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On Some Integral Invariants, Lefschetz Numbers and Induction Maps

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Dedicated to Professor Hiroshi Toda on his 60th birthday

§1. Introduction.

Let M be a compact complex manifold, G any compact subgroup of the complex Lie group of all holomorphic automorphisms of M and $\mathfrak{G}$ the Lie algebra of G which consists of holomorphic vector fields on M. In [5], the first author defined a character $f: \mathfrak{G} \to \boldsymbol{C}$ (more generally defined a $\boldsymbol{C}$-character of the complex Lie algebra of all holomorphic vector fields on M) which depends only on the complex structure of M and vanishes if M admits a Kaehler-Einstein metric. In this paper, we first see that characters of this kind appear naturally in the Lefschetz numbers. More precisely, let $\mathscr{D}$ be the Dolbeault complex of M with values in a certain holomorphic vector bundle over M and H^i the i-th cohomology group of $\mathscr{D}$. Then the Lefschetz number $L(g)$, for $g \in G$, is by definition

$$L(g) = \sum_i (-1)^i \operatorname{tr}(g|_{H^i}) .$$

In Theorem 4.3, we show that $f(X)$, for $X \in \mathfrak{G}$, coincides up to constant with *the second term of the Taylor expansion of* $L(\exp tX)$ whose first term is of course the arithmetic genus of $\mathscr{D}$. Then it becomes clear that f depends only on the complex structure of M and that $f(\mathrm{Ad}(g)X) = f(X)$ for any $g \in G$.

Now we wish to put this view point into a single diagram. Let G and H be compact Lie groups with Lie algebras $\mathfrak{G}$ and $\mathfrak{H}$. Let M be a compact oriented manifold of dimension $2m$ and P a principal right H-bundle over M. Suppose that G acts on $P \to M$ on the left as bundle automorphisms and that the action of G on M is orientation-preserving. Let θ be a G-invariant connection of P. Then, as in [4], an H-equivariant

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$\mathfrak{H}$-valued 0-form $J(X)$ on P is defined for $X \in \mathfrak{G}$ by

$$(1.1) \qquad J(X)(p)=\theta(X^*_p) \qquad \text{for} \quad p \in P$$

where X^* denotes the vector field on P induced by the flow $\exp tX$ and $\mathscr{F}(\phi)\colon \otimes^k \mathfrak{G} \to C$ is defined for an H-invariant polynomial ϕ of degree $m+k$ by

$$(1.2) \qquad \mathscr{F}(\phi)(X_1, \cdots, X_k)=\binom{m+k}{k}\left(\frac{\sqrt{-1}}{2\pi}\right)^m \int_M \phi(J(X_1)\wedge \cdots \wedge J(X_k)\wedge(\wedge^m \Theta))$$

where $X_i \in \mathfrak{G}$ and Θ denotes the curvature form of θ. It can be seen from the left G-invariance of θ that $\mathscr{F}(\phi)$ is Ad(G)-invariant and thus we get a C-linear map $\mathscr{F}\colon I^{m+k}(H) \to I^k(G)$ where $I^l(G)$ denotes the set of all G-invariant polynomials of degree l with C-coefficients. Clearly, when $\deg \phi = m$, $\mathscr{F}(\phi)$ is a characteristic number and when $\deg \phi = m+1$, $\mathscr{F}(\phi)$ is a character of $\mathfrak{G}$ into C. In Theorem 3.11, we give relations between $\mathscr{F}$ and the Lefschetz number (or the Atiyah-Singer index). In particular when M is a compact complex manifold, G is a compact subgroup of the (biholomorphic) automorphism group of M and P is the unitary frame bundle with respect to a G-invariant Hermitian metric, $f\colon \mathfrak{G} \to C$ coincides up to constant with $\mathscr{F}(c_1^{m+1})$ where c_1 is the first Chern polynomial (this can be seen using Yau's solution to the Calabi conjecture, see [6]) and Theorem 4.3 follows from Theorem 3.11.

With these understood, it would be clear that Theorem 3.11 can be applied to other geometric cases such as the signature complex for an oriented manifold and the Dirac operator for a spin manifold. Note that, in these cases, G may be any closed subgroup of the automorphism group Aut(M) of M because Aut(M) itself is compact. In Section 5, we shall apply Theorem 3.11 to the homogeneous space G/H (where H is a closed subgroup of a compact Lie group G) and get an induction homomorphism $I^*(H) \to I^*(G)$ which is expressed by the integration over G/H and coincides with the induced representation $R(H) \to R(G)$ (where $R(H)$ denotes the representation ring of H) (see Segal [9]) and with the transfer map $H^*(BH) \to H^*(BG)$ (see Becker-Gottlieb [3]) under the natural correspondence between R, I^* and H^*.

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§2. Chern characters.

Let $R(G)$ be the representation ring of a compact Lie group G. Namely, $R(G)=K_G(\mathrm{pt})$ is the Grothendieck construction of the commutative semiring of all complex G-modules. $R(G)$ is isomorphic to a free $\boldsymbol{Z}$-module with the irreducible complex G-modules A_i's or the irreducible characters χ_i's of A_i's as its basis and is also regarded as a subring of the ring of all real analytic functions on G generated by χ_i's.

