TOKYO J. MATH.
VOL. 13, NO. 2, 1990

Pitman Type Theorem for One-Dimensional Diffusion Processes

Yasumasa SAISHO* and Hideki TANEMURA

Kumamoto University and Chiba University
(Communicated by M. Obata)

Introduction.

For a regular diffusion X in $\boldsymbol{R}$ starting from 0 with generator $\mathscr{L}=d/dm\cdot d/ds$, $s(0)=0$, and for any fixed $r>0$, define

$$\tau_1=\inf\{t>0 : X(t)=r\} ,$$
$$\tau_2=\sup\{t>0 : X(t)=0,\ t<\tau_1\} .$$

Then Williams's theorem ([4]) states that $\{X(\tau_2+t) : 0\leqq t\leqq \tau_1-\tau_2\}$ is identical in distribution to $\{\tilde{X}(t) : 0\leqq t\leqq \tilde{\tau}\}$, where $\tilde{X}$ is a diffusion process starting from 0 with generator $\tilde{\mathscr{L}}f=(1/s)\mathscr{L}(sf)$ and $\tilde{\tau}=\inf\{t>0 : \tilde{X}(t)=r\}$. In the case where X is a one-dimensional Brownian motion B with $B(0)=0$, $\tilde{X}$ is a Bessel process with index 3 (the radial part of a three-dimensional Brownian motion) and Pitman [2] proved that

$$(1)\qquad \{\tilde{X}(t),\ t\geqq 0\}\overset{\mathrm{d}}{=}\{B(t)+2L(t),\ t\geqq 0\} ,\qquad L(t)=-\min_{0\leqq u\leqq t} B(u) ,$$

where "$\overset{\mathrm{d}}{=}$" means the equality in distribution.

In this paper we consider the case where X is the one-dimensional diffusion process defined by the stochastic differential equation (abbreviated : SDE)

$$(2)\qquad X(t)=\int_0^t \sigma(X(u))dB(u)+\int_0^t b(X(u))du ,$$

and will prove that $\tilde{X}$ admits a representation similar to (1) (see Theorem 1.1′). The assumptions for the coefficients σ and b are that they are Lipschitz continuous and $\sigma(x)>0$, $\forall x\in \boldsymbol{R}$. To state our result more pre-

Received January 29, 1990
Revised April 16, 1990

* The first author was partially supported by Grant-in-Aid for Scientific Research (No. 01740131), Ministry of Education, Science and Culture.

cisely, we consider the SDE of Skorohod type with reflecting boundary condition at 0 (see §1 for precise formulation):

$$(3) \qquad Y(t)=\int_0^t \sigma(Y(u)+L(u))dB(u)+\int_0^t b(Y(u)+L(u))du+L(t) .$$

It can be proved that this equation has a unique solution (Y, L) (Proposition 2.1). Put

$$(4) \qquad \begin{cases} Z(t)=Y(t)+L(t) , \\ \tau=\inf\{t>0 : Z(t)=r\} , \qquad r>0 . \end{cases}$$

Then the main result in this paper is that the process Z is a diffusion process with generator $\tilde{\mathscr{L}}f=(1/s)\mathscr{L}(sf)$ and that $\{Z(t) : 0\leqq t\leqq \tau\}\stackrel{d}{=}\{X(\tau_2+t) : 0\leqq t\leqq \tau_1-\tau_2\}$.

In §1 we state our problem and results precisely. In §2 we give some preliminary results on one-dimensional Skorohod equations and equations with singular drift coefficients. The proof of the main result is given in §3. In §4 we state some remarks.

§1. Formulation of the problem and the results.

