymbol{R}_+^1\times \boldsymbol{R}_+^1\times \boldsymbol{R}_+^1
\end{cases}
$$

where $\gamma_j (j=1, \cdots, 5)$, b and c are complex constants.

We assume that the quadratic equation

$$
(2.9)\qquad (c+1)z^2+2bz+(c-1)=0
$$

has roots in $\bar{D}=\{z\in \boldsymbol{C}|\ |z|\leqq 1\}$ if they are different and in $D=\{z\in \boldsymbol{C}|\ |z|<1\}$ if they are equal.

Case (I). The equation (2.9) has different roots in $\bar{D}$.

We set

$$U=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix}=\begin{pmatrix} u_t-(u_x+\gamma_5 u)+z_1 u_y \\ z_1(u_t+u_x+\gamma_5 u)+u_y \\ u_t-(u_x+\gamma_5 u)+z_2 u_y \\ z_2(u_t+u_x+\gamma_5 u)+u_y \\ u \end{pmatrix} \tag{2.10}$$

where z_1 and z_2 are two different roots of the equation (2.9). Then, we have

$$\operatorname{rank}\begin{pmatrix} 1 & -1 & z_1 \\ z_1 & z_1 & 1 \\ 1 & -1 & z_2 \\ z_2 & z_2 & 1 \end{pmatrix}=3 . \tag{2.11}$$

Therefore, the problem (2.8) is transformed into the system:

$$\begin{cases} U_t=\begin{pmatrix} -1 & & & & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} U_x+\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 0 \end{pmatrix} U_y+D_2U+F_2 \\ \quad =A_2U_x+B_2U_y+D_2U+F_2 \\ U(0, x, y)=U_0(x, y) \\ P_2U|_{x=0}=G_2 \\ (t, x, y)\in \boldsymbol{R}^1_+\times\boldsymbol{R}^1_+\times\boldsymbol{R}^1_+ \end{cases} \tag{2.12}$$

where D_2 is a 5×5 constant matrix

$$P_2=\begin{pmatrix} 1 & z_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & z_1 & 0 \end{pmatrix}, \quad F_2={}^t(f, z_1f, f, z_2f, 0)$$

and

$$G_2={}^t\left(-\frac{2g}{c+1}, -\frac{2g}{c+1}\right).$$

Here, by the assumption and simple calculations, we obtain

$$((A_2U, U))\geqq 0 \qquad \text{for any} \quad U\in \operatorname{Ker} P_2 . \tag{2.13}$$

Especially, for $b=c=0$, we are concerned with the mixed problem with

Neumann boundary condition. Then, we have $z_1=1$ and $z_2=-1$.

Case (II). The equation (2.9) has a non zero double root z_0 in D.

We set

$$U=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}=\begin{pmatrix} u_t-(u_x+\gamma_5 u)+\sqrt{1-\varepsilon^2}z_1 u_y \\ z_1(u_t+u_x+\gamma_5 u)+\sqrt{1-\varepsilon^2}u_y \\ \varepsilon\sqrt{1+|z_1|^2}u_y \\ u \end{pmatrix} \tag{2.14}$$

where ε is a sufficiently small positive constant, z_1 and z_2 are solutions of the equation

$$\sqrt{1-\varepsilon^2}(c+1)z^2+2bz+\sqrt{1-\varepsilon^2}(c-1)=0 \tag{2.15}$$

and

$$z_1=\frac{1-\varepsilon}{\sqrt{1-\varepsilon^2}}z_0\,,\quad z_2=\frac{1+\varepsilon}{\sqrt{1-\varepsilon^2}}z_0\,,\quad 0<|z_i|<1\quad (i=1,2)\,.$$

Then we have

$$\operatorname{rank}\begin{pmatrix} 1 & -1 \\ z_1 & z_1 \end{pmatrix}=2\,. \tag{2.16}$$

Therefore, the problem (2.8) is transformed into the system:

(2.17)

$$\begin{cases} U_t=\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \dfrac{1-|z_1|^2}{1+|z_1|^2} & \\ 0 & & & 1 \end{pmatrix}U_x+\begin{pmatrix} 0 & \sqrt{1-\varepsilon^2} & \dfrac{\varepsilon}{\sqrt{1+|z_1|^2}} & 0 \\ \sqrt{1-\varepsilon^2} & 0 & \dfrac{\varepsilon z_1}{\sqrt{1+|z_1|^2}} & 0 \\ \dfrac{\varepsilon}{\sqrt{1+|z_1|^2}} & \dfrac{\varepsilon\bar{z}_1}{\sqrt{1+|z_1|^2}} & -\dfrac{2\sqrt{1-\varepsilon^2}}{1+|z_1|^2}\operatorname{Re} z_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}U_y \\ \qquad +D_3U+F_3 \\ \quad =A_3U_x+B_3U_y+D_3U+F_3 \\ U(0,x,y)=U_0(x,y) \\ P_3U|_{x=0}=G_3 \\ (t,x,y)\in \boldsymbol{R}^1_+\times\boldsymbol{R}^1_+\times\boldsymbol{R}^1_+ \end{cases}$$

where D_3 is a 4×4 constant matrix,

$$P_3=(1,z_2,0,0)\,,\quad F_3={}^t(f,zf_1,0,0)$$

and

$$G_3 = -\frac{2g}{c+1}\,.$$

By the assumption and simple calculations, we have

$$((A_3U,\ U)) \geqq C((U,\ U)) \quad \text{for any} \quad U \in \operatorname{Ker} P_3 \tag{2.18}$$

where C is a positive constant.

Case (III). The equation (2.9) has zero double root i.e. $b=0$ and $c=1$. We set

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} u_t - (u_x + \gamma_5 u) + \sqrt{1-\varepsilon^2}\, z_1 u_y \\ z_1(u_t + u_x + \gamma_5 u) + \sqrt{1-\varepsilon^2}\, u_y \\ \varepsilon\sqrt{1+|z_1|^2}\, u_y \\ u \end{pmatrix} \tag{2.19}$$

where $z_1 = 1/(2(c+1)) = 1/4$, $z_2 = 2(c-1) = 0$ and ε is a positive constant smaller than 1. Then, we get

$$\operatorname{rank}\begin{pmatrix} 1 & -1 \\ z_1 & z_1 \end{pmatrix} = 2\,. \tag{2.20}$$

Hence, the problem (2.8) is transformed into the system:

(2.21)

$$\begin{cases} U_t = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \dfrac{1-|z_1|^2}{1+|z_1|^2} & \\ 0 & & & 1 \end{pmatrix} U_x + \begin{pmatrix} 0 & \sqrt{1-\varepsilon^2} & \dfrac{\varepsilon}{\sqrt{1+|z_1|^2}} & 0 \\ \sqrt{1-\varepsilon^2} & 0 & \dfrac{\varepsilon z_1}{\sqrt{1+|z_1|^2}} & 0 \\ \dfrac{\varepsilon}{\sqrt{1+|z_1|^2}} & \dfrac{\varepsilon \bar{z}_1}{\sqrt{1+|z_1|^2}} & -\dfrac{2\sqrt{1-\varepsilon^2}}{1+|z_1|^2}\operatorname{Re} z_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} U_y \\ \qquad + D_4U + F_4 \\ \qquad = A_4U_x + B_4U_y + D_4U + F_4 \\ U(0,\ x,\ y) = U_0(x,\ y) \\ P_4U|_{x=0} = G_4 \\ (t,\ x,\ y) \in \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \end{cases}$$

where D_4 is a 4×4 constant matrix,

$$P_4 = (1,\ z_2,\ z_3,\ 0)\,, \quad z_2 = 0\,, \quad z_3 = -\frac{1}{\sqrt{17}}\,\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}$$

and

$$F_4={}^t(f,\ z_1 f,\ 0,\ 0)\ ,\quad G_4=-\frac{2g}{c+1}(=-g)\ .$$

For $\varepsilon(\varepsilon\fallingdotseq 1)$, we have

$$((A_4U,\ U))\geqq C((U,\ U))\qquad \text{for any}\quad U\in \operatorname{Ker} P_4 \tag{2.22}$$

where C is a positive constant.

REMARK 2. The problem (2.8) is L^2-well-posed if and only if the equation (2.9) has roots in $\bar{D}=\{z\in C|\ |z|\leqq 1\}$ if they are different and in D if they are equal (see [6] and [10]).

REMARK 3. Let M be the same operator in (1.6). If M is regularly hyperbolic with respect to t, then we have

$$\begin{cases}\beta_{11}+\alpha_1^2>0\\ \sum_{i,j=2}^{n}\beta_{ij}\eta_i\eta_j+\Big(\sum_{j=2}^{n}\alpha_j\eta_j\Big)^2-(\beta_{11}+\alpha_1^2)\Big(\sum_{j=2}^{n}\beta_{1j}\eta_j+\alpha_1\sum_{j=2}^{n}\alpha_j\eta_j\Big)^2>0\end{cases} \tag{2.23}$$

and

$$\sigma(M)=\tilde{\xi}_*^2+\tilde{d}_*(\eta')^2-\tilde{\tau}_*^2 \tag{2.24}$$

where

$$\begin{cases}\tilde{\tau}_*=\tau-\alpha_1\xi-\sum_{j=2}^{n}\alpha_j\eta_j\\ \tilde{\xi}_*=(\beta_{11}+\alpha_1^2)^{1/2}\Big\{\xi+(\beta_{11}+\alpha_1^2)^{-1}\Big(\sum_{j=2}^{n}\beta_{1j}\eta_j+\alpha_1\sum_{j=2}^{n}\alpha_j\eta_j\Big)\Big\}\\ \tilde{d}_*(\eta')=\Big[\sum_{i,j=2}^{n}\beta_{ij}\eta_i\eta_j+\Big(\sum_{j=2}^{n}\alpha_j\eta_j\Big)^2\\ \qquad\qquad -(\beta_{11}+\alpha_1^2)^{-1}\Big(\sum_{j=2}^{n}\beta_{1j}\eta_j+\alpha_1\sum_{j=2}^{n}\alpha_j\eta_j\Big)^2\Big]^{1/2}\end{cases} \tag{2.25}$$

for $(\xi,\ \eta')=(\xi,\ \eta_2,\ \cdots,\ \eta_n)\in \boldsymbol{R}^n$. This representation is used to obtain the energy estimate for the Cauchy problem. If $\beta_{11}>0$, we have $\tilde{d}(\eta')=\tilde{d}_*(\eta')$ (see [10]).

§3. Mixed problem with zero initial data.

In this section, we consider the mixed problem

$$(3.1)\quad \begin{cases} L[u]=\dfrac{\partial^2 u}{\partial t^2}-2t^k\displaystyle\sum_{j=1}^n h_j(t,x)\dfrac{\partial^2 u}{\partial t\partial x_j}-t^{2k}\sum_{i,j=1}^n a_{ij}(t,x)\dfrac{\partial^2 u}{\partial x_i\partial x_j}+a_0(t,x)\dfrac{\partial u}{\partial t} \\ \qquad +t^{k-1}\displaystyle\sum_{j=1}^n a_j(t,x)\dfrac{\partial u}{\partial x_j}+d(t,x)u=f_1(t,x) \\ u(0,x)=0\ ,\quad u_t(0,x)=0 \\ B[u]|_S=t^k\Big(\sqrt{\displaystyle\sum_{i,j=1}^n a_{ij}(t,s)\nu_i\nu_j}\dfrac{\partial u}{\partial \nu}+X(t,s)u\Big)-\beta(t,s)\dfrac{\partial u}{\partial t}+\gamma(t,s)u|_S \\ \qquad =g_1(t,s) \\ (t,x)\in(0,T)\times\Omega\ . \end{cases}$$

Now, we shall prove

THEOREM 3.1. *Let m be a non-negative integer. Then, we can find an integer $N(>m)$ such that for any $f_1\in\mathscr{E}_t^\infty(H^\infty(\Omega))$ and $g_1\in\mathscr{E}_t^\infty(H^\infty(S))$ satisfying $(\partial/\partial t)^i f_1|_{t=0}=0$ $(i=0,1,\cdots,N)$ and $(\partial/\partial t)^j g_1|_{t=0}=0$ $(j=0,\cdots,N+1)$, there exists a unique solution $u(t,x)$ of the mixed problem* (3.1) *which belongs to $\bigcap_{j=0}^{m+2}\mathscr{E}_t^j(H^{m+2-j}(\Omega))$.*

To obtain the above theorem, we treat the following problem

$$(3.2)\quad \begin{cases} L_\delta[u]=\dfrac{\partial^2 u}{\partial t^2}-2(t+\delta)^k\displaystyle\sum_{j=1}^n h_j(t,x)\dfrac{\partial^2 u}{\partial t\partial x_j}-(t+\delta)^{2k}\sum_{i,j=1}^n a_{ij}(t,x)\dfrac{\partial^2 u}{\partial x_i\partial x_j} \\ \qquad +a_0(t,x)\dfrac{\partial u}{\partial t}+(t+\delta)^{k-1}\displaystyle\sum_{j=1}^n a_j(t,x)\dfrac{\partial u}{\partial x_j}+d(t,x)u=f_1(t,x) \\ u(0,x)=0\ ,\quad u_t(0,x)=0 \\ B_\delta[u]\Big|_S=\Big\{(t+\delta)^k\Big(\sqrt{\displaystyle\sum_{i,j=1}^n a_{ij}(t,s)\nu_i\nu_j}\dfrac{\partial u}{\partial \nu}+X(t,s)u\Big)-\beta(t,s)\dfrac{\partial u}{\partial t} \\ \qquad +\gamma(t,s)u\Big\}\Big|_S=g_1(t,s) \\ (t,x)\in(0,T)\times\Omega \end{cases}$$

where $0<\delta\ll 1$, $f_1\in\mathscr{E}_t^\infty(H^\infty(\Omega))$, $g_1\in\mathscr{E}_t^\infty(H^\infty(S))$, and

$$(3.3)\quad \begin{cases} \Big(\dfrac{\partial}{\partial t}\Big)^i f_1|_{t=0}=0 \quad (i=0,1,\cdots,N_1) \\ \Big(\dfrac{\partial}{\partial t}\Big)^j g_1|_{t=0}=0 \quad (j=0,1,\cdots,N_1+1)\ . \end{cases}$$

By the assumptions (A.I) and (A.II), and the fact that the operator L_δ is regularly hyperbolic, we obtain a unique solution $u_\delta(t,x)$ of the mixed problem (3.2)-(3.3) which belongs to $\cap_{j=0}^{N_1+3}\mathscr{E}_t^j(H^{N_1+3-j}(\Omega))$ (see [11]).