For any $R(G) \ni z = \sum_i n_i A_i$ (finite sum), $n_i \in \boldsymbol{Z}$, the Chern character $\mathrm{ch}(z) \in H_G^{**} = H^{**}(BG; \boldsymbol{C}) = \prod_{k=0}^{\infty} H^{2k}(BG; \boldsymbol{C})$ is defined by $\mathrm{ch}(z) = \sum_i n_i \mathscr{C}\mathscr{h}(EG \times_G A_i)$ where $\mathscr{C}\mathscr{h}$ is the usual Chern character of a complex vector bundle $EG \times_G A_i$ over BG.

On the other hand, a counterpart in $I^{**}(G) = \prod_{k=0}^{\infty} I^k(G)$ is defined as follows. For any $R(G) \ni z = \sum_i n_i A_i$, an $\mathrm{Ad}(G)$-invariant real analytic function α_z on $\mathfrak{G}$ is defined by

$$\alpha_z(X) = z(\exp X) = \mathrm{tr}(\exp X|_z) = \sum_i n_i \,\mathrm{tr}(\exp X|_{A_i}) \qquad \text{for} \quad X \in \mathfrak{G}\,.$$

DEFINITION 2.1. Let $\mathrm{ch}(z) \in I^{**}(G)$ denote the image of α_z under the invariant Taylor homomorphism of [7]. Namely, $\{\mathrm{ch}(z)\}_{(k)} \in I^k(G)$ (where $\Psi_{(k)} \in I^k(G)$ denotes the degree k term of $\Psi \in I^{**}(G)$) is characterized by the Taylor expansion

$$\alpha_z(tX) = \sum_{k=0}^{\infty} \{\mathrm{ch}(z)\}_{(k)}(X, \cdots, X)t^k \qquad \text{for any} \quad X \in \mathfrak{G}\,.$$

Thus $\mathrm{ch}\colon R(G) \to H_G^{**}$ and $\mathrm{ch}\colon R(G) \to I^{**}(G)$ are defined and easily seen to be ring homomorphisms. These two ch's correspond to each other under the Weil homomorphism $W_G\colon I^k(G) \to H_G^{2k}$. Throughout this paper, the Weil homomorphism always means the modified Weil homomorphism of [7]. Namely, W_G is normalized by the property that, for $G = U(1) = S^1$, $q^* W_{U(1)}(x) = \{(\sqrt{-1}/2\pi)\Omega\}$ where $x\colon \mathfrak{u}(1) \to \boldsymbol{C}$ is an invariant polynomial on the Lie algebra $\mathfrak{u}(1) = \sqrt{-1}\boldsymbol{R}$ given by the inclusion map and Ω is a curvature form in the principal bundle $q\colon EU(1) \to BU(1)$.

The next lemma follows immediately from the definition of ch's and [7, p. 453].

LEMMA 2.2. *The following diagram is commutative.*

$$\begin{tikzcd} & R(G) \arrow[dl, "\mathrm{ch}"'] \arrow[dr, "\mathrm{ch}"] & \\ I^{**}(G) \arrow[rr, "W_G"] & & H_G^{**} \end{tikzcd}$$

ch's are obviously extended to $\boldsymbol{C}$-algebra homomorphisms, and if G is connected, then $R(G) \otimes \boldsymbol{C}$, $I^{**}(G)$ and H_G^{**} may be identified under the above diagram in the sense of the next lemma.

LEMMA. *If G is connected (and compact), then*

(2.3) $W_G\colon I^{**}(G) \to H_G^{**}$ *is a* C*-algebra isomorphism,*

(2.4) ch: $R(G)\otimes C \to I^{**}(G)$ *is an injective* C*-algebra homomorphism,*

(2.5) *for any finite sum* $\phi \in \sum_{k=0}^{N} I^k(G)$, *there exists* $z \in R(G)\otimes C$ *such that* $\mathrm{ch}(z)=\phi$ *modulo higher terms of degree* $\geqq N+1$.

(2.4) and (2.5) assert that $R(G)\otimes C$ becomes a "dense" subalgebra of $I^{**}(G)\cong H_G^{**}$ under ch.

PROOF OF (2.4). For $R(G)\otimes C \ni z=\sum_i c_i\chi_i$, $c_i \in C$, $\mathrm{ch}(z)=0$ means that $\sum_i c_i\chi_i(\exp X)=0$ for any $X\in\mathfrak{G}$. This means that $\sum_i c_i\chi_i=0$ because G is connected.

PROOF OF (2.3) AND (2.5). Let $T^r\subset G$ be a maximal torus and W the Weyl group which acts on T^r as inner automorphisms. Then (2.3) follows immediately from the well-known facts that both $I^k(G)$ and H_G^{2k} consist of W-invariant elements of $I^k(T^r)$ and $H_{T^r}^{2k}$ respectively and that W_G maps $I^k(T^r)$ isomorphically onto $H_{T^r}^{2k}$ commuting with the W-action.

For the proof of (2.5), it suffices to show that for any $\phi\in I^k(G)$ there exists $z\in R(G)\otimes C$ such that $\mathrm{ch}(z)=\phi$ modulo terms of degree $\geqq k+1$. As described above, ϕ is a W-invariant element of $I^k(T^r)=\{$polynomials of degree k in $C[x_1,\cdots,x_r]\}$ where x_i is an $\mathrm{Ad}(T^r)$-invariant polynomial of degree 1 given by

(2.6) $x_i(X)=\sqrt{-1}\theta_i\in C$ for an element $X=(\sqrt{-1}\theta_1,\cdots,\sqrt{-1}\theta_r)$ of the Lie algebra of T^r.