Let σ, $b : \boldsymbol{R}\to\boldsymbol{R}$ be Lipschitz continuous functions, namely, assume that there exists a positive constant K such that

$$(1.1) \qquad |\sigma(x)-\sigma(y)|+|b(x)-b(y)|\leqq K|x-y| , \qquad \forall x, y\in \boldsymbol{R} ;$$

we also assume $\sigma(x)>0$, $\forall x\in\boldsymbol{R}$. Let X be the unique solution of the SDE:

$$(1.2) \qquad X(t)=\int_0^t \sigma(X(u))dB(u)+\int_0^t b(X(u))du , \qquad t\geqq 0 ,$$

where $B\equiv\{B(t) : t\geqq 0\}$ is a one-dimensional Brownian motion starting from 0 defined on a probability space $(\Omega, \mathscr{F}, P)$. We denote the proper reference family of B by $(\mathscr{F}_t)$, i.e.,

$$\mathscr{F}_t=\bigcap_{\varepsilon>0}\sigma\{B(u) : u\in[0, t+\varepsilon]\} .$$

Let $\mathscr{L}$ be the generator of X, i.e.,

$$\mathscr{L}=\frac{1}{2}\sigma(x)^2\frac{d^2}{dx^2}+b(x)\frac{d}{dx}=\frac{d}{dm}\frac{d}{ds} ,$$

where

$$m(dx)=2\sigma(x)^{-2}e^{\beta(x)}dx\ ,$$
$$s(x)=\int_0^x e^{-\beta(y)}dy\ ,\quad \beta(x)=2\int_0^x b(u)/\sigma^2(u)du\ .$$

For any fixed $r>0$, define

$$\tau_1=\inf\{t>0: X(t)=r\}\ ,$$
$$\tau_2=\sup\{t>0: X(t)=0,\ t<\tau_1\}\ .$$

We consider the SDE of Skorohod type

$$Y(t)=\int_0^t \sigma(Y(u)+L(u))dB(u) + \int_0^t b(Y(u)+L(u))du+L(t)\ ,\qquad t\geqq 0\ , \tag{1.3}$$

where Y and L should be found under the conditions:

(1.4) Y is $(\mathscr{F}_t)$-adapted, continuous and $Y(t)\geqq 0$, $\forall t\geqq 0$,

(1.5) L is continuous, nondecreasing, $L(0)=0$ and

$$L(t)=\int_0^t \mathbf{1}_{\{0\}}(Y(u))dL(u)\ .$$

In the above (and also in the sequel) $\mathbf{1}_{\{0\}}$ denotes the indicator function of the set $\{0\}$. It will be proved that the equation (1.3) can be solved uniquely (Proposition 2.1). Put

$$Z(t)=Y(t)+L(t)\ ,\qquad t\geqq 0\ . \tag{1.6}$$

Then our main theorem is stated as follows.

THEOREM 1.1. *Suppose that σ and b are Lipschitz continuous and $\sigma(x)>0$, $\forall x\in \boldsymbol{R}$. Then, Z is a diffusion process on $\boldsymbol{R}_+\equiv[0,\infty)$ starting from 0 with generator $\tilde{\mathscr{L}}$ and is identical in distribution to the unique nonnegative solution of the SDE*

$$Z(t)=\int_0^t \sigma(Z(u))dB(u)+\int_0^t \tilde{b}(Z(u))du\ ,\qquad t\geqq 0 \tag{1.7}$$

with the singular drift coefficient $\tilde{b}$ defined by

$$\tilde{b}(x)=b(x)+\sigma(x)^2 s'(x)/s(x)\ ,\qquad x>0\ .$$

By virtue of Williams's theorem stated in the introduction, the above theorem may be rephrased as follows.

THEOREM 1.1$'$. *$\{X(\tau_2+t):0\leqq t\leqq \tau_1-\tau_2\}$ is identical in distribution to $\{Z(t):0\leqq t\leqq \tau\}$, where $\tau=\inf\{t>0:Z(t)=r\}$.*

If $\sigma(x)\equiv 1$ and $b(x)\equiv 0$, Theorem 1.1 is nothing but Pitman's theorem stated in the introduction. Furthermore, in case $\sigma(x)\equiv 1$ and $b(x)\equiv b$ (constant), $X(t)$ is equal to $B(t)+bt$ and the solution of the Skorohod equation (1.3) is written as $Y(t)=X(t)+L(t)$, where $L(t)=-\min_{0\leqq u\leqq t} X(u)$ (see §2). Theorem 1.1 then implies that $X(t)-2\min_{0\leqq u\leqq t} X(u)$ is a diffusion process on $\boldsymbol{R}_+$ with generator $\frac{1}{2}d^2/dx^2+b\cdot\coth(bx)\cdot d/dx$ (see Rogers and Pitman [3]).