LEMMA 3.2. *Let m be a non-negative integer. Then, there exists a integer $N_2(>m)$ such that for any solution $u_\delta(t, x)\in\cap_{j=0}^{m+3}\mathscr{E}_t^j(H^{m+3-j}(\Omega))$ of the mixed problem* (3.2)-(3.3), *the estimate*

$$\sum_{j=0}^{m+3}\left\|\left(\frac{\partial}{\partial t}\right)^j u_\delta(t, \cdot)\right\|_{m+3-j,\Omega}^2 \leq C_{T,m}\int_0^t\left\{\left\|\left(\frac{\partial}{\partial \tau}\right)^{N_2+1} f_1(\tau, \cdot)\right\|_{m+2,\Omega}^2+\left\|\left(\frac{\partial}{\partial \tau}\right)^{N_2+1} g_1(\tau, \cdot)\right\|_{m+2,S}^2\right\}d\tau \tag{3.4}$$

holds for any $t\in[0, T]$ provided that $f_1(t, x)\in\mathscr{E}_t^\infty(H^\infty(\Omega))$ and $g_1(t, s)\in\mathscr{E}_t^\infty(H^\infty(S))$ satisfying $(\partial/\partial t)^i f_1|_{t=0}=0$ $(i=0, 1, \cdots, N_2)$ and $(\partial/\partial t)^j g_1|_{t=0}=0$ $(j=0, 1, \cdots, N_2+1)$ where $C_{T,m}$ is a positive constant independent of δ, f_1 and g_1.

PROOF. Since the proof is fairly lengthy, we separate it into three steps.

First step. We transform the mixed problem (3.2) into the one for symmetric hyperbolic pseudo differential systems of first order with positive boundary condition.

Let $\{V_j\}_{j=1}^l$ be an open covering of $S=\partial\Omega$ such that for any j, there exists a smooth coordinate transformation $\Psi_j=(\phi_{j1}, \cdots, \phi_{jn})$ from V_j onto W_j in $\boldsymbol{R}^n$ with properties

$$\begin{cases}\Psi_j(V_j\cap\Omega)=W_j\cap\boldsymbol{R}_+^n\ , & \boldsymbol{R}_+^n=\{y=(y_1, \cdots, y_n)\,|\,y_1>0\}\\ \Psi_j(V_j\cap S)=W_j\cap\boldsymbol{R}^{n-1}\ ,\end{cases} \tag{3.5}$$

and, L and B in $V_j\cap\bar{\Omega}$ are transformed into $\tilde{L}$ and $\tilde{B}$ in $W_j\cap\overline{\boldsymbol{R}_+^n}$ which are the same operators in (1.2) and (1.3). Next, let $\{\varphi_j(x)\}_{j=1}^l$ be a partition of unity in a neighborhood of $S=\partial\Omega$ corresponding to the open covering $\{V_j\}_{j=1}^l$. Without loss of generality, we have only to treat the case that the support of $\varphi(x)$ is contained in a small neighborhood V_* of $s_*\in S$ and there exists a smooth function $\rho(x)$ which satisfies the following condition

$$\begin{cases}\rho(V_*\cap S)=0\\ \dfrac{\partial\rho}{\partial x_1}(V_*)\neq 0\ .\end{cases} \tag{3.6}$$

We consider the localized problem of (3.2)

$$
(3.7)\qquad \begin{cases} L_\delta[\varphi\cdot u_\delta]=\varphi\cdot f_1+[L_\delta,\varphi]u_\delta \\ (\varphi\cdot u_\delta)(0,x)=0\ ,\quad (\varphi u_\delta)_t(0,x)=0 \\ B_\delta[u]|_S=\varphi\cdot g_1+[B_\delta,\varphi]u_\delta|_S \\ (t,x)\in(0,T)\times\Omega\ . \end{cases}
$$

By the transformation

$$
(3.8)\qquad y_1=\rho(x)\quad\text{and}\quad y_j=x_j\qquad (j=2,\cdots,n)
$$

we have

$$
(3.9)\qquad \begin{cases} \tilde{L}_\delta[\bar{\varphi}\cdot\bar{u}_\delta]=\bar{\varphi}\cdot\bar{f}_1+[\tilde{L}_\delta,\bar{\varphi}]\bar{u}_\delta \\ (\bar{\varphi}\cdot\bar{u}_\delta)(0,y)=0,\ (\bar{\varphi}\cdot\bar{u}_\delta)_t(0,y)=0 \\ \tilde{B}_\delta[\bar{\varphi}\cdot\bar{u}_\delta]|_{y_1=0}=\bar{\varphi}\cdot\bar{g}_1+[\tilde{B}_\delta,\bar{\varphi}]\bar{u}_\delta|_{y_1=0} \\ (t,y)\in(0,T)\times\boldsymbol{R}^n_+ \end{cases}
$$

where $\boldsymbol{R}^n_+=\{y=(y_1,y_2,\cdots,y_n)|y_1>0\}$, $y'=(y_2,\cdots,y_n)$,

$$
(3.10)\qquad \begin{cases} \tilde{L}_\delta=\dfrac{\partial^2}{\partial t^2}-2(t+\delta)^k\sum_{j=1}^n\tilde{h}_j(t,y)\dfrac{\partial^2}{\partial t\partial y_j}-(t+\delta)^{2k}\sum_{i,j=1}^n\tilde{a}_{ij}(t,y)\dfrac{\partial^2}{\partial y_i\partial y_j} \\ \qquad +\tilde{a}_0(t,y)\dfrac{\partial}{\partial t}+(t+\delta)^{k-1}\sum_{j=1}^n\tilde{a}_j(t,y)\dfrac{\partial}{\partial y_j}+\tilde{d}(t,y) \\ \tilde{B}_\delta=\dfrac{1}{\sqrt{\tilde{a}_{11}(t,o,y')}}\Big\{(t+\delta)^k\Big[\tilde{a}_{11}(t,0,y')\dfrac{\partial}{\partial y_1}+\sum_{j=2}^n\tilde{a}_{1j}(t,0,y')\dfrac{\partial}{\partial y_j}\Big] \\ \qquad +\tilde{h}_1(t,0,y')\dfrac{\partial}{\partial t}\Big\}+(t+\delta)^k\sum_{j=2}^n\tilde{\alpha}_j(t,y')\dfrac{\partial}{\partial y_j} \\ \qquad -\tilde{\beta}(t,y')\Big(1+\dfrac{\tilde{h}_1(t,0,y')^2}{\tilde{a}_{11}(t,0,y')}\Big)^{1/2}\Big\{\dfrac{\partial}{\partial t}-(t+\delta)^k\Big(1+\dfrac{\tilde{h}_1(t,0,y')^2}{\tilde{a}_{11}(t,0,y')}\Big)^{-1} \\ \qquad \cdot\Big[\sum_{j=2}^n\tilde{h}_j(t,0,y')\dfrac{\partial}{\partial y_j}-\dfrac{\tilde{h}_1(t,0,y')}{\tilde{a}_{11}(t,o,y')}\sum_{j=2}^n\tilde{a}_{1j}(t,0,y')\dfrac{\partial}{\partial y_j}\Big]\Big\}+\tilde{\gamma}(t,y')\ , \end{cases}
$$

and for the function $w(x)$ defined in a domain $V_*\cap\Omega$, we denote by $\bar{w}(y)$ the function defined in $\boldsymbol{R}^n_+$ ($\bar{w}(y)=\bar{w}(\rho(x),x_2,\cdots,x_n)=w(x)$). The functions $\tilde{h}_j$, $\tilde{a}_{ij}$, $\tilde{a}_0$, $\tilde{a}_j$ and $\tilde{d}$ have extensions $\tilde{\tilde{h}}_j$, $\tilde{\tilde{a}}_{ij}$, $\tilde{\tilde{a}}_0$, $\tilde{\tilde{a}}_j$ and $\tilde{\tilde{d}}$ which belong to $\mathscr{B}([0,T]\times\overline{\boldsymbol{R}^n_+})$ and are constant outside a compact set in $[0,T]\times\overline{\boldsymbol{R}^n_+}$. The functions, $\tilde{\alpha}_j$, $\tilde{\beta}$ and $\tilde{\gamma}$ have extensions $\tilde{\tilde{\alpha}}_j$, $\tilde{\tilde{\beta}}$ and $\tilde{\tilde{\gamma}}$ such that $\tilde{\tilde{\alpha}}_j$ and $\tilde{\tilde{\gamma}}$ belong to $C_0^\infty([0,T]\times\boldsymbol{R}^{n-1})$, and $\tilde{\tilde{\beta}}$ belongs to $\mathscr{B}([0,T]\times\boldsymbol{R}^{n-1})$ and is a positive constant outside a compact set in $[0,T]\times\boldsymbol{R}^{n-1}$. And we are able to have that these extended functions satisfy the following conditions:

(i) $$\frac{\partial^2}{\partial t^2}-2\sum_{j=1}^{n}\tilde{\tilde{h}}_j(t, y)\frac{\partial^2}{\partial t\partial y_j}-\sum_{i,j=1}^{n}\tilde{\tilde{a}}_{ij}(t, y)\frac{\partial^2}{\partial y_i\partial y_j}$$

is regularly hyperbolic on $[0, T]\times\overline{\boldsymbol{R}^n_+}$ and $\tilde{\tilde{a}}_{11}(t, y)>0$.

(ii) The quadratic equation

$$(\tilde{c}+1)z^2+2\tilde{b}z+(\tilde{c}-1)=0 \tag{3.11}$$

has roots in $D=\{z\in \boldsymbol{C}|\ |z|<1\}$ where

$$\begin{cases}\tilde{b}=\sum_{j=2}^{n}\tilde{\tilde{\alpha}}_j(t, y')\eta_j/d(\eta')\\ \tilde{c}=\tilde{\tilde{\beta}}(t, y')\\ d(\eta')=\Big[\sum_{i,j=2}^{n}\tilde{\tilde{a}}_{ij}(t, 0, y')\eta_i\eta_j-\dfrac{1}{\tilde{\tilde{a}}_{11}(t, 0, y')}\Big(\sum_{j=2}^{n}\tilde{\tilde{a}}_{1j}(t, 0, y')\eta_j\Big)^2\\ \qquad+\Big(1+\dfrac{\tilde{\tilde{h}}_1(t, 0, y')^2}{\tilde{\tilde{a}}_{11}(t, 0, y')}\Big)^{-1}\cdot\Big(\sum_{j=2}^{n}h_j(t, 0, y')\eta_j\\ \qquad-\dfrac{\tilde{\tilde{h}}_1(t, 0, y')}{\tilde{\tilde{a}}_{11}(t, 0, y')}\sum_{j=2}^{n}\tilde{\tilde{a}}_{1j}(t, 0, y')\eta_j\Big)^2\Big]^{1/2}.\end{cases} \tag{3.12}$$

We denote again these extended functions $\tilde{\tilde{\theta}}$ by $\tilde{\theta}$.