On the other hand, it is also well-known that $R(G)\otimes C$ consists of W-invariant elements of $R(T^r)\otimes C=C[t_1, t_1^{-1},\cdots,t_r,t_r^{-1}]$ where t_i is an irreducible character of T^r given by $t_i(g)=e^{\sqrt{-1}\theta_i}$ for $g=(e^{\sqrt{-1}\theta_1},\cdots,e^{\sqrt{-1}\theta_r})\in T^r$. Now it is easy to see from the definition of ch that

$$\mathrm{ch}(t_i)=e^{x_i}=1+x_i+\frac{1}{2}x_i^2+\cdots\in I^{**}(T^r)\,.$$

Furthermore, it is clear that ch commutes with the W-action because the W-action is induced from an automorphism of T^r. Now, for $\phi=\phi(x_1,\cdots,x_r)$, put $z=|W|^{-1}\sum_{\sigma\in W}\sigma\cdot\phi(t_1-1,\cdots,t_r-1)$ where $|W|$ is the order of W. Then it follows from the properties of ch described above and the W-invariance of ϕ that

$$\begin{aligned}\mathrm{ch}(z)&=|W|^{-1}\sum_{\sigma\in W}\sigma\cdot\phi(e^{x_1}-1,\ \cdots,\ e^{x_r}-1)\\&=|W|^{-1}\sum_{\sigma\in W}\sigma\cdot\phi(x_1+\text{higher},\ \cdots,\ x_r+\text{higher})\\&=|W|^{-1}\sum_{\sigma\in W}\sigma\cdot\phi(x_1,\ \cdots,\ x_r)+\text{higher}\\&=\phi(x_1,\ \cdots,\ x_r)+\text{higher}\,.\end{aligned}$$

Q.E.D.

COROLLARY 2.7. *For any compact (not necessarily connected) Lie group* G, W_G *is injective.*

PROOF. Let G_0 be the identity component of G and $i\colon G_0\to G$ the inclusion. Then $i^*\colon I^{**}(G)\to I^{**}(G_0)$ is clearly injective. Thus the corollary follows from (2.3) because $i'\circ W_G$ coincides with $W_{G_0}\circ i^*$ for $i'=i^*\colon H_G^{**}\to H_{G_0}^{**}$. Q.E.D.

The next lemma is an obvious consequence of (2.5).

LEMMA 2.8. *If* G *is connected (and compact), then for any* $\Psi\in I^{**}(G)$ *such that* $\Psi_{(0)}\neq 0$ *and any* $\phi\in I^{m+k}(G)$, *there exists* $z\in R(G)\otimes C$ *such that* $\{\Psi\cdot\mathrm{ch}(z)\}_{(m+k)}=\phi$.

§3. $\mathscr{T}$ and the Atiyah-Singer index.

Now we come back to the situation of Section 1. Let $M_G=EG\times_G M$ be the associated oriented M-fiber bundle over BG with projection $\pi_M\colon M_G\to BG$ and $P_G=EG\times_G P$ the associated P-fiber bundle over BG with projection $\pi_P\colon P_G\to BG$. Note that, in arguments of this section, BG, EG, M_G and P_G are regarded as compact smooth manifolds by considering their finite skeletons. P_G is also regarded as a principal H-bundle over M_G with classifying map $c\colon M_G\to BH$. Then

DEFINITION 3.1. A C-linear map $\mathscr{G}\colon H_H^{2m+2k}\to H_G^{2k}$ is defined to be the composition of $c^*\colon H_H^{2m+2k}\to H^{2m+2k}(M_G;\,C)$ and the Gysin homomorphism (i.e. the integration over the fiber) $\pi_{M*}\colon H^{2m+2k}(M_G;\,C)\to H_G^{2k}$.

The next proposition is a generalization of a result in [6] and is proved similarly by use of the G-invariance of θ.

PROPOSITION 3.2. *The following diagram is commutative.*

$$\begin{array}{ccc} I^{m+k}(H) & \xrightarrow{\mathscr{T}} & I^k(G)\\ {\scriptstyle W_H}\big\downarrow & & \big\downarrow{\scriptstyle W_G}\\ H_H^{2m+2k} & \xrightarrow{\mathscr{G}} & H_G^{2k}\end{array}$$

PROOF. A fixed connection ω in $EG \to BG$ (with curvature form Ω) defines a splitting of tangent spaces $T_{(e,p)}P_G = T^V_{(e,p)}P_G \oplus H_{(e,p)}$ for any $(e, p) \in EG \times_G P = P_G$ ($(e, p) = (e \cdot g^{-1}, g \cdot p)$ in P_G for any $g \in G$) where $H_{(e,p)}$ denotes the horizontal subspace and $T^V P_G \cong EG \times_G TP$ denotes the vertical subbundle of the tangent bundle TP_G. This splitting defines the vertical projection $\kappa_{(e,p)}\colon T_{(e,p)}P_G \to T^V_{(e,p)}P_G$. Furthermore, θ defines a splitting $T_pP = \mathfrak{H} \oplus \hat{H}_p$ for any $p \in P$ with $\hat{H}_p$ as its horizontal subspace, and this splitting defines the vertical projection $\theta_p\colon T_pP \to \mathfrak{H}$. Here $T^V_{(e,p)}P_G$ is naturally identified with T_pP and θ_p defines $\tilde{\theta}_{(e,p)}\colon T^V_{(e,p)}P_G \to \mathfrak{H}$. By use of the left G-invariance of θ, it is easy to verify that $\tilde{\theta}_{(e,p)}$ is independent of the choice of the identification $T^V_{(e,p)}P_G = T_pP$ (i.e. the choice of the representative $(e, p) \in EG \times_G P = P_G$). Thus a $\mathfrak{H}$-valued 1-form ψ on P_G is defined by $\psi = \tilde{\theta} \circ \kappa$ and is easily verified to give a connection in principal H-bundle $P_G \to M_G$. Straightforward calculations show that the curvature form Ψ of ψ is given by