§2. One-dimensional Skorohod equations and equations with singular drifts.

In this section we state some results on one-dimensional Skorohod equations and equations with singular drifts.

Let $\mathscr{W}$ (resp. $\mathscr{W}_+$) be the space of all $\boldsymbol{R}$ (resp. $\boldsymbol{R}_+$) valued continuous functions defined on $\boldsymbol{R}_+$. For $w\in\mathscr{W}$ and $t\geqq 0$ the notation $\|w\|_t$ stands for $\sup\{|w(u)|:0\leqq u\leqq t\}$. *Given $w\in\mathscr{W}$ with $w(0)\geqq 0$ we call the equation*

$$(2.1)\qquad \eta(t)=w(t)+l(t)\,,\qquad t\geqq 0\,,$$

the Skorohod equation (for w) and the pair (η, l) its solution if the following conditions (2.2) and (2.3) are satisfied:

$$(2.2)\qquad \eta\in\mathscr{W}_+\,,$$

$$(2.3)\qquad l\in\mathscr{W}_+,\ l \text{ is nondecreasing, } l(0)=0 \text{ and}$$
$$l(t)=\int_0^t \mathbf{1}_{\{0\}}(\eta(u))dl(u)\,.$$

It is well known that there exists a unique solution of the Skorohod equation (2.1) given by

$$(2.4)\qquad \begin{cases}\eta(t)=w(t)-\min\limits_{0\leqq u\leqq t}\{w(u)\wedge 0\}\,,\\ l(t)=-\min\limits_{0\leqq u\leqq t}\{w(u)\wedge 0\}\,.\end{cases}$$

The following lemma is immediate from (2.4).

LEMMA 2.1. (i) *Let $w_i\in\mathscr{W}$ with $w_i(0)\geqq 0$ and (η_i, l_i) be the unique solutions of the Skorohod equations*

$$\eta_i(t)=w_i(t)+l_i(t)\,,\quad t\geqq 0\,,\qquad i=1,\,2\,,$$

respectively. Then we have

$$
\begin{cases} \|\eta_1-\eta_2\|_t \leqq 2\|w_1-w_2\|_t \,, \\ \|l_1-l_2\|_t \leqq \|w_1-w_2\|_t \,, \qquad t\geqq 0 \,. \end{cases} \tag{2.5}
$$

(ii) *Let* (η, l) *be the unique solution of* (2.1). *Then, for* $0\leqq t_1<t_2<\infty$ *we have*

$$
\begin{cases} |\eta(t_2)-\eta(t_1)| \leqq 2\max\{|w(u)-w(t_1)| : t_1\leqq u\leqq t_2\} \,, \\ |l(t_2)-l(t_1)| \leqq \max\{|w(u)-w(t_1)| : t_1\leqq u\leqq t_2\} \,. \end{cases} \tag{2.6}
$$

PROPOSITION 2.1. *The SDE* (1.3) *of Skorohod type has a unique solution.*

PROOF. We construct a solution of (1.3) by the iteration method. Let $T>0$ be any finite fixed time and define a sequence of Skorohod equations by

$$
\begin{aligned}
&Y_0(t)\equiv 0 \,, \qquad L_0(t)\equiv 0 \,, \\
&Y_n(t)=\int_0^t \sigma(Y_{n-1}(u)+L_{n-1}(u))dB(u) \\
&\qquad\quad +\int_0^t b(Y_{n-1}(u)+L_{n-1}(u))du+L_n(t) \,, \qquad n\geqq 1 \,.
\end{aligned}
$$