LEMMA 3.3. *Let α and β be complex constatns such that $\alpha=c+b$, $\beta=c-b$, $c=c_1+ic_2$, $b=b_1+ib_2$ and, b_1, b_2, c_1 and c_2 are real constants.*

(i) *The following conditions are equivalent.*

① $$\begin{pmatrix}2\,\mathrm{Re}\,\alpha & \mathrm{Im}(\alpha\cdot\bar{\beta})\\ \mathrm{Im}(\alpha\cdot\bar{\beta}) & 2\,\mathrm{Re}\,\beta\end{pmatrix}>0\,.$$

② *The quadratic equation*

$$(c+1)z^2+2bz+(c-1)=0$$

has roots in $D=\{z\in \boldsymbol{C}|\ |z|<1\}$.

(ii) *If the condition* (i)-① *holds, we have*

$$\begin{pmatrix}2\,\mathrm{Re}\,\alpha' & \mathrm{Im}\,(\alpha'\cdot\bar{\beta}')\\ \mathrm{Im}\,(\alpha'\cdot\bar{\beta}') & 2\,\mathrm{Re}\,\beta'\end{pmatrix}>0\,,$$

where $\alpha'=c+kb$, $\beta'=c-kb$ and k is any constant such that $0\leqq k\leqq 1$.

PROOF. By the Hermite theorem in [10] and the conformal mapping from the upper half plane to the unit disk, we have Lemma 3.3.

Q.E.D.

By Lemma 3.3, the quadratic equation

$$(\tilde{c}+1)z^2+2\tilde{b}_\mu z+(\tilde{c}-1)=0 \tag{3.13}$$

has roots in $D=\{z\in C|\ |z|<1\}$ where

$$\begin{cases}\tilde{b}_\mu=\sum_{j=2}^{n}\tilde{\alpha}_j(t, y')\eta_j/d_\mu(\eta') \\ d_\mu(\eta')=\sqrt{d(\eta')^2+\mu}\end{cases} \tag{3.14}$$

and μ is a positive constant.

For $(t_0, y_0', \eta_0', \sigma_0)\in[0, T]\times R^{n-1}\times\bar{\Sigma}_+(\bar{\Sigma}_+=\{(\eta', \sigma)|\mu=\sigma^2, \sigma\geqq 0, \eta'=(\eta_2, \cdots, \eta_n), |\eta'|^2+\sigma^2=1\})$, we choose z_1 and z_2 in the following ways:

Case (I). If the equation (3.13) has different roots in $D=\{z\in C|\ |z|<1\}$, we choose z_1 and z_2 as the solutions of the equation (3.13).

Case (II). If the equations (3.13) has a non-zero double root in $D=\{z\in C|\ |z|<1\}$, we choose z_1 and z_2 as the solutions of equation

$$\sqrt{1-\varepsilon^2}(\tilde{c}+1)z^2+2\tilde{b}_\mu z+\sqrt{1-\varepsilon^2}(\tilde{c}-1)=0 \tag{3.15}$$

where ε is a sufficiently small positive constant.

Case (III). If the equation (3.13) has zero double root, we determine

$$z_1=\frac{1}{2(\tilde{c}+1)} \quad\text{and}\quad z_2=2(\tilde{c}-1)\,.$$

Let $\{\omega_j\}_{j=1}^p$ be an open covering of the set $[0, T]\times R^{n-1}\times\bar{\Sigma}_+$ such that for any j,

(i) ω_j is an open set in $[0, T]\times R^{n-1}\times\bar{\Sigma}_+$

(ii) for any $(t, y', \eta', \sigma)\in\omega_j$,

① $z_1\neq z_2$,

② $\mathrm{Re}\, z_1>\mathrm{Re}\, z_2$ or $\mathrm{Im}\, z_1>\mathrm{Im}\, z_2$,

③ $z_1, z_2\in D=\{z\in C|\ |z|<1\}$

where z_1 and z_2 are determined by above arguments.

Now, we consider a partition of unity

$$\begin{cases}\sum_{j=1}^{p}q_j(t, y', \eta': \sigma)=1 \quad\text{on}\quad [0, T]\times R^{n-1}\times\bar{\Sigma}_+ \\ q_j\in C^\infty([0, T]\times R^{n-1}\times\bar{\Sigma}_+)\end{cases} \tag{3.16}$$

corresponding to the open covering $\{\omega_j\}_{j=1}^p$. We extend each q_j keeping homogeneity of order zero with respect to (η', σ) such that $q_j\in S^0(\mu)$. For any j, we determine smooth functions $\zeta_j\in\mathscr{B}([0, T]\times R^{n-1}\times\bar{\Sigma}_+)$ such that supp $\zeta_j \subset \omega_j$, $0 \leqq \zeta_j \leqq 1$ and $\zeta_j = 1$ on supp $[q_j|_{\bar{\Sigma}_+}]$, and extend each ζ_j keeping homogeneity of order zero with respect to (η', σ) such that $\zeta_j \in S^0(\mu)$.

Operating the pseudo differential operator $q_j(t, y', (1/i)D_{y'}: \sigma)$ with respect to $y' = (y_2, \cdots, y_n)$ on (3.9), we have

$$\text{(3.17)} \qquad \begin{cases} \tilde{L}_\delta[v] = f_2 \\ v(0, y) = 0, \ v_t(0, y) = 0 \\ \tilde{B}_\delta[v]|_{y_1=0} = g_2 \\ (t, y) \in (0, T) \times \boldsymbol{R}^n_+ \end{cases}$$

where $q_j(\bar{\varphi} \cdot \bar{u}_\delta) = v$, $\psi \bar{u}_\delta = w$, $\psi \in C_0^\infty(\boldsymbol{R}^n)$, $\psi = 1$ on supp $[\bar{\varphi}]$,

$$\text{(3.18)} \qquad \begin{aligned} f_2 &= q_j\{\bar{\varphi} \cdot \bar{f}_1 + [\tilde{L}_\delta, \bar{\varphi}]\psi \bar{u}_\delta\} + [\tilde{L}_\delta, q_j](\bar{\varphi} \cdot \psi \bar{u}_\delta) \\ &= q_j(\bar{\varphi} \cdot \bar{f}_1) + T_0^{(1)} w_t + (t+\delta)^k \sum_{j=1}^n T_j^{(1)} \frac{\partial w}{\partial y_j} + T_{00}^{(1)} w \\ &\qquad (T_0^{(1)}, \ T_j^{(1)}, \ T_{00}^{(1)} \in \mathcal{S}^0(1)) \end{aligned}$$

and

$$\text{(3.19)} \qquad \begin{aligned} g_2 &= q_j\{\bar{\varphi} \cdot \bar{g}_1 + [\tilde{B}_\delta, \bar{\varphi}]\psi \bar{u}_\delta|_{y_1=0}\} + [\tilde{B}_\delta, q_j](\bar{\varphi} \cdot \psi \bar{u}_\delta)|_{y_1=0} \\ &= q_j(\bar{\varphi} \cdot \bar{g}_1) + T_{00}^{(2)} w|_{y_1=0} \quad (T_{00}^{(2)} \in \mathcal{S}^0(1)) \ . \end{aligned}$$

From now on, we transform the mixed problem (3.17) into the one for first order system. Let Q_0 and Q_1 be

$$\text{(3.20)} \qquad \begin{aligned} Q_0 = \Big(1 + \frac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\Big)^{1/2} \Big\{ \frac{\partial}{\partial t} - \Big(1 + \frac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\Big)^{-1} (t+\delta)^k \ . \\ \cdot \Big(\sum_{j=2}^n \tilde{h}_j(t, y) \frac{\partial}{\partial y_j} - \frac{\tilde{h}_1(t, y)}{\tilde{a}_{11}(t, y)} \sum_{j=2}^n \tilde{a}_{1j}(t, y) \frac{\partial}{\partial y_j} \Big) \Big\} \end{aligned}$$

and

$$\text{(3.21)} \qquad \begin{aligned} Q_1 = \frac{1}{\sqrt{\tilde{a}_{11}(t, y)}} \Big[\tilde{a}_{11}(t, y) \frac{\partial}{\partial y_1} + \sum_{j=2}^n \tilde{a}_{1j}(t, y) \frac{\partial}{\partial y_j} \\ + \frac{1}{(t+\delta)^k} \tilde{h}_1(t, y) \frac{\partial}{\partial t} \Big] + \frac{1}{(t+\delta)^k} \tilde{\gamma}(t, y') \end{aligned}$$

respectively. Also, let Q_2 be a pseudo differential operator with respect to $y' = (y_2, \cdots, y_n)$ with the symbol

$$\text{(3.22)} \qquad \sigma(Q_2) = i \Big[\sum_{i,j=2}^n \tilde{a}_{ij}(t, y) \eta_i \eta_j - \frac{1}{\tilde{a}_{11}(t, y)} \Big(\sum_{j=2}^n \tilde{a}_{1j}(t, y) \eta_j \Big)^2$$

$$+\left(1+\frac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\right)^{-1}\left(\sum_{j=2}^{n}\tilde{h}_j(t, y)\eta_j-\frac{\tilde{h}_1(t, y)}{\tilde{a}_{11}(t, y)}\sum_{j=2}^{n}\tilde{a}_{1j}(t, y)\eta_j\right)^2+\mu\Bigg]^{1/2}.$$

Then, the symbol $\sigma(Q_2)$ satisfies the inequality

$$C\left(\sum_{j=2}^{n}\eta_j^2+\mu\right)\geqq|\sigma(Q_2)|^2\geqq C^{-1}\left(\sum_{j=2}^{n}\eta_j^2+\mu\right) \tag{3.23}$$

for any $(t, y, \eta', \mu)\in[0, T]\times\overline{R^n_+}\times R^{n-1}\times\overline{R^1_+}$ where C is a positive constant. We notice that

$$\begin{aligned}\sigma_0(\tilde{L}_\delta)&=-\tau^2+2(t+\delta)^k\left(\tilde{h}_1(t, y)\xi+\sum_{j=2}^{n}\tilde{h}_j(t, y)\eta_j\right)\tau\\&\quad+(t+\delta)^{2k}\left\{\tilde{a}_{11}(t, y)\xi^2+2\left(\sum_{j=2}^{n}\tilde{a}_{1j}(t, y)\eta_j\right)\xi+\sum_{i,j=2}^{n}\tilde{a}_{ij}(t, y)\eta_i\eta_j\right\}\\&=\{\sigma_0(Q_0)\}^2-\{(t+\delta)^k\sigma_0(Q_1)\}^2-(t+\delta)^{2k}\{\sigma(Q_2)^2+\mu\}\end{aligned}$$

where $\sigma_0(X)$ is the principal symbol of X.

Case (I) of first step. The equation (3.13) has different roots in $D=\{z\in C|\ |z|<1\}$ for any $(t, y', \eta', \sigma)\in\omega_j$.

We set

$$U=\begin{pmatrix}U_1\\U_2\\U_3\\U_4\\U_5\end{pmatrix}=\begin{pmatrix}Q_0v-(t+\delta)^kQ_1v+(t+\delta)^k\tilde{z}_1Q_2v\\ \tilde{z}_1(Q_0v+(t+\delta)^kQ_1v)+(t+\delta)^kQ_2v\\ Q_0v-(t+\delta)^kQ_1v+(t+\delta)^k\tilde{z}_2Q_2v\\ \tilde{z}_2(Q_0v+(t+\delta)^kQ_1v)+(t+\delta)^kQ_2v\\ \sqrt{\mu}\,v\end{pmatrix} \tag{3.24}$$

where $\tilde{z}_1$ and $\tilde{z}_2$ are pseudo differential operators with respect to y' whose symbols satisfy

$$\begin{cases}\tilde{z}_1(t, y', \eta': \sigma)=\zeta_j(t, y', \eta': \sigma)z_1(t, y', \eta': \sigma)+\dfrac{1}{2}(1-\zeta_j(t, y', \eta': \sigma))\\ \tilde{z}_2(t, y', \eta': \sigma)=\zeta_j(t, y', \eta': \sigma)z_2(t, y', \eta': \sigma)\\ |\tilde{z}_j(t, y', \eta': \sigma)|_{\bar{\Sigma}_+}\leqq C_1<1\quad(j=1, 2)\\ C_2\geqq|\tilde{z}_1(t, y', \eta': \sigma)-\tilde{z}_2(t, y', \eta': \sigma)|_{\bar{\Sigma}_+}\geqq C_2^{-1},\end{cases} \tag{3.25}$$