$$\Psi = \tilde{\Theta} \circ \kappa \otimes \kappa + J(\Omega) \tag{3.3}$$

where $\tilde{\Theta}\colon T^V P_G \otimes T^V P_G \to \mathfrak{H}$ is given by the curvature form Θ and $J(\Omega)$ is an H-equivariant $\mathfrak{H}$-valued ω-horizontal 2-form on P_G defined as follows. For any $A \in T_bBG$, $b \in BG$, let $A^\sharp$ denote the right G-invariant ω-horizontal lift of A on EG. Then, for any $A, B \in T_{(e,p)}P_G$, put

$$J(\Omega)_{(e,p)}(A, B) = J(\Omega_e((\pi_{P*}A)^\sharp, (\pi_{P*}B)^\sharp))(p) \in \mathfrak{H} .$$

Using the right G-invariance of $(\pi_{P*}A)^\sharp$, $(\pi_{P*}B)^\sharp$ and the property of J that $J(\mathrm{Ad}(g)X)(p) = J(X)(g^{-1} \cdot p)$ for any $g \in G$, any $X \in \mathfrak{G}$, it is easy to verify that $J(\Omega)_{(e,p)}$ is independent of the choice of the representative $(e, p) \in EG \times_G P = P_G$.

Now it follows from the definition of π_{M*}, $\mathscr{F}$ and the (modified) Weil homomorphisms that

$$\begin{aligned}
\mathscr{G} \circ W_H(\phi) &= \pi_{M*} \circ c^* \circ W_H(\phi) = \pi_{M*}\left(\left(\frac{\sqrt{-1}}{2\pi}\right)^{m+k} \phi(\wedge^{m+k}\Psi)\right) \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^k \binom{m+k}{k} \left(\frac{\sqrt{-1}}{2\pi}\right)^m \pi_{M*}\{\phi((\wedge^k J(\Omega)) \wedge (\wedge^m \tilde{\Theta} \circ \kappa \otimes \kappa))\} \\
&= W_G \circ \mathscr{F}(\phi)
\end{aligned}$$

for any $\phi \in I^{m+k}(H)$. Q.E.D.

For the rest of this paper, we shall work under the following assumption.

ASSUMPTION 3.4. There exists a real oriented H-module V with a representation $\rho: H \to SO_R(V) \cong SO(2m)$ such that $\rho^* e \in H_H^{2m}$ does not vanish for the Euler class $e \in H_{SO(2m)}^{2m}$ and that $P \times_H V$ is isomorphic to the tangent bundle TM.

Assumption 3.4 means that P is an oriented H-structure over M, and then G acts on M as oriented H-structure preserving automorphisms.

Now, let $W^0, \cdots, W^N$ be a sequence of complex H-modules and let $\sigma_i: V \to \mathrm{Hom}(W^{i-1}, W^i)$ $(1 \leqq i \leqq N)$ be H-equivariant maps (i.e. $\sigma_i(h \cdot \xi) = h \cdot \sigma_i(\xi) \cdot h^{-1}$ for any $h \in H$, $\xi \in V$) such that, for $V \ni \xi \neq 0$,

$$0 \longrightarrow W^0 \xrightarrow{\sigma_1(\xi)} W^1 \longrightarrow \cdots \xrightarrow{\sigma_N(\xi)} W^N \longrightarrow 0$$

is exact. Then the universal elliptic symbol class $v \in K_H(V)$ is defined by the compactly supported complex

$$\begin{array}{ccccc} \{0 \to \cdots \to & V \times W^{i-1} & \to & V \times W^i & \to \cdots \to 0\} \\ & \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid & \\ & (\xi, w) & & (\xi, \sigma_i(\xi)(w)) & \end{array} \tag{3.5}$$

on V. And, for a fixed universal elliptic symbol class $v \in K_H(V)$, the v-index class $\mathscr{I}_v \in H_H^{**}$ is defined as in [1, p. 559] by

$$\mathscr{I}_v = (-1)^m \left\{ \frac{\sum_{i=0}^{N} (-1)^i \mathscr{C}h(EH \times_H W^i)}{\rho^* e} \cdot \rho^* \mathscr{I} \right\} \tag{3.6}$$

where $\mathscr{I} \in H_{SO(2m)}^{**}$ is the index class of [1, p. 555].

On the other hand, let $P \times_H : R(H) \to K_G(M)$ denote the homomorphism defined by the associating construction $z \to P \times_H z$ for an H-module z,

$$\alpha_P : K_H(V) \xrightarrow{q_1^*} K_{G \times H}(V) \xrightarrow{q_2^*} K_{G \times H}(P \times V) = K_G(P \times_H V) = K_G(TM)$$

the homomorphism defined by projections $q_1: G \times H \to H$, $q_2: P \times V \to V$ and, for $v \in K_H(V)$, $\beta_v: K_G(M) \to K_G(TM)$ the homomorphism defined by the multiplication $u \to \tau^* u \cdot \alpha_P(v)$ for $u \in K_G(M)$ where $\tau: TM \to M$ is the projection. Then

DEFINITION 3.7. For a fixed universal elliptic symbol class v, a homomorphism $\mathscr{E}_v: R(H) \to R(G)$ is defined to be the composition of $P \times_H$, β_v and the equivariant index homomorphism $K_G(TM) \to R(G)$. Namely, $\mathscr{E}_v(z)$ is the G-equivariant index of $P \times_H z$-valued v-elliptic complex on M for $z \in R(H)$. Note that $\mathscr{E}_v$ is obviously extended to a C-linear map $\mathscr{E}_v: R(H) \otimes C \to R(G) \otimes C$.

This $\mathscr{E}_v$ is related to $\mathscr{G}$ as follows.

PROPOSITION 3.8. *The following diagram is commutative;*

$$\begin{tikzcd} R(H) \arrow[r, "\mathscr{E}_v"] \arrow[d, "\mathscr{I}_v\text{-ch}"'] & R(G) \arrow[d, "\mathrm{ch}"] \\ H_H^{**} \arrow[r, "\mathscr{G}"'] & H_G^{**} \end{tikzcd}$$

where $\mathscr{I}_v$-ch *is defined by the multiplication* $z \to \mathrm{ch}(z)\cdot\mathscr{I}_v$ *for* $z \in R(H)$.

PROOF. Let $TM_G = EG\times_G TM$ be the tangent bundle along the fibers of M_G (i.e. $TM_G = T^V M_G$ in the notation of the proof of Proposition 3.2). TM_G is an oriented vector bundle over M_G and is also an oriented TM-fiber bundle over BG. Let Ψ: $H^{**}(M_G; C) \to H_c^{**}(TM_G; C)$ be the Thom isomorphism (where H_c^{**} denotes the cohomology with compact supports) and π_{TM*}: $H_c^{**}(TM_G; C) \to H_G^{**}$ the Gysin homomorphism for the projection π_{TM}: $TM_G \to BG$. Let τ_G: $TM_G \to M_G$ be the projection given by $\tau_G(e, \xi) = (e, \tau(\xi))$ for $(e, \xi) \in EG\times_G TM = TM_G$, τ: $TM \to M$. Then, since $\pi_{M*}\circ\tau_{G*} = \pi_{TM*}$ for τ_{G*}: $H_c^{**}(TM_G; C) \to H^{**}(M_G; C)$ and Ψ is equal to $\tau_{G*}{}^{-1}$, it suffices for the proof of the proposition to show the commutativity of the following diagram:

$$\begin{tikzcd} R(H) \arrow[r, "P\times_H"] \arrow[d, "EH\times_H"'] \arrow[dr, phantom, "\text{(i)}"] & K_G(M) \arrow[r, "\beta_v"] \arrow[d, "EG\times_G"'] \arrow[dr, phantom, "\text{(ii)}"] & K_G(TM) \arrow[r, "G\text{-ind}"] \arrow[d, "EG\times_G"'] \arrow[dr, phantom, "\text{(iii)}"] & R(G) \arrow[d, "EG\times_G"'] \\ K(BH) \arrow[r, "c^*"] \arrow[d, "\delta_v"'] \arrow[dr, phantom, "\text{(iv)}"] & K(M_G) \arrow[r, "\gamma_v"] \arrow[d, "\varepsilon_v"'] \arrow[dr, phantom, "\text{(v)}"] & K(TM_G) \arrow[r, "f\text{-ind}"] \arrow[d, "\zeta"'] \arrow[dr, phantom, "\text{(vi)}"] & K(BG) \arrow[d, "\mathscr{Ch}"'] \\ H_H^{**} \arrow[r, "c^*"] & H^{**}(M_G; C) \arrow[r, "\Psi"] & H_c^{**}(TM_G; C) \arrow[r, "\pi_{TM*}"] & H_G^{**} \end{tikzcd}$$

where $EH\times_H$, $EG\times_G$ are the homomorphisms defined by the associating construction, G-ind is the G-equivariant index, f-ind is the index of families over BG, γ_v is the homomorphism defined by $u \to \tau_G{}^*u\cdot EG\times_G \alpha_P(v)$ for $u \in K(M_G)$, δ_v is the homomorphism defined by $u \to \mathscr{Ch}(u)\cdot\mathscr{I}_v$ for $u \in K(BH)$, ε_v is the homomorphism defined by $u \to \mathscr{Ch}(u)\cdot c^*\mathscr{I}_v$ for $u \in K(M_G)$ and ζ is the homomorphism defined by $u \to (-1)^m \tau_G{}^*\mathscr{I}(TM_G)\cdot\mathscr{Ch}(u)$ for $u \in K(TM_G)$. Note that the index class $\mathscr{I}(TM_G) \in H^{**}(M_G; C)$ is equal to $c^*\rho^*\mathscr{I}$ because $\rho\circ c$: $M_G \to BSO(2m)$ is the classifying map of $TM_G = P_G\times_H V$.