C_1 and C_2 are positive constants independent of $(t, y', \eta': \sigma)\in[0, T]\times R^{n-1}\times\bar{\Sigma}_+$. Then, by the same method as the one in §2, the problem (3.17) is transformed into the problem

$$(3.26)\qquad \begin{cases} M_1 U_t = A_1 U_{y_1} + \sum\limits_{j=2}^{n} B_{1j} U_{y_j} + D_1 Q_2 U \\ \qquad + \left(E_{11} + \dfrac{1}{t+\delta} E_{12}\right) U + K_{11} V_{11} + K_{12} V_{12} + F_1 \\ U(0, y) = 0 \\ P_1 U|_{y_1=0} = G_1 \\ (t, y) \in (0, T) \times \boldsymbol{R}_+^n \end{cases}$$

where

$$M_1 = \left(1 + \frac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\right)^{1/2} \cdot I - \frac{\tilde{h}_1(t, y)}{\sqrt{\tilde{a}_{11}(t, y)}} \begin{pmatrix} -1 & & & & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}$$

$$A_1 = (t+\delta)^k \sqrt{\tilde{a}_{11}(t, y)} \begin{pmatrix} -1 & & & & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}$$

$$B_{1j} = \frac{\tilde{a}_{1j}(t, y)}{\tilde{a}_{11}(t, y)} A_1 + (t+\delta)^k \left(1 + \frac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\right)^{-1/2} \cdot \left(\tilde{h}_j(t, y) - \frac{\tilde{h}_1(t, y)}{\tilde{a}_{11}(t, y)} \tilde{a}_{1j}(t, y)\right) I$$

$$(j = 2, 3, \cdots, n)$$

$$D_1 = (t+\delta)^k \left(\begin{array}{cc|cc|c} 0 & 1 & & & \\ 1 & 0 & & & \\ \hline & & 0 & 1 & \\ & & 1 & 0 & \\ \hline & & & & 0 \end{array}\right)$$

E_{11} $\cdots$ a 5×5 pseudo differential system which has the property that for $\sigma(E_{11}) = (e_{ij})$, the following conditions hold:

(i) $e_{ij}(t, y, \eta' : \mu) \in C^\infty([0, T] \times \overline{\boldsymbol{R}_+^n} \times \boldsymbol{R}^{n-1})$,

(ii) for any $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$, there is a positive constant $C_\theta^{(i,j)}$ independent of μ such that

$$\left|\left(\frac{\partial}{\partial t}\right)^{\theta_1}\left(\frac{\partial}{\partial y_1}\right)^{\theta_2}\left(\frac{\partial}{\partial y'}\right)^{\theta_3}\left(\frac{\partial}{\partial \eta'}\right)e_{ij}\right| \leqq C_\theta^{(i,j)}\mu^{1/2}\langle\eta'\rangle^{-|\theta_4|} \tag{3.27}$$

$$\text{where } \langle\eta'\rangle=\sqrt{\sum_{j=2}^{n}\eta_j^2+1} \text{ and } \mu\geqq 1 .$$

E_{12} $\cdots$a 5×5 pseudo differential system which has the same property as E_{11} with $C_\theta^{(i,j)}$ instead of $C_\theta^{(i,j)}\mu^{1/2}$ in (3.27).

K_{11} $\cdots$a 5×2 pseudo differential system which has the same property as E_{11} with $C_\theta^{(i,j)}\mu^{-1/2}$ instead of $C_\theta^{(i,j)}\mu^{1/2}$ in (3.27).

K_{12} $\cdots$a $5\times(n+2)$ pseudo differential system which has the same property as E_{11} with $C_\theta^{(i,j)}$ instead of $C_\theta^{(i,j)}\mu^{1/2}$ in (3.27).

$$V_{11}={}^t(v_t,\ (t+\delta)^k v_{y_1}) .$$

$$V_{12}={}^t(w,\ w_t,\ (t+\delta)^k w_{y_1},\ \cdots,\ (t+\delta)^k w_{y_n}) .$$

$$F_1={}^t(q_j(\bar{\varphi}\cdot\bar{f}_1),\ \tilde{z}_1 q_j(\bar{\varphi}\cdot\bar{f}_1),\ q_j(\bar{\varphi}\cdot\bar{f}_1),\ \tilde{z}_2 q_j(\bar{\varphi}\cdot\bar{f}_1),\ 0) .$$

$$P_1=\begin{pmatrix}1 & \tilde{z}_2 & 0 & 0 & 0\\ 0 & 0 & 1 & \tilde{z}_1 & 0\end{pmatrix}(\in \mathcal{S}^0(\mu)) ,$$

and

$$\begin{aligned}G_1={}&{}^t((1+\tilde{z}_2\tilde{z}_1)Q_0 v-(t+\delta)^k(1-\tilde{z}_2\tilde{z}_1)Q_1 v\\ &+(t+\delta)^k(\tilde{z}_1+\tilde{z}_2)Q_2 v\ ,\quad (1+\tilde{z}_1\tilde{z}_2)Q_0 v\\ &-(t+\delta)^k(1-\tilde{z}_1\tilde{z}_2)Q_1 v+(t+\delta)^k(\tilde{z}_1+\tilde{z}_2)Q_2 v)\\ ={}&{}^t\Big(-\frac{2g_2}{\tilde{c}+1}+T_0^{(3)}v+T_1^{(3)}v_t+T_2^{(3)}(t+\delta)^k v_{y_1}\ ,\\ &-\frac{2g_2}{\tilde{c}+1}+T_0^{(4)}v+T_1^{(4)}v_t+T_2^{(4)}(t+\delta)^k v_{y_1}\Big)\\ &(T_0^{(3)},\ T_0^{(4)}\in \mathcal{S}^0(\mu),\ T_1^{(3)},\ T_2^{(3)},\ T_1^{(4)},\ T_2^{(4)}\in S^{-1}(\mu)) .\end{aligned}$$

By (3.25) and the result for the pseudo differential operator with parameter in [2], we have

$$\langle A_1 U,\ U\rangle \geqq C(t+\delta)^k\langle U,\ U\rangle \tag{3.28}$$

for any $U\in \operatorname{Ker} P_1\cap L^2(\boldsymbol{R}_{y'}^{n-1})$ where C is a positive constant independent of $\mu\in[\mu_1,\ \infty)$ and μ_1 is a positive constant.

We explain the representation (3.26) by an example. For the first component $\mathring{U}_1$ of $(M_1 U_t-A_1 U_{y_1}-\sum_{j=2}^{n} B_{1j}U_{y_j}-D_1 Q_2 U)$, we obtain

$$\begin{aligned}\mathring{U}_1={}&Q_0\{Q_0 v-(t+\delta)^k Q_1 v+(t+\delta)^k\tilde{z}_1 Q_2 v\}\\ &+(t+\delta)^k Q_1\{Q_0 v-(t+\delta)^k Q_1 v+(t+\delta)^k\tilde{z}_1 Q_2 v\}\\ &-(t+\delta)^k Q_2\{\tilde{z}_1(Q_0 v+(t+\delta)^k Q_1 v)+(t+\delta)^k Q_2 v\}\end{aligned} \tag{3.29}$$

and its principal part is

$$v_{tt}-2(t+\delta)^k\sum_{j=1}^n \tilde{h}_j(t,y)\frac{\partial^2 v}{\partial t\partial y_j}-(t+\delta)^{2k}\sum_{i,j=1}^n \tilde{a}_{ij}(t,y)\frac{\partial^2 v}{\partial y_i\partial y_j}. \tag{3.30}$$

Using the method of Kumano-go [9] (p. 69, Lemma 2.4), we get

$$\begin{cases} R_0=(R_0\triangle Q_2^{-1})Q_2+R_1 \\ \dfrac{\partial}{\partial t}(Q_2v)-Q_2v_t=Q_{2t}v=(Q_{2t}\triangle Q_2^{-1})Q_2v+R_2v \\ \dfrac{\partial}{\partial y_1}(Q_2v)-Q_2v_{y_1}=Q_{2y_1}v=(Q_{2y_1}\triangle Q_2^{-1})Q_2v+R_3v \\ 1=(\tilde{z}_1-\tilde{z}_2)^{-1}\cdot(\tilde{z}_1-\tilde{z}_2)+\mu^{-1/2}R_4 \\ \mu=\mu^{1/2}(\mu^{1/2}\triangle Q_2^{-1})Q_2+\mu^{1/2}R_5 \\ (R_0\in S^1(\mu)\,,\quad R_1,\cdots,R_5\in S^0(\mu)) \end{cases} \tag{3.31}$$

where $T_1\triangle T_2(T_1\triangle T_2^{-1})$ is a pseudo differential operator with respect to y' with the symbol $\sigma(T_1\triangle T_2)=\sigma(T_1)\times\sigma(T_2)$ $(\sigma(T_1\triangle T_2^{-1})=\sigma(T_1)\times\sigma(T_2)^{-1})$. Hence, we have the representation (3.26).

Case (II) of first step. The equation (3.13) has a non-zero double root in $D=\{z\in C|\ |z|<1\}$ for some $(t,y',\eta',\sigma)\in\omega_j$.

We set

$$U=\begin{pmatrix}U_1\\U_2\\U_3\\U_4\end{pmatrix}=\begin{pmatrix}Q_0v-(t+\delta)^kQ_1v+\sqrt{1-\varepsilon^2}(t+\delta)^k\tilde{z}_1Q_2v \\ \tilde{z}_1(Q_0v+(t+\delta)^kQ_1v)+\sqrt{1-\varepsilon^2}(t+\delta)^kQ_2v \\ \varepsilon(t+\delta)^k\sqrt{1+|\tilde{z}_1|^2}Q_2v \\ \sqrt{\mu}\,v\end{pmatrix} \tag{3.32}$$

where $0<\varepsilon\ll 1$, $\tilde{z}_1$ is a pseudo differential operator with respect to y' whose symbol satisfies

$$\tilde{z}_1(t,y',\eta':\sigma)=\begin{cases}\zeta_j(t,y',\eta':\sigma)z_1(t,y',\eta':\sigma)+\dfrac{1}{2}(1-\zeta_j(t,y',\eta':\sigma)) \\ \qquad (\mathrm{Re}\,z_1>\mathrm{Re}\,z_2\quad\text{or}\quad \mathrm{Im}\,z_1\neq 0,\ \mathrm{Im}\,z_2\neq 0) \\ \zeta_j(t,y',\eta':\sigma)z_1(t,y',\eta':\sigma)-\dfrac{1}{2}(1-\zeta_j(t,y',\eta':\sigma)) \\ \qquad (\mathrm{Re}\,z_2<\mathrm{Re}\,z_1<0)\end{cases} \tag{3.33}$$

and

$$C_1\leqq|\tilde{z}_1(t,y',\eta':\sigma)|_{\bar{\Sigma}_+}\leqq C_2<1 \tag{3.34}$$

for positive constants C_1 and C_2 independent of $(t, y', \eta', \sigma) \in [0, T] \times R^{n-1} \times \bar{\Sigma}_+$. Then, by the same method as the one in §2, the problem (3.17) is transformed into the problem

$$
(3.35) \qquad \begin{cases} M_2 U_t = A_2 U_{y_1} + \sum_{j=2}^{n} B_{2j} U_{y_j} + D_2 Q_2 U \\ \qquad + \left(E_{21} + \dfrac{1}{t+\delta} E_{22} \right) U + K_{21} V_{21} + K_{22} V_{22} + F_2 \\ U(0, y) = 0 \\ P_2 U|_{y_1=0} = G_2 \\ \quad (t, y) \in (0, T) \times R_+^n \end{cases}
$$

where

$$
M_2 = \left(1 + \frac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\right)^{1/2} \cdot I - \frac{\tilde{h}_1(t, y)}{\sqrt{\tilde{a}_{11}(t, y)}} \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \dfrac{1-|\tilde{z}_1|^2}{1+|\tilde{z}_1|^2} & \\ 0 & & & 0 \end{pmatrix}
$$

$$
A_2 = (t+\delta)^k \sqrt{\tilde{a}_{11}(t, y)} \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \dfrac{1-|\tilde{z}_1|^2}{1+|\tilde{z}_1|^2} & \\ 0 & & & 1 \end{pmatrix}
$$

$$
\begin{aligned} B_{2j} = & \frac{\tilde{a}_{1j}(t, y)}{\tilde{a}_{11}(t, y)} A_2 \\ & + (t+\delta)^k \left(1 + \frac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\right)^{-1/2} \left(\tilde{h}_j(t, y) - \frac{\tilde{h}_1(t, y)}{\tilde{a}_{11}(t, y)} \tilde{a}_{1j}(t, y) \right) \cdot I \\ & (j = 2, \cdots, n) \end{aligned}
$$

$$
D_2 = (t+\delta)^k \begin{pmatrix} 0 & \sqrt{1-\varepsilon^2} & \dfrac{\varepsilon}{\sqrt{1+|\tilde{z}_1|^2}} & 0 \\ \sqrt{1-\varepsilon^2} & 0 & \dfrac{\varepsilon \tilde{z}_1}{\sqrt{1+|\tilde{z}_1|^2}} & 0 \\ \dfrac{\varepsilon}{\sqrt{1+|\tilde{z}_1|^2}} & \dfrac{\varepsilon \bar{\tilde{z}}_1}{\sqrt{1+|\tilde{z}_1|^2}} & -\dfrac{2\sqrt{1-\varepsilon^2}}{1+|\tilde{z}_1|^2} \operatorname{Re} \tilde{z}_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

E_{21} $\cdots$a 4×4 pseudo differential system with the same property as E_{11}.