Now, the commutativities of (i), (ii), (iii), (iv) are obvious and the commutativity of (vi) follows from the index theorem for families [2, Theorem (5.1)]. Here the orientation of TM_G in the definition of Ψ differs from that in [2] and the correct sign is $(-1)^{2m(2m+1)/2}=(-1)^m$ (see [1, (2.13)]). The commutativity of (v) is verified as follows. Let $q\colon P_G\times V\to V$ be the proper projection and $\alpha_{P_G}=q^*\colon K_H(V)\to K_H(P_G\times V)=K(P_G\times_H V)=K(TM_G)$. Then $EG\times_G\alpha_P(v)\in K(TM_G)$ is clearly equal to $\alpha_{P_G}(v)$. Hence it follows that, for any $u\in K(M_G)$,

$$\begin{aligned}\zeta\circ\gamma_v(u)&=(-1)^m\tau_G{}^*\mathscr{I}(TM_G)\cdot\mathscr{C}\!h(\gamma_v(u))\\&=(-1)^m\tau_G{}^*c^*\rho^*\mathscr{I}\cdot\tau_G{}^*\mathscr{C}\!h(u)\cdot\mathscr{C}\!h(\alpha_{P_G}(v))\\&=\tau_G{}^*(\mathscr{C}\!h(u)\cdot(-1)^m c^*\rho^*\mathscr{I})\cdot\Psi\Psi^{-1}\mathscr{C}\!h(\alpha_{P_G}(v))\\&=\Psi\{\mathscr{C}\!h(u)\cdot(-1)^m\Psi^{-1}\mathscr{C}\!h(\alpha_{P_G}(v))\cdot c^*\rho^*\mathscr{I}\}\ .\end{aligned}$$

Here it follows from [1, (2.16)] that

$$\Psi^{-1}\mathscr{C}\!h(\alpha_{P_G}(v))=c^*\left\{\frac{\sum_{i=0}^{N}(-1)^i\mathscr{C}\!h(EH\times_H W^i)}{\rho^*e}\right\}$$

and therefore $\zeta\circ\gamma_v(u)$ is equal to $\Psi(\mathscr{C}\!h(u)\cdot c^*\mathscr{I}_v)=\Psi\circ\varepsilon_v(u)$. Q.E.D.

Now, let $e\in I^m(SO(2m))$ denote the Euler polynomial and $\mathscr{I}\in I^{**}(SO(2m))$ the index polynomial which correspond to $e\in H^{2m}_{SO(2m)}$ and $\mathscr{I}\in H^{**}_{SO(2m)}$ respectively under the Weil isomorphism $W_{SO(2m)}$ (see (2.3)). Then the v-index polynomial $\mathscr{I}_v\in I^{**}(H)$ is defined by

$$\mathscr{I}_v=(-1)^m\left\{\frac{\mathrm{ch}(W)}{\rho^*e}\cdot\rho^*\mathscr{I}\right\} \tag{3.9}$$

where $R(H)\ni W=\sum_{i=0}^N(-1)^iW^i$. Note that $\mathscr{I}_v$ of (3.9) corresponds to $\mathscr{I}_v$ of (3.6) under the Weil homomorphism W_H (see Lemma 2.2).

DEFINITION 3.10. For a fixed universal elliptic symbol class v, $\mathscr{F}_v\colon I^{**}(H)\to I^{**}(G)$ is defined by $\mathscr{F}_v(\phi)=\mathscr{F}(\mathscr{I}_v\cdot\phi)$ for $\phi\in I^{**}(H)$.

Now, we state our first theorem.

THEOREM 3.11. *The following diagram is commutative.*

$$\begin{array}{ccc}R(H)&\xrightarrow{\mathscr{E}_v}&R(G)\\ {\scriptstyle\mathrm{ch}}\big\downarrow&&\big\downarrow{\scriptstyle\mathrm{ch}}\\ I^{**}(H)&\xrightarrow{\mathscr{F}_v}&I^{**}(G)\end{array}$$

PROOF. Consider the diagram;

$$\begin{tikzcd}
 & R(H) \arrow[rr, "\mathscr{E}_v"] \arrow[dl, "\mathscr{T}_v\text{-ch}"'] \arrow[dd, "\mathscr{T}_v\text{-ch}"] & & R(G) \arrow[dl, "\mathrm{ch}"'] \arrow[dd, "\mathrm{ch}"] \\
I^{**}(H) \arrow[rr, "\mathscr{T}"'] \arrow[dr, "W_H"'] & & I^{**}(G) \arrow[dr, "W_G"] & \\
 & H_H^{**} \arrow[rr, "\mathscr{G}"'] & & H_G^{**}
\end{tikzcd}$$

where $\mathscr{T}_v$-ch: $R(H)\to I^{**}(H)$ is defined by the multiplication $z\to \mathrm{ch}(z)\cdot\mathscr{T}_v$ for $z\in R(H)$. Now the theorem follows from Lemma 2.2, Corollary 2.7, Proposition 3.2, Proposition 3.8 and the above diagram. Q.E.D.

§4. Infinitesimal Lefschetz numbers.

Let M be a compact complex manifold of complex dimension m, $H(M)$ the complex Lie group (see [8]) of all holomorphic automorphisms of M and G any compact subgroup of $H(M)$. Let P be the principal $U(m)$-bundle of unitary frames with respect to a G-invariant Hermitian metric on M. Then G acts on P on the left and the holomorphic tangent bundle TM is isomorphic to $P\times_H V$ for $H=U(m)$, $V=\boldsymbol{C}^m$, $\rho=$inclusion: $U(m)\to GL(m;\boldsymbol{C})$. Here ρ is identified with the standard inclusion $\rho\colon U(m)\to SO(2m)$ and Assumption 3.4 is satisfied. Furthermore, the Hermitian connection θ is left G-invariant and

$$\mathscr{T}: I^{m+k}(U(m))\longrightarrow I^k(G)$$

is defined by (1.2).