E_{22} $\cdots$a 4×4 pseudo differential system with the same property as E_{12}.

E_{21} $\cdots$a 4×2 pseudo differential system with the same property as K_{11}.

E_{22} $\cdots$a $4\times(n+2)$ pseudo differential system with the same property as K_{12}.

$V_{21}=V_{11}$, $V_{22}=V_{12}$.

$F_2={}^t(q_j(\overline{\varphi}\cdot\overline{f}_1),\ \tilde{z}_1 q_j(\overline{\varphi}\cdot\overline{f}_1),\ 0,\ 0)$.

$P_2=(1,\ \tilde{z}_2,\ 0,\ 0)$ $(\in \mathcal{S}^0(\mu))$.

$\tilde{z}_2(t,\ y',\ \eta'\colon\sigma)=\zeta_j(t,\ y',\ \eta'\colon\sigma)z_2(t,\ y',\ \eta'\colon\sigma)$ and $|\tilde{z}_2(t,\ y',\ \eta'\colon\sigma)|_{\bar{\Sigma}_+}\leqq C_1<1$ where C_1 is a positive constant independently of

$$(t,\ y',\ \eta',\ \sigma)\in[0,\ T]\times \boldsymbol{R}^{n-1}\times\bar{\Sigma}_+ .$$

and

$$\begin{aligned}G_2&=(1+\tilde{z}_2\tilde{z}_1)Q_0v-(t+\delta)^k(1-\tilde{z}_2\tilde{z}_1)Q_1v\\&\quad+\sqrt{1-\varepsilon^2}(t+\delta)^k(\tilde{z}_1+\tilde{z}_2)Q_2v\\&=-\frac{2g_2}{\tilde{c}+1}+T_0^{(5)}v+T_1^{(5)}v_t+T_2^{(5)}(t+\delta)^k v_{y_1}\\&\qquad(T_0^{(5)}\in\mathcal{S}^0(\mu),\ T_1^{(5)},\ T_2^{(5)}\in\mathcal{S}^{-1}(\mu)) .\end{aligned}$$

By the fact that $\tilde{z}_2(t,\ y',\ \eta'\colon\sigma)$ is independent of $(t,\ y')$ outside a compact set in $[0,\ T]\times\boldsymbol{R}^{n-1}$ and the result in [2], we obtain

$$\operatorname{Re}\langle A_2U,\ U\rangle\geqq C(t+\delta)^k\langle U,\ U\rangle \tag{3.36}$$

for any $U\in\operatorname{Ker}P_2\cap L^2(\boldsymbol{R}^{n-1}_{y'})$ where C is a positive constant independent of $\mu\in[\mu_2,\ \infty)$ and μ_2 is a positive constant.

Case (III) *of first step.* The equation (3.13) has zero double root for some $(t,\ y',\ \eta',\ \sigma)\in\omega_j$.

There is a sufficiently small positive constant θ such that

$$\begin{cases}|\tilde{c}-1|\leqq\theta\\|\tilde{b}_\mu|\leqq\theta\end{cases} \tag{3.37}$$

for any $(t,\ y',\ \eta',\ \sigma)\in\omega_j$.

We set

$$(3.38)\qquad U=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}=\begin{pmatrix} Q_0v-(t+\delta)^kQ_1v+\sqrt{1-\varepsilon^2}(t+\delta)^k\tilde{z}_1Q_2v \\ \tilde{z}_1(Q_0v+(t+\delta)^kQ_1v)+\sqrt{1-\varepsilon^2}(t+\delta)^kQ_2v \\ \varepsilon(t+\delta)^k\sqrt{1+|\tilde{z}_1|^2}Q_2v \\ \sqrt{\mu}\,v \end{pmatrix}$$

where $\tilde{z}_1=1/2(\tilde{c}+1)$, $C\leqq|\tilde{z}_1|\leqq(1/2)$ and C is a positive constant independent of $(t, y')\in[0, T]\times R^{n-1}$. Then, by the same method as the one in §2, the problem (3.17) is transformed into the problem

$$(3.39)\qquad \begin{cases} M_3U_t=A_3U_{y_1}+\sum_{j=2}^{n}B_{3j}U_{y_j}+D_3Q_2U \\ \qquad +\left(E_{31}+\dfrac{1}{t+\delta}E_{32}\right)U+K_{31}V_{31}+K_{32}V_{32}+F_3 \\ U(0, y)=0 \\ P_3U|_{y_1=0}=G_3 \\ \quad (t, y)\in(0, T)\times R^n_+ \end{cases}$$

where

$$M_3=\left(1+\frac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\right)^{1/2}\cdot I-\frac{\tilde{h}_1(t, y)}{\sqrt{\tilde{a}_{11}(t, y)}}\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \dfrac{1-|\tilde{z}_1|^2}{1+|\tilde{z}_1|^2} & \\ 0 & & & 0 \end{pmatrix}$$

$$A_3=(t+\delta)^k\sqrt{\tilde{a}_{11}(t, y)}\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \dfrac{1-|\tilde{z}_1|^2}{1+|\tilde{z}_1|^2} & \\ 0 & & & 1 \end{pmatrix}$$

$$B_{3j}=\frac{\tilde{a}_{1j}(t, y)}{\tilde{a}_{11}(t, y)}A_3 +(t+\delta)^k\left(1+\frac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\right)^{-1/2}\left(\tilde{h}_j(t, y)-\frac{\tilde{h}_1(t, y)}{\tilde{a}_{11}(t, y)}\tilde{a}_{1j}(t, y)\right)\cdot I \quad (j=2, \cdots, n)$$

$$D_3=(t+\delta)^k\begin{pmatrix} 0 & \sqrt{1-\varepsilon^2} & \dfrac{\varepsilon}{\sqrt{1+|\tilde{z}_1|^2}} & 0 \\ \sqrt{1-\varepsilon^2} & 0 & \dfrac{\varepsilon\tilde{z}_1}{\sqrt{1+|\tilde{z}_1|^2}} & 0 \\ \dfrac{\varepsilon}{\sqrt{1+|\tilde{z}_1|^2}} & \dfrac{\varepsilon\bar{\tilde{z}}_1}{\sqrt{1+|\tilde{z}_1|^2}} & -\dfrac{2\sqrt{1-\varepsilon^2}}{1+|\tilde{z}_1|^2}\operatorname{Re}\tilde{z}_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$E_{31}\cdots$a 4×4 pseudo differential system with the same property as E_{11}.
$E_{32}\cdots$a 4×4 pseudo differential system with the same property as E_{12}.
$K_{31}\cdots$a 4×2 pseudo differential system with the same property as K_{11}.
$K_{32}\cdots$a $4\times(n+2)$ pseudo differential system with the same property as K_{12}.

$V_{31}=V_{11}$, $V_{32}=V_{12}$.
$F_3={}^t(q_j(\bar{\varphi}\cdot\bar{f}_1), \tilde{z}_1 q_j(\bar{\varphi}\cdot\bar{f}_1), 0, 0)$.
$P_3=(1, \tilde{z}_2, \tilde{z}_3, 0) \quad (\in \mathcal{S}^0(\mu))$.
$\tilde{z}_2(t, y', \eta' : \sigma)=2(\tilde{c}(t, y')-1)\zeta_j(t, y', \eta' : \sigma)$

$$\tilde{z}_3(t, y', \eta' : \sigma)=-\frac{1}{\varepsilon\sqrt{1+|\tilde{z}_1|^2}}\left\{\frac{2\tilde{b}_\mu}{\tilde{c}+1}+\sqrt{1-\varepsilon^2}\frac{(4\tilde{c}^2-3)}{2(\tilde{c}+1)}\right\}\zeta_j(t, y', \eta' : \sigma)$$

and

$$\begin{aligned} G_3&=(1+\tilde{z}_2\tilde{z}_1)Q_0 v-(t+\delta)^k(1-\tilde{z}_2\tilde{z}_1)Q_1 v \\ &\quad+\{\sqrt{1-\varepsilon^2}(t+\delta)^k(\tilde{z}_1+\tilde{z}_2)+\varepsilon(t+\delta)^k\tilde{z}_3\sqrt{1+|\tilde{z}_1|^2}\}Q_2 v \\ &=-\frac{2g_2}{\tilde{c}+1}+T_0^{(6)}v+T_1^{(6)}v_t+(t+\delta)^k T_2^{(6)}v_{y_1} \\ &\qquad (T_0^{(6)}\in\mathcal{S}^0(\mu),\ T_1^{(6)},\ T_2^{(6)}\in\mathcal{S}^{-1}(\mu)) . \end{aligned}$$

Then, we can choose ε, z_1 and z_2 as

$$\text{(3.40)}\qquad \begin{cases} 1-\theta_1\leqq\varepsilon<1 \\ |\tilde{z}_j(t, y', \eta' : \sigma)|\leqq\theta_2 \quad (j=2, 3) \end{cases}$$

for any $(t, y', \eta', \sigma)\in\omega_j$ where θ_1 and θ_2 are sufficiently small positive constants. Therefore, by the results in [2], we obtain

$$\text{(3.41)}\qquad \langle A_3 U, U\rangle\geqq C(t+\delta)^k\langle U, U\rangle$$

for any $U\in\operatorname{Ker}P_3\cap L^2(R_{y'}^{n-1})$ where C is a positive constant independent of $\mu\in[\mu_3, \infty)$ and μ_3 is a positive constant.

Second step. We shall prove the following inequality

$$\sum_{j=0}^{1}\left\|\left(\frac{\partial}{\partial t}\right)^{j} u_{\delta}(t, \cdot)\right\|^{2}_{1-j,\mu,\Omega} \leqq C_{T} t^{2N_{3}-2k+2}\int_{0}^{t}\left\{\left\|\left(\frac{\partial}{\partial \tau}\right)^{N_{3}+1} f_{1}(\tau, \cdot)\right\|^{2}_{0,\Omega} + \left\|\left(\frac{\partial}{\partial \tau}\right)^{N_{3}+1} g_{1}(\tau, \cdot)\right\|^{2}_{0,S}\right\} d\tau \tag{3.42}$$

where C_T is a positive constant independent of δ, f_1 and g_1, and N_3 is a positive integer.

We treat the energy estimate for the Case (II) of first step because we can obtain the one for other cases by the same method. We set

$$\Phi(t) = \mathrm{Re}\,(e^{-\mu t} M_2 U, e^{-\mu t} U)\,. \tag{3.43}$$

Then, we have

$$\begin{aligned}
\frac{d}{dt}\Phi(t) &= -2\mu\Phi(t) + \mathrm{Re}\,(e^{-\mu t} M_2 U_t, e^{-\mu t} U) \\
&\quad + \mathrm{Re}\,(e^{-\mu t} M_2 U, e^{-\mu t} U_t) + \mathrm{Re}\,(e^{-\mu t} M_{2t} U, e^{-\mu t} U) \\
&= -2\mu\Phi(t) + \mathrm{Re}\left(e^{-\mu t}\left(A_2 U_{y_1} + \sum_{j=2}^{n} B_{2j} U_{y_j} + D_2 Q_2 U\right.\right. \\
&\quad \left.\left. + E_{21} U + \frac{1}{t+\delta} E_{22} U + K_{21} V_{21} + K_{22} V_{22} + F_2\right), e^{-\mu t} U\right) \\
&\quad + \mathrm{Re}\left(e^{-\mu t} U, e^{-\mu t}\left(A_2 U_{y_1} + \sum_{j=2}^{n} B_{2j} U_{y_j} + D_2 Q_2 U\right.\right. \\
&\quad \left.\left. + E_{21} U + \frac{1}{t+\delta} E_{22} U + K_{21} V_{21} + K_{22} V_{22} + F_2\right)\right) \\
&\quad + \mathrm{Re}\,(e^{-\mu t} M_{2t} U, e^{-\mu t} U) + \mathrm{Re}(e^{-\mu t} U, e^{-\mu t} H U_t) \\
&\leqq -C_{11}\mu\Phi(t) + C_{12}\left\{\frac{1}{t}\Phi(t) + \frac{1}{\mu}\|F\|^{2}_{0,\mu}\right. \\
&\quad \left. + \frac{1}{\mu}\|V_{22}\|^{2}_{0,\mu}\right\} - \mathrm{Re}\langle A_2 e^{-\mu t} U, e^{-\mu t} U\rangle
\end{aligned} \tag{3.44}$$

for any $\mu \geqq \mu_4$ where μ_4, C_{11} and C_{12} are positive constants and

$$H = \begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & & \\ & & h_{33} & \\ 0 & & & 0 \end{pmatrix} \in \mathcal{S}^{-1}(\mu)\,. \tag{3.45}$$

By (3.36), we obtain

$$\text{(3.46)} \qquad \operatorname{Re}\langle A_2 e^{-\mu t}U,\ e^{-\mu t}U\rangle \geqq C_1(t+\delta)^k\langle e^{-\mu t}U,\ e^{-\mu t}U\rangle \\ -C_2(t+\delta)^k\langle e^{-\mu t}G_2,\ e^{-\mu t}G_2\rangle$$

for any $\mu \geqq \mu_5$ where C_1 and C_2 are positive constants independent of μ and μ_5 is a positive constant. Therefore, we have

$$\text{(3.47)} \quad \frac{d}{dt}\Phi(t) \leqq -C_{11}\mu\Phi(t) + C_{12}\left\{\frac{1}{t}\Phi(t) + \frac{1}{\mu}\|F_2\|^2_{0,\mu} + \frac{1}{\mu}\|V_{22}\|^2_{0,\mu}\right\} \\ -C_{13}(t+\delta)^k\langle e^{-\mu t}U,\ e^{-\mu t}U\rangle + C_{14}(t+\delta)^k\langle e^{-\mu t}G_2,\ e^{-\mu t}G_2\rangle\ .$$

Then, we get

$$\text{(3.48)} \quad \begin{aligned} \frac{d}{dt}(t^{-\alpha}\Phi(t)) &= -\alpha t^{-(\alpha+1)}\Phi(t) + t^{-\alpha}\Phi'(t) \\ &\leqq (C_{12}-\alpha)t^{-(\alpha+1)}\Phi(t) - C_{11}\mu t^{-\alpha}\Phi(t) \\ &\quad + \frac{C_{12}}{\mu}t^{-\alpha}\{\|F_2\|^2_{0,\mu} + \|V_{22}\|^2_{0,\mu}\} \\ &\quad + t^{-\alpha}(t+\delta)^k\{-C_{13}\langle\!\langle U\rangle\!\rangle^2_{0,\mu} + C_{14}\langle\!\langle G_2\rangle\!\rangle^2_{0,\mu}\} \end{aligned}$$

for any $\mu \geqq \mu_6$ where μ_6 is a positive constant. When we choose f_1 and N_3 as

$$\text{(3.49)} \quad \begin{cases} \left(\dfrac{\partial}{\partial t}\right)^j f_1|_{t=0} = 0 \quad (j=0, 1, \cdots, N_3) \\ N_3 \geqq \left[\dfrac{\alpha}{2}\right] + \left[\dfrac{n+1}{2}\right] + 3\ , \end{cases}$$

we have $t^{-\alpha}\Phi(t)|_{t=0}=0$. By (3.48), we obtain

$$\text{(3.50)} \quad \begin{aligned} t^{-\alpha}\Phi(t) &+ C_{11}\mu\int_0^t \tau^{-\alpha}\Phi(\tau)d\tau + C_{13}\int_0^t \tau^{-\alpha}(\tau+\delta)^k\langle\!\langle U\rangle\!\rangle^2_{0,\mu}d\tau \\ &\leqq \frac{C_{12}}{\mu}\int_0^t \tau^{-\alpha}\{\|F_2\|^2_{0,\mu} + \|V_{22}\|^2_{0,\mu}\}d\tau \\ &\quad + C_{14}\int_0^t \tau^{-\alpha}(\tau+\delta)^k\langle\!\langle G_2\rangle\!\rangle^2_{0,\mu}d\tau \end{aligned}$$

for any $\alpha \geqq \alpha_1$ where α_1 is a positive constant. Hence, we get

$$\text{(3.51)} \quad \begin{aligned} \Phi(t) &+ C_{11}\mu\int_0^t \left(\frac{t}{\tau}\right)^\alpha \Phi(\tau)d\tau + C_{13}\int_0^t \left(\frac{t}{\tau}\right)^\alpha (\tau+\delta)^k\langle\!\langle U\rangle\!\rangle^2_{0,\mu}d\tau \\ &\leqq \frac{C_{12}}{\mu}\int_0^t \left(\frac{t}{\tau}\right)^\alpha \{\|F_2\|^2_{0,\mu} + \|V_{22}\|^2_{0,\mu}\}d\tau \end{aligned}$$

$$+C_{14}\int_0^t\left(\frac{t}{\tau}\right)^{\alpha}(\tau+\delta)^k\langle\!\langle G_2\rangle\!\rangle_{0,\mu}^2 d\tau\ .$$

When $\{\varphi_j\}_{j=1}^l$ is a partition of unity in a neighborhood of $S=\partial\Omega$, we set $\varphi_0=1-\sum_{j=1}^l\varphi_j$. By the fact that φ_0 vanishes in a neighborhood of S and Remark 3 in §2, we have the same type inequality without the boundary estimate and $G_2\equiv 0$ as in (3.51) (i.e. the energy estimate for the Cauchy problem). Using a partition of unity in a neighborhood of $\bar{\Omega}$, (3.18), (3.19) and the same result for the Cauchy problem as (3.51), we obtain

$$\begin{aligned}\|W(t)\|_{0,\mu,\Omega}^2&+C_{21}\mu\int_0^t\left(\frac{t}{\tau}\right)^{\alpha}\|W(\tau)\|_{0,\mu,\Omega}^2 d\tau\\&+C_{22}\int_0^t\left(\frac{t}{\tau}\right)^{\alpha}(\tau+\delta)^k\|W(\tau)\|_{0,\mu,S}^2 d\tau\\ \leqq&\frac{C_{23}}{\mu}\int_0^t\left(\frac{t}{\tau}\right)^{\alpha}\|f_1(\tau,\cdot)\|_{0,\mu,\Omega}^2 d\tau\\&+C_{24}\int_0^t\left(\frac{t}{\tau}\right)^{\alpha}(\tau+\delta)^k\|g_1(\tau,\cdot)\|_{0,\mu,S}^2 d\tau\end{aligned}\tag{3.52}$$

for any $\mu\geqq\mu_7$ and any $\alpha\geqq\alpha_2$ where $W(t)={}^t(u_\delta(t), u_{\delta t}(t), (t+\delta)^k u_{\delta x_1}, \cdots, (t+\delta)^k u_{\delta x_n})$, $C_{21}, \cdots, C_{24}, \mu_7$ and α_2 are positive constants independent of δ, f_1 and g_1. On the other hand, since $(\partial/\partial t)^i f_1|_{t=0}=0(i=0, \cdots, N_3)$ and $(\partial/\partial t)^j g_1|_{t=0}=0(j=0, \cdots, N_3+1)$, we can represent as follows

$$\begin{cases}f_1(t,x)=\dfrac{1}{N_3!}\displaystyle\int_0^t(t-\tau)^{N_3}\left(\frac{\partial}{\partial\tau}\right)^{N_3+1}f_1(\tau,x)d\tau\\ g_1(t,s)=\dfrac{1}{N_3!}\displaystyle\int_0^t(t-\tau)^{N_3}\left(\frac{\partial}{\partial\tau}\right)^{N_3+1}g_1(\tau,s)d\tau\ .\end{cases}\tag{3.53}$$

Therefore, by Schwarz inequality, we have

$$\begin{cases}\|f_1(t,\cdot)\|_{0,\Omega}^2\leqq t^{2N_3+1}\displaystyle\int_0^t\left\|\left(\frac{\partial}{\partial\tau}\right)^{N_3+1}f_1(\tau,\cdot)\right\|_{0,\Omega}^2 d\tau\\ \|g_1(t,\cdot)\|_{0,S}^2\leqq t^{2N_3+1}\displaystyle\int_0^t\left\|\left(\frac{\partial}{\partial\tau}\right)^{N_3+1}g_1(\tau,\cdot)\right\|_{0,S}^2 d\tau\end{cases}\tag{3.54}$$

and

$$\int_0^t\left(\frac{t}{\tau}\right)^{\alpha}\tau^{2N_3+1}\int_0^\tau\left\|\left(\frac{\partial}{\partial\nu}\right)^{N_3+1}\theta(\nu,\cdot)\right\|_{0,\omega}^2 d\nu e^{-2\mu\tau}d\tau\tag{3.55}$$

$$\leqq \int_0^t \left\| \left(\frac{\partial}{\partial \nu}\right)^{N_3+1} \theta(\nu,\ \cdot) \right\|_{0,\omega}^2 d\nu\, t^\alpha \int_0^t \tau^{2N_3+1-\alpha} d\tau$$

$$\leqq t^{2N_3+2} \int_0^t \left\| \left(\frac{\partial}{\partial \tau}\right)^{N_3+1} \theta(\tau,\ \cdot) \right\|_{0,\omega}^2 d\tau\ .$$

Combining (3.52), (3.54) and (3.55), we obtain

$$\begin{aligned}(3.56)\qquad & \|W(t)\|_{0,\mu,\Omega}^2 + C_{21}\mu \int_0^t \left(\frac{t}{\tau}\right)^\alpha \|W(\tau)\|_{0,\mu,\Omega}^2 d\tau \\ & + C_{22} \int_0^t \left(\frac{t}{\tau}\right)^\alpha (\tau+\delta)^k \|W(\tau)\|_{0,\mu,S}^2 d\tau \\ & \leqq \frac{C_{23}}{\mu} t^{2N_3+2} \int_0^t \left\| \left(\frac{\partial}{\partial \tau}\right)^{N_3+1} f_1(\tau,\ \cdot) \right\|_{0,\Omega}^2 d\tau \\ & + C_{25} t^{2N_3+2} \int_0^t \left\| \left(\frac{\partial}{\partial \tau}\right)^{N_3+1} g_1(\tau,\ \cdot) \right\|_{0,S}^2 d\tau\ .\end{aligned}$$

Hence, we have (3.42).

Third step. We shall prove the inequality

$$\begin{aligned}(3.57)\qquad & \sum_{j=0}^m \left\| \left(\frac{\partial}{\partial t}\right)^j u_\delta(t,\ \cdot) \right\|_{m-j,\mu,\Omega}^2 \\ & \leqq C'_{T,m} \mu^{2m-3} t^{2N_4} \int_0^t \left\{ \left\| \left(\frac{\partial}{\partial \tau}\right)^{N_4+\beta} f_1(\tau,\ \cdot) \right\|_{m-1,\Omega}^2 \right. \\ & \left. + \left\| \left(\frac{\partial}{\partial \tau}\right)^{N_4+\beta} g_1(\tau,\ \cdot) \right\|_{m-1,S}^2 \right\} d\tau\end{aligned}$$

where $m \geqq 2$, $C'_{T,m}$ and β are positive constants independent of δ, f_1 and g_1, and N_4 is a positive integer.

It suffices to show the inequality for $m=2$.

We treat the energy estimate for the Case (II) of first step because we can obtain the one for other cases by the same method.

Differentiating all the term in (3.35) with respect to t and using $f_1(0, x)=0$, we have

$$\left\{\begin{aligned} M_2(U_t)_t = {} & A_2(U_t)_{y_1} + \sum_{j=2}^n B_{2j}(U_t)_{y_j} + D_2 Q_2 U_t \\ & + E_{21} U_t + \frac{1}{t+\delta} E_{22} U_t + K_{21} V_{21t} + K_{22} V_{22t} + F_{2t} \\ & - M_{2t} U_t + \left(A_{20t} + \frac{k}{t+\delta} A_{20}\right)(t+\delta)^k U_{y_1} \end{aligned}\right.$$

$$
(3.58)\quad \begin{cases} \qquad +\sum_{j=2}^{n} B_{2jt}U_{y_j}+(D_2Q_2)_t U+E_{21t}U+\left(\frac{1}{t+\delta}E_{22}\right)_t U \\ \qquad +K_{21t}V_{21}+K_{22t}V_{22} \\ U_t(0,y)=0 \\ P_2 U_t|_{y_1=0}=G_{2t}-P_{2t}U|_{y_1=0} \\ \quad (t,y)\in(0,T)\times \boldsymbol{R}_+^n \end{cases}
$$

where

$$
(3.59)\qquad A_{20}=\sqrt{\tilde{a}_{11}(t,y)}\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \frac{1-|\tilde{z}_1|^2}{1+|\tilde{z}_1|^2} & \\ 0 & & & 1 \end{pmatrix}.
$$