Now, let $v\in K_{U(m)}(\boldsymbol{C}^m)$ be the universal elliptic symbol class of the Dolbeault complex so that $W^i=\wedge^i\boldsymbol{C}^m$ $(0\leqq i\leqq m)$. Then it follows from the same calculation as in [1, §4] that $\mathscr{T}_v\in I^{**}(U(m))$ is given by the Todd polynomial

$$\mathscr{T}=\prod_{i=1}^{m}\frac{x_i}{1-e^{-x_i}}$$

where x_i's are those of (2.6), namely, $x_1(X)$, $x_2(X),\cdots, x_m(X)$ are eigenvalues of a skew-Hermitian matrix $X\in\mathfrak{u}(m)$.

On the other hand, $\mathscr{E}_v$: $R(U(m))\to R(G)$ is expressed as follows. It is well-known that $R(U(m))$ is isomorphic to the polynomial ring $\boldsymbol{Z}[\wedge^1\boldsymbol{C}^m,\cdots,\wedge^m\boldsymbol{C}^m,\wedge^{-m}\boldsymbol{C}^m]$ and any complex $U(m)$-module A corresponds uniquely to a holomorphic $GL(m;\boldsymbol{C})$-module. Thus a holomorphic vector bundle ξ_A over M is defined by

$$\xi_A = P' \times_{GL(m;C)} A \ (\cong P \times_{U(m)} A)$$

for principal $GL(m; \boldsymbol{C})$-bundle P' of holomorphic frames, and the ξ_A-valued Dolbeault operator (or complex)

$$\mathscr{D}_A : 0 \longrightarrow \Omega^{0,0}(\xi_A) \xrightarrow{\bar\partial_A^1} \Omega^{0,1}(\xi_A) \longrightarrow \cdots \xrightarrow{\bar\partial_A^m} \Omega^{0,m}(\xi_A) \longrightarrow 0$$

is defined where $\Omega^{0,q}(\xi_A)$ is the ξ_A-valued $(0, q)$-forms on M. Since this ξ_A-valued Dolbeault complex is an $H(M)$-equivariant elliptic complex, the j-th cohomology group $H_A^j = \mathrm{Ker}\ \bar\partial_A^{j+1}/\mathrm{Im}\ \bar\partial_A^j$ is a finite dimensional G-module. Then, for any $z = \sum_i n_i A_i \in R(U(m))$, $\mathscr{E}_v(z) \in R(G)$ is given by

$$\mathscr{E}_v(z) = \sum_i n_i \sum_{j=0}^{m} (-1)^j H_{A_i}^j \tag{4.1}$$

which is the equivariant index of the elliptic complex $\sum_i n_i \mathscr{D}_{A_i}$. Hence, by the definition of ch, $\{\mathrm{ch} \circ \mathscr{E}_v(z)\}_{(k)} \in I^k(G)$ is characterized by

$$\{\mathrm{ch} \circ \mathscr{E}_v(z)\}_{(k)}(X, \cdots, X) = \frac{1}{k!}\left[\left(\frac{d}{dt}\right)^k \mathrm{Lf}_z(\exp tX)\right]_{t=0}$$

for any $X \in \mathfrak{G}$ where Lf_z is the Lefschetz number of the elliptic complex $\sum_i n_i \mathscr{D}_{A_i}$ given by

$$\mathrm{Lf}_z(g) = \sum_i n_i \,\mathrm{Lf}_i(g)$$

where

$$\mathrm{Lf}_i(g) = \sum_{j=0}^{m} (-1)^j \,\mathrm{tr}(g|_{H_{A_i}^j}) \qquad \text{for} \quad g \in G\,. \tag{4.2}$$

Now, the next theorem follows from Theorem 3.11 and Lemma 2.8 because $\mathscr{T}_{(0)} = 1 \neq 0$.

THEOREM 4.3. *For any $\phi \in I^{m+k}(U(m))$, there exists $z = \sum_i c_i A_i \in R(U(m)) \otimes \boldsymbol{C}$ such that $\mathscr{T}(\phi) \in I^k(G)$ is characterized by*

$$\mathscr{T}(\phi)(X, \cdots, X) = \frac{1}{k!}\left[\left(\frac{d}{dt}\right)^k \sum_i c_i \,\mathrm{Lf}_i(\exp tX)\right]_{t=0} \tag{4.4}$$

for any $X \in \mathfrak{G}$ where Lf_i *is the Lefschetz number of the $P \times_{U(m)} A_i$-valued Dolbeault complex $\mathscr{D}_{A_i}$ given by* (4.2).

REMARK 4.5. It is well-known that $I^*(U(m))$ is isomorphic to the polynomial ring $\boldsymbol{C}[c_1, \cdots, c_m]$ of the Chern polynomials c_i. When the above $\phi \in I^{m+k}(U(m))$ is contained in $\boldsymbol{Z}[c_1, \cdots, c_m]$, it is proved that the above

$z \in R(U(m)) \otimes C$ can be taken to be an element $z = \sum_i n_i A_i$ of $R(U(m))$, and therefore $\mathscr{F}(\phi)$ is expressed by the k-th derivative of the Lefschetz number of the *single* elliptic complex $\sum_i n_i \mathscr{D}_{A_i}$.

COROLLARY 4.6. *If $P \times_{U(m)} A_i$-valued Dolbeault cohomology groups $H^j_{A_i}$'s in* (4.1) *can be embedded in some de Rham cohomology groups and hence have the homotopy-invariance, then the right-hand term of* (4.4) *and hence $\mathscr{F}(\phi)$ vanish for $k \geqq 1$. In particular, $\mathscr{F}(\mathscr{T}_{(m+k)})$ which corresponds to $z = 1 \in R(U(m))$ vanishes for $k \geqq 1$ if M is a Kaehler manifold.*

§5. Induced invariant polynomials.

Let G be a compact Lie group with Lie algebra $\mathfrak{G}$, H a closed subgroup with Lie algebra $\mathfrak{H} \subset \mathfrak{G}$ and $i: H \to G$ the inclusion map. Here we assume that $\dim G - \dim H = 2m$. Let $\rho: H \to O_R(\mathfrak{G}/\mathfrak{H}) \cong O(2m)$ be the isotropy representation on the tangent space to the homogeneous space G/H at the identity coset which is induced by the adjoint representation. We assume that

(5.1) the image of ρ is contained in $SO(2m)$
(i.e. $\det \rho(h) > 0$ for any $h \in H$).

Then G/H has a G-invariant orientation and $G \times_{(H,\rho)} \boldsymbol{R}^{2m}$ is isomorphic to the oriented tangent bundle $T(G/H)$. Moreover, we assume that $H^{2m}_H \ni \rho^* e$ does not vanish for the Euler class $e \in H^{2m}_{SO(2m)}$. If $\rho^* e$ vanishes or $\dim G - \dim H$ is odd, then $i_!$ in Definition 5.2 should be defined to be the zero-mapping.

Now, put $P = G$, $M = G/H$ and let θ be a left G-invariant connection in the principal H-bundle $P \to M$. Then $\mathscr{F}: I^{m+k}(H) \to I^k(G)$ is defined by (1.2). Here, let $v \in K_H(\boldsymbol{R}^{2m}) = K_H(\mathfrak{G}/\mathfrak{H})$ be the universal elliptic symbol class of the de Rham operator so that $W^0 = \bigoplus_{i:\text{even}} \wedge^i \boldsymbol{R}^{2m} \otimes C$, $W^1 = \bigoplus_{i:\text{odd}} \wedge^i \boldsymbol{R}^{2m} \otimes C$ (see [9, p. 119]). Then $\mathscr{I}_v \in I^{**}(H)$ is equal to $\rho^* e \in I^m(H)$ for the Euler polynomial $e \in I^m(SO(2m))$ and $\mathscr{F}_v: I^{**}(H) \to I^{**}(G)$ is given by $\mathscr{F}_v(\phi) = \mathscr{F}(\rho^* e \cdot \phi)$ for $\phi \in I^{**}(H)$.

DEFINITION 5.2. $i_!: I^k(H) \to I^k(G)$ is defined by $i_!(\phi) = \mathscr{F}_v(\phi) = \mathscr{F}(\rho^* e \cdot \phi)$ for $\phi \in I^k(H)$.

Note that $i_!$ turns out to be independent of the choice of the G-invariant orientations of G/H.

Now, it follows from Proposition 3.2 that $\mathscr{F}: I^{m+k}(H) \to I^k(G)$ corresponds to the Gysin homomorphism $\pi_*: H^{2m+2k}_H \to H^{2k}_G$ under the Weil homomorphisms for $\pi=i\colon BH\to BG$. Hence it can be verified from [3, Theorem 4.3] that $i_!$ in Definition 5.2 corresponds to the transfer map $i_!\colon H_H^{2k}\to H_G^{2k}$ of Becker-Gottlieb [3].

On the other hand, for any $z\in R(H)$, $\mathscr{E}_v(z)\in R(G)$ is equal to the index of $G\times_H z$-valued de Rham operator which coincides with $i_!(z)$ for the induced representation $i_!\colon R(H)\to R(G)$ (see Segal [9]).

Thus the next theorem follows from Lemma 2.2, Proposition 3.2, Proposition 3.8 and Theorem 3.11.

THEOREM 5.3. *The following diagram is commutative.*

$$\begin{tikzcd}
& R(H) \arrow[rr, "i_!"] \arrow[dl, "\mathrm{ch}"'] \arrow[dd, "\mathrm{ch}"] & & R(G) \arrow[dl, "\mathrm{ch}"'] \arrow[ddr, "\mathrm{ch}"] & \\
I^{**}(H) \arrow[rr, "i_!"'] \arrow[dr, "W_H"'] & & I^{**}(G) \arrow[drr, "W_G"] & & \\
& H_H^{**} \arrow[rrr, "i_!"'] & & & H_G^{**}
\end{tikzcd}$$

Now the next corollaries follow from Corollary 2.7, Theorem 5.3 and the properties of the transfer map $i_!\colon H_H^{**}\to H_G^{**}$.

COROLLARY 5.4. *$i_!\colon I^{**}(H)\to I^{**}(G)$ is independent of the choice of the G-invariant connection θ.*

COROLLARY 5.5. *Let K be a closed subgroup of H and $j\colon K\to H$ the inclusion map. Then $i_!\circ j_!=(i\circ j)_!\colon I^{**}(K)\to I^{**}(G)$.*

COROLLARY 5.6. *$i_!\circ i^*(\phi)=\chi(G/H)\phi$ for any $\phi\in I^k(G)$ where $\chi(G/H)$ is the Euler characteristic of G/H. In particular, if* rank $H=$ rank G, *then $i_!\colon I^k(H)\to I^k(G)$ is surjective for any k.*

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