Differentiating all the term in (3.35) with respect to $y_r(r=2,\cdots,n)$ and multipling $(t+\delta)^k$, we obtain

$$
(3.60)\quad \begin{cases} M_2((t+\delta)^k U_{y_r})_t=A_2((t+\delta)^k U_{y_r})_{y_1}+\sum_{j=2}^{n} B_{2j}((t+\delta)^k U_{y_r})_{y_j} \\ \qquad +D_2Q_2((t+\delta)^k U_{y_r})+E_{21}((t+\delta)^k U_{y_r})+\frac{E_{22}}{t+\delta}((t+\delta)^k U_{y_r}) \\ \qquad +K_{21}(t+\delta)^k V_{21y_r}+K_{22}(t+\delta)^k V_{22y_r}+(t+\delta)^k F_{2y_r} \\ \qquad -(t+\delta)^k M_{2y_r}U_t+\frac{k}{t+\delta}M_2((t+\delta)^k U_{y_r}) \\ \qquad +A_{20y_r}(t+\delta)^{2k}U_{y_1}+\sum_{j=2}^{n} B_{2jy_r}(t+\delta)^k U_{y_j} \\ \qquad +(D_2Q_2)_{y_r}(t+\delta)^k U+(t+\delta)^k\Big\{E_{21y_r}U \\ \qquad +\frac{1}{t+\delta}E_{22y_r}U+K_{21y_r}V_{21}+K_{22y_r}V_{22}\Big\} \\ ((t+\delta)^k U_{y_r})(0,y)=0 \\ P_2((t+\delta)^k U_{y_r})|_{y_1=0}=(t+\delta)^k[G_{2y_r}-P_{2y_r}U|_{y_1=0}] \\ \quad (t,y)\in(0,T)\times \boldsymbol{R}_+^n . \end{cases}
$$

Operating the pseudo differential operator $(t+\delta)^k Q_2$ for (3.35), we have

$$(3.61)\quad \begin{cases} M_2((t+\delta)^kQ_2U)_t = A_2((t+\delta)^kQ_2U)_{y_1} + \sum_{j=2}^n B_{2j}((t+\delta)^kQ_2U)_{y_j} \\ \quad + DQ_2((t+\delta)^kQ_2U) + E_{21}((t+\delta)^kQ_2U) + \dfrac{E_{22}}{t+\delta}((t+\delta)^kQ_2U) \\ \quad + K_{21}((t+\delta)^kQ_2V_{21}) + K_{22}((t+\delta)^kQ_2V_{22}) + (t+\delta)^kQ_2F_2 \\ \quad -(t+\delta)^k[Q_2, M_2]U_t + \dfrac{k}{t+\delta}M_2(t+\delta)^kQ_2U \\ \quad +(t+\delta)^kM_2Q_{2t}U + (t+\delta)^k[Q_2, A_2]U_{y_1} - (t+\delta)^kA_2Q_{2y_1}U \\ \quad +(t+\delta)^k\left\{\sum_{j=2}^n [Q_2, B_{2j}]U_{y_j} + [Q_2, D_2]Q_2U\right\} \\ \quad +(t+\delta)^k\sum_{j=2}^n B_{2j}\left[Q_2, \dfrac{\partial}{\partial y_j}\right]U + (t+\delta)^k\left\{[Q_2, E_{21}]U\right. \\ \quad \left. +\dfrac{1}{t+\delta}[Q_2, E_{22}]U + [Q_2, K_{21}]V_{21} + [Q_2, K_{22}]V_{22}\right\} \\ ((t+\delta)^kQ_2U)(0, y) = 0 \\ P_2((t+\delta)^kQ_2U)|_{y_1=0} = (t+\delta)^k[Q_2G_2 - [Q_2, P_2]U|_{y_1=0}] \\ \quad (t, y) \in (0, T)\times \boldsymbol{R}^n_+ . \end{cases}$$

Also, by (3.35), we get

$$(3.62)\quad \begin{aligned}(t+\delta)^kU_{y_1} = (A_{201}+A_{202})\Big(&M_2U_t - \sum_{j=2}^n B_{2j}U_{y_j} - D_2Q_2U \\ &-E_{21}U - \frac{1}{t+\delta}E_{22}U - K_{21}V_{21} - K_{22}V_{22} - F_2\Big)\end{aligned}$$

where

$$(3.63)\quad \begin{aligned} A_{201} &= \frac{1}{\sqrt{\tilde{a}_{11}(t, y)}}\begin{pmatrix} -1 & & & & 0 \\ & 1 & & & \\ & & \dfrac{1+|\tilde{z}_1|^2}{1-|\tilde{z}_1|^2} & & \\ 0 & & & & 0 \end{pmatrix} \quad (\in \mathcal{S}^0(\mu)) \\ A_{202} &= \begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & & \\ & & a_{202} & \\ 0 & & & 1 \end{pmatrix} \quad (\in \mathcal{S}^{-1}(\mu)) . \end{aligned}$$

We set

$$(3.64)\quad \tilde{U} = {}^t(U_t, (t+\delta)^kU_{y_2}, \cdots, (t+\delta)^kU_{y_n}, (t+\delta)^kQ_2U) .$$

By (3.58), (3.60), (3.61) and (3.62), we obtain

$$\text{(3.65)}\quad \begin{cases} \tilde{M}_2\tilde{U}_t=\tilde{A}_2\tilde{U}_{y_1}+\sum_{j=2}^{n}\tilde{B}_{2j}\tilde{U}_{y_j}+\tilde{D}_2Q_2\tilde{U}+\tilde{E}_{21}\tilde{U} \\ \qquad +\frac{1}{t+\delta}\tilde{E}_{22}\tilde{U}+\tilde{K}_{21}\tilde{V}_{21}+\tilde{K}_{22}\tilde{V}_{22}+\tilde{F}_2+\tilde{H}_2 \\ \tilde{U}(0, y)=0 \\ \tilde{P}_2\tilde{U}|_{y_1=0}=\tilde{G}_2 \\ \quad (t, y)\in(0, T)\times \boldsymbol{R}^n_+ \end{cases}$$

where

$$\tilde{M}_2=\begin{pmatrix} M_2 & & 0 \\ & \ddots & \\ 0 & & M_2 \end{pmatrix}, \quad \tilde{A}_2=\begin{pmatrix} A_2 & & 0 \\ & \ddots & \\ 0 & & A_2 \end{pmatrix}$$

$$\tilde{B}_{2j}=\begin{pmatrix} B_{2j} & & 0 \\ & \ddots & \\ 0 & & B_{2j} \end{pmatrix}, \quad \tilde{D}_2=\begin{pmatrix} D_2 & & 0 \\ & \ddots & \\ 0 & & D_2 \end{pmatrix}$$

$$(j=2, \cdots, n)$$

$$\tilde{K}_{21}=\begin{pmatrix} K_{21} & & 0 \\ & \ddots & \\ 0 & & K_{21} \end{pmatrix}, \quad \tilde{K}_{22}=\begin{pmatrix} K_{22} & & 0 \\ & \ddots & \\ 0 & & K_{22} \end{pmatrix}$$

$$\tilde{P}_2=\begin{pmatrix} P_2 & & \\ & \ddots & \\ & & P_2 \end{pmatrix}$$

$\tilde{E}_{21}\cdots$a $(4n+4)\times(4n+4)$ pseudo differential system with the same property as E_{11}.

$\tilde{E}_{22}\cdots$a $(4n+4)\times(4n+4)$ pseudo differential system with the same property as E_{12}.

$$\tilde{F}_2={}^t(F_{2t}, (t+\delta)^kF_{2y_2}, \cdots, (t+\delta)^kF_{2y_n}, (t+\delta)^kQ_2F_2)\ .$$

$$\tilde{G}_2={}^t(G_{2t}-P_{2t}U|_{y_1=0}, (t+\delta)^k[G_{2y_2}-P_{2y_2}U|_{y_1=0}]\ ,$$
$$\cdots, (t+\delta)^k[G_{2y_n}-P_{2y_n}U|_{y_1=0}], (t+\delta)^k[Q_2G_2-[Q_2, P_2]U|_{y_1=0}])\ .$$

$$\tilde{V}_{21}={}^t(V_{21t}, (t+\delta)^kV_{21y_2}, \cdots, (t+\delta)^kV_{21y_n}, (t+\delta)^kQ_2V_{21})$$

$$\tilde{V}_{22}={}^t(V_{22t}, (t+\delta)^kV_{22y_2}, \cdots, (t+\delta)^kV_{22y_n}, (t+\delta)^kQ_2V_{22})$$

$$\tilde{H}_2={}^t(H_{20}, H_{22}, \cdots, H_{2n}, H_{2(n+1)})$$

$$
\begin{aligned}
H_{20} = & \Big(A_{20t}+\frac{k}{t+\delta}A_{20}\Big)A_{202}\Big(M_2U_t-\sum_{j=2}^{n}B_{2j}U_{y_j}-D_2Q_2U\Big) \\
& -\Big(A_{20t}+\frac{k}{t+\delta}A_{20}\Big)(A_{201}+A_{202})\Big(E_{21}U+\frac{1}{t+\delta}E_{22}U \\
& +K_{21}V_{21}+K_{22}V_{22}+F_2\Big)+D_2\{Q_{2t}-(Q_{2t}\triangle Q_2^{-1})Q_2\}U \\
& +E_{21t}U+\Big(\frac{E_{22}}{t+\delta}\Big)_t U+K_{21t}V_{21}+K_{22t}V_{22}\ .
\end{aligned}
$$

$$
\begin{aligned}
H_{2r} = & (t+\delta)^k A_{20y_r}A_{202}\Big(M_2U_t-\sum_{j=2}^{n}B_{2j}U_{y_j}-D_2Q_2U\Big) \\
& -(t+\delta)^k A_{20y_r}(A_{201}+A_{202})\Big(E_{21}U+\frac{1}{t+\delta}E_{22}U \\
& +K_{21}V_{21}+K_{22}V_{22}+F_2\Big)+D_2\{Q_{2y_r}-(Q_{2y_r}\triangle Q_2^{-1})Q_2\}(t+\delta)^kU \\
& +(t+\delta)^k\Big\{E_{21y_r}U+\frac{1}{t+\delta}E_{22y_r}U+K_{21y_r}V_{21}+K_{22y_r}V_{22}\Big\} \\
& \qquad\qquad (r=2,\cdots,n)\ .
\end{aligned}
$$

$$
\begin{aligned}
H_{2,(n+1)} = & (t+\delta)^k M_2\{Q_{2t}-(Q_{2t}\triangle Q_2^{-1})Q_2\}U \\
& +[Q_2,\ A_2]A_{202}\Big(M_2U_t-\sum_{j=2}^{n}B_{2j}U_{y_j}-D_2Q_2U\Big) \\
& -(t+\delta)^k A_2\{Q_{2y_1}-(Q_{2y_1}\triangle Q_2^{-1})Q_2\}U \\
& -[Q_2,\ A_2](A_{201}+A_{202})\Big\{E_{21}U+\frac{1}{t+\delta}E_{22}U+K_{21}V_{21} \\
& +K_{22}V_{22}+F_2\Big\}+(t+\delta)^k\sum_{j=2}^{n}B_{2j}\Big\{\Big[Q_2,\ \frac{\partial}{\partial y_j}\Big] \\
& -\Big(\Big[Q_2,\ \frac{\partial}{\partial y_j}\Big]\triangle Q_2^{-1}\Big)Q_2\Big\}U+(t+\delta)^k\Big\{[Q_2,\ E_{21}]U \\
& +\frac{1}{t+\delta}[Q_2,\ E_{22}]U+[Q_2,\ K_{21}]V_{21}+[Q_2,\ K_{22}]V_{22}\Big\}
\end{aligned}
$$

and

(3.66) $$\mathrm{Re}\langle\tilde{A}_2\tilde{U},\ \tilde{U}\rangle \geqq C(t+\delta)^k\langle\tilde{U},\ \tilde{U}\rangle$$

for any $\tilde{U}\in\operatorname{Ker}\tilde{P}_2\cap L^2(R_{y'}^{n-1})$ and any $\mu\geqq\mu_8$ where C and μ_8 are positive constants.

We set

(3.67) $$\tilde{\Phi}(t)=\mathrm{Re}(e^{-\mu t}\tilde{M}_2\tilde{U},\ e^{-\mu t}\tilde{U})\ .$$

By the same method as the one to obtain (3.51), we have

$$\tilde{\Phi}(t)+C_{31}\mu\int_0^t\left(\frac{t}{\tau}\right)^\alpha\tilde{\Phi}(\tau)d\tau+C_{32}\int_0^t\left(\frac{t}{\tau}\right)^\alpha(\tau+\delta)^k\langle\!\langle\tilde{U}\rangle\!\rangle_{0,\mu}^2d\tau \tag{3.68}$$
$$\leqq\frac{C_{33}}{\mu}\int_0^t\left(\frac{t}{\tau}\right)^\alpha\{\|\tilde{F}_2\|_{0,\mu}^2+\|\tilde{V}_{22}\|_{0,\mu}^2+\|\tilde{H}_2\|_{0,\mu}^2\}d\tau$$
$$+C_{34}\int_0^t\left(\frac{t}{\tau}\right)^\alpha(\tau+\delta)^k\langle\!\langle\tilde{G}_2\rangle\!\rangle_{0,\mu}^2d\tau$$

for any $\mu\geqq\mu_9$ and any $\alpha\geqq\alpha_8$ where $C_{31},\cdots,C_{34}$, μ_9 and α_8 are positive constants. By (3.56), (3.62) and (3.68), we obtain

$$\|(t+\delta)^kU_{y_1}\|_{0,\mu}^2+C_{31}\mu\int_0^t\left(\frac{t}{\tau}\right)^\alpha\|(\tau+\delta)^kU_{y_1}\|_{0,\mu}^2d\tau \tag{3.69}$$
$$+C_{32}\int_0^t\left(\frac{t}{\tau}\right)^\alpha(\tau+\delta)^k\langle\!\langle(\tau+\delta)^kU_{y_1}\rangle\!\rangle_{0,\mu}^2d\tau$$
$$\leqq\frac{C_{43}}{\mu}\int_0^t\left(\frac{t}{\tau}\right)^\alpha\{\|\tilde{F}_2\|_{0,\mu}^2+\|\tilde{V}_{22}\|_{0,\mu}^2+\|\tilde{H}_2\|_{0,\mu}^2\}d\tau$$
$$+C_{44}\int_0^t\left(\frac{t}{\tau}\right)^\alpha(\tau+\delta)^k\langle\!\langle\tilde{G}_2\rangle\!\rangle_{0,\mu}^2d\tau$$
$$+C_{45}\mu+t^{2N_3+1}\int_0^t\left\{\left\|\left(\frac{\partial}{\partial\tau}\right)^{N_3+1}f_1(\tau,\cdot)\right\|_{0,\Omega}^2+\left\|\left(\frac{\partial}{\partial\tau}\right)^{N_3+1}g_1(\tau,\cdot)\right\|_{0,S}^2\right\}d\tau$$
$$+C_{46}\|F_2\|_{0,\mu}^2+C_{47}\mu\int_0^t\left(\frac{t}{\tau}\right)^\alpha\|F_2\|_{0,\mu}^2d\tau+C_{48}\int_0^t\left(\frac{t}{\tau}\right)^\alpha(\tau+\delta)^k\langle\!\langle F_2\rangle\!\rangle_{0,\mu}^2d\tau\,.$$

Hence, by the same method of the second step, (3.56), (3.68) and (3.69), we have

$$\|\tilde{W}(t)\|_{0,\mu,\Omega}^2+C_{51}\mu\int_0^t\left(\frac{t}{\tau}\right)^\alpha\|\tilde{W}(\tau)\|_{0,\mu,\Omega}^2d\tau \tag{3.70}$$
$$+C_{52}\int_0^t\left(\frac{t}{\tau}\right)^\alpha(\tau+\delta)^k\langle\!\langle\tilde{W}(\tau)\rangle\!\rangle_{0,\mu,S}^2d\tau$$
$$\leqq C'_{T,2}\mu^{3/2}t^{2N_4}\int_0^t\left\{\left\|\left(\frac{\partial}{\partial\tau}\right)^{N_4+\beta}f_1(\tau,\cdot)\right\|_{1,\Omega}^2\right.$$
$$\left.+\left\|\left(\frac{\partial}{\partial\tau}\right)^{N_4+\beta}g_1(\tau,\cdot)\right\|_{1,S}^2\right\}d\tau$$

for any $\mu\geqq\mu_{10}$ and any $\alpha\geqq\alpha_4$ where

$$\tilde{W}(t)={}^t(u_{\delta t},(t+\delta)^ku_{\delta x_1},\cdots,(t+\delta)^ku_{\delta x_n},u_{\delta tt}, \tag{3.71}$$
$$(t+\delta)^ku_{\delta tx_1},\cdots,(t+\delta)^ku_{\delta tx_n},(t+\delta)^{2k}u_{\delta x_1x_1},\cdots,$$
$$(t+\delta)^{2k}u_{\delta x_nx_n})$$

μ_{10}, α_4, β and $C'_{T,2}$ are positive constants independent of δ, f_1 and g_1, and N_4 is a positive integer. So, we obtain (3.57). Therefore, we have Lemma 3.2. Q.E.D.

DEFINITION. Ω has the ordinary cone property if there is a cone C such that for each point $x \in \Omega$, there is a cone $C_i \subset \Omega$ with vertex x congruent to C. Ω has the restricted cone property if $\partial\Omega$ has a locally finite open covering $\{O_i\}$ and corresponding cones $\{C_i\}$ with vertices at the origin and the property that $x + C_i \subset \Omega$ for $x \in \Omega \cap O_i$.

LEMMA 3.4. *Let Ω be a bounded domain having the restricted cone property. Then, every bounded sequence in $H_m(\Omega)$ has a subsequence which converges in $H_j(\Omega)$ if $j < m$.*

PROOF. See [1].

PROOF OF THEOREM 3.1. By the fact that the domain $(0, T) \times \Omega$ has the restricted cone property, Lemma 3.2 and Lemma 3.4, we have a solution $u(t, x)$ of the problem (3.1) which satisfies

$$\sum_{j=0}^{m+2} \left\| \left(\frac{\partial}{\partial t} \right)^j u(t, \cdot) \right\|^2_{m+2-j,\Omega} \leqq C_{T,m} \int_0^t \left\{ \left\| \left(\frac{\partial}{\partial \tau} \right)^{N_2+1} f_1(\tau, \cdot) \right\|^2_{m+2,\Omega} + \left\| \left(\frac{\partial}{\partial \tau} \right)^{N_2+1} g_1(\tau, \cdot) \right\|^2_{m+2,S} \right\} d\tau . \tag{3.72}$$

Let $\beta + |\gamma| \leqq m+2$. Then, we can write

$$D_t^\beta D_x^\gamma u_\delta(t, x) - D_t^\beta D_x^\gamma u_\delta(t', x) = \int_{t'}^t D_t^{\beta+1} D_x^\gamma u_\delta(\tau, x) d\tau \tag{3.73}$$

for almost all $x \in \Omega$ since $D_t^\beta D_x^\gamma u_\delta(t, x)$ and $D_t^{\beta+1} D_x^\gamma u_\delta(t, x) \in L^1_{\mathrm{loc}}((0, T) \times \Omega)$. Using Schwarz inequality, we obtain

$$\int_\Omega |D_t^\beta D_x^\gamma u_\delta(t, x) - D_t^\beta D_x^\gamma u_\delta(t', x)|^2 dx \leqq |t-t'| \int_0^T \int_\Omega |D_t^{\beta+1} D_x^\gamma u_\delta(\tau, x)|^2 dx d\tau \leqq C|t-t'| \tag{3.74}$$

for $\beta + |\gamma| \leqq m+2$ by Lemma 3.2 where C is a positive constant. As $\delta \to 0$, we get

$$\|D_t^\beta u(t, x) - D_t^\beta u(t', x)\|_{m+2-\beta,\Omega} \leqq C' |t-t'|^{1/2} \tag{3.75}$$

for the solution $u(t, x)$ of the problem (3.1) where C' is a positive constant. Let us define $D_t^\beta u(T, x)$ by

(3.76) $$D_t^\beta u(T, x)=\lim_{t\to T-0} D_t^\beta u(t, x) \quad \text{in} \quad H^{m+2-\beta}(\Omega)\ .$$

Then, the mapping $t\to D_t^\beta u(t, x)\in H^{m+2-\beta}(\Omega)$ is strongly continuous on $[0, T]$, which proves $u(t, x)\in \cap_{j=0}^{m+2} \mathscr{E}_t^j(H^{m+2-j}(\Omega))$. By the the same method used to obtain (3.4), we can get the similar estimate of the solution $u(t, x)$ of the problem (3.1). Therefore, the uniqueness of the solution holds. Hence, we have Theorem 3.1. Q.E.D.

§4. The existence of the solution.

In this section, we shall prove the Main Theorem.

For any fixed non-negative integer m, we take as N in Theorem 3.1. Let $u_{r+2}(x)$ $(r=0, 1, \cdots, N)$ be defined by (1.9) and put

(4.1) $$w(t, x)=\sum_{j=0}^{N+2}\frac{t^j}{j!}u_j(x)\ .$$

Then, we have

(4.2) $$\begin{cases} f_1(t, x)=f(t, x)-L[w]\in \mathscr{E}_t^\infty(H^\infty(\Omega)) \\ g_1(t, s)=g(t, s)-B[w]|_S\in \mathscr{E}_t^\infty(H^\infty(S)) \end{cases}$$

and

(4.3) $$\begin{cases} \left(\dfrac{\partial}{\partial t}\right)^i f_1|_{t=0}=0 \quad (i=0, \cdots, N) \\ \left(\dfrac{\partial}{\partial t}\right)^j g_1|_{t=0}=0 \quad (j=0, \cdots, N+1) \end{cases}$$

by (1.9) and (1.11). Therefore, by Theorem 3.1, there exists a unique solution $v(t, x)\in \cap_{j=0}^{m+2} \mathscr{E}_t^j(H^{m+2-j}(\Omega))$ of the problem

(4.4) $$\begin{cases} L[v]=f_1 \\ v(0, x)=0,\ v_t(0, x)=0 \\ B[v]|_S=g_1 \\ \quad (t, x)\in(0, T)\times\Omega\ . \end{cases}$$

If we set

(4.5) $$u(t, x)=v(t, x)+w(t, x)\ ,$$

this is the unique solution of the problem (1.1). Hence, we have the Main Theorem. Q.E.D.

REMARK 4. By our method, we can treat the following problem

$$(4.6)\quad \begin{cases} L[u]=\dfrac{\partial^2 u}{\partial t^2}-2t^k\sum_{j=1}^{n} h_j(t, x)\dfrac{\partial^2 u}{\partial t\partial x_j}-t^{2k}\sum_{i,j=1}^{n} a_{ij}(t, x)\dfrac{\partial^2 u}{\partial x_i\partial x_j} \\ \qquad +a_0(t, x)\dfrac{\partial u}{\partial t}+t^{k-1}\sum_{j=1}^{n} a_j(t, x)\dfrac{\partial u}{\partial x_j}+d(t, x)u=f(t, x) \\ u(0, x)=u_0(x),\ u_t(0, x)=u_1(x) \\ B[u]|_S=u|_S=g(t, s) \\ \quad (t, x)\in(0, T)\times\Omega\ . \end{cases}$$

REMARK 5. By our method, we can treat the following problem

$$(4.7)\quad \begin{cases} L[u]=\dfrac{\partial^2 u}{\partial t^2}-2t^k\sum_{j=1}^{n} h_j(t, x)\dfrac{\partial^2 u}{\partial t\partial x_j}-t^{2k}\sum_{i,j=1}^{n} a_{ij}(t, x)\dfrac{\partial^2 u}{\partial x_i\partial x_j} \\ \qquad +a_0(t, x)\dfrac{\partial u}{\partial t}+t^{k-1}\sum_{j=1}^{n} a_j(t, x)\dfrac{\partial u}{\partial x_j}+d(t, x)u=f(t, x) \\ u(0, x)=u_0(x)\ ,\quad u_t(0, x)=u_1(x) \\ B_N[u]|_S=0 \\ \quad (t, x)\in(0, T)\times\Omega \end{cases}$$

where

$$(4.8)\quad \tilde{B}_N=\frac{1}{\sqrt{\tilde{a}_{11}(t, 0, y')}}\left\{t^k\left[\tilde{a}_{11}(t, 0, y')\frac{\partial}{\partial y_1}+\sum_{j=2}^{n}\tilde{a}_{1j}(t, 0, y')\frac{\partial}{\partial y_j}\right]\right. \\ \left. +\tilde{h}_1(t, 0, y')\frac{\partial}{\partial t}\right\}+\tilde{r}(t, y')$$

instead of $\tilde{B}$ in (1.3). Then, we have $l=1$ for (3.16) and choose U in (3.24) as

$$(4.9)\quad U=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix}=\begin{pmatrix} Q_0v-(t+\delta)^kQ_{11}w+(t+\delta)^kQ_2v \\ Q_0v+(t+\delta)^kQ_{11}w+(t+\delta)^kQ_2v \\ Q_0v-(t+\delta)^kQ_{11}w-(t+\delta)^kQ_2v \\ -Q_0v-(t+\delta)^kQ_{11}w+(t+\delta)^kQ_2v \\ v \end{pmatrix}$$

where

$$(4.10)\quad \begin{cases} Q_{11}w=\bar{\varphi}\cdot Q_1w \\ w=\psi\bar{u}_\delta\ . \end{cases}$$

REMARK 6. By the similar systemization in § 2 and § 3 and the theory of the pseudo differential operator, we can treat the L^2-well-posed mixed problem for regularly hyperbolic equations of second order with variable coefficients.

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Present Address:
5-18-19 KAMIMEGURO
MEGURO-KU, TOKYO 153