By Lemma 2.1 (i), we have

$$
\begin{aligned}
E\{\|Y_n-Y_{n-1}\|_t^2+\|L_n-L_{n-1}\|_t^2\}
&\leqq 10E\Big[\Big\|\int_0^{\cdot}\{\sigma(Y_{n-1}(u)+L_{n-1}(u))-\sigma(Y_{n-2}(u)+L_{n-2}(u))\}dB(u)\Big\|_t^2\Big] \\
&\quad +10E\Big[\Big\|\int_0^{\cdot}\{b(Y_{n-1}(u)+L_{n-1}(u))-b(Y_{n-2}(u)+L_{n-2}(u))\}du\Big\|_t^2\Big] \\
&\leqq 40K^2E\Big\{\int_0^t\|Y_{n-1}+L_{n-1}-Y_{n-2}-L_{n-2}\|_u^2du\Big\} \\
&\quad +10K^2TE\Big\{\int_0^t\|Y_{n-1}-Y_{n-2}+L_{n-1}-L_{n-2}\|_u^2du\Big\} \\
&\leqq (80+20T)K^2\int_0^t E\{\|Y_{n-1}-Y_{n-2}\|_u^2+\|L_{n-1}-L_{n-2}\|_u^2\}du \,.
\end{aligned} \tag{2.7}
$$

On the other hand, by Lemma 2.1 (ii), it is easy to see that

$$
\begin{aligned}
&E\{\|L_1\|_t^2\}\leqq 8\sigma(0)^2t+2b(0)^2t^2 \,, \\
&E\{\|Y_1\|_t^2\}\leqq 32\sigma(0)^2t+8b(0)^2t^2 \,.
\end{aligned}
$$

Therefore, by a routine argument we see that $Y_n(t)$ and $L_n(t)$ converge uniformly in $t\in[0, T]$ as n tends to ∞. If we denote the limits by Y and L respectively, and if we put

$$W_n(t)=\int_0^t\sigma(Y_{n-1}(u)+L_{n-1}(u))dB(u)+\int_0^t b(Y_{n-1}(u)+L_{n-1}(u))du ,$$
$$W(t)=\int_0^t\sigma(Y(u)+L(u))dB(u)+\int_0^t b(Y(u)+L(u))du ,$$

then $W_n\to W$ (uniformly in $t\in[0, T]$) and

$$\begin{cases} Y_n(t)=W_n(t)-\min\limits_{0\leq u\leq t} W_n(u) , \\ L_n(t)=-\min\limits_{0\leq u\leq t} W_n(u) \end{cases} \tag{2.8}$$

implies that (2.8) holds without suffix n. Thus the pair (Y, L) is a solution of the SDE (1.3) of Skorohod type. The proof of the uniqueness of the solution is also routine and so it is omitted. □

REMARK 2.1. In the case where $\sigma\equiv1$, we can also prove that the equation (1.3) has a unique solution for every continuous function $B(t)$ with $B(0)=0$.

Next, we consider an equation with a singular drift

$$\xi(t)=w(t)+\int_0^t\alpha(\xi(u))du , \qquad t\geqq0 , \tag{2.9}$$

where $w\in\mathscr{W}$ with $w(0)\geqq0$ is given and α is a continuous function defined on $(0, \infty)$ satisfying the following conditions:

(C.1) $\lim\limits_{x\downarrow0}\alpha(x)=\infty$,

(C.2) $\alpha(x)$ is nonincreasing and $\alpha(x)\geqq0$, $\forall x>0$,

(C.3) for each $\varepsilon>0$, there exists a positive constant K_ε such that

$$|\alpha(x)-\alpha(y)|\leqq K_\varepsilon|x-y| \qquad \text{for} \quad x, y\in[\varepsilon, \infty) .$$

LEMMA 2.2 ([1]). *For any given $w\in\mathscr{W}$ with $w(0)\geqq0$, the equation (2.9) has a unique nonnegative solution. Furthermore, if ξ_i is the nonnegative solution of*

$$\xi_i(t)=w_i(t)+\int_0^t\alpha(\xi_i(u))du , \quad t\geqq0 , \quad i=1, 2 ,$$

respectively, we have

$$\|\xi_1-\xi_2\|_t\leqq2\|w_1-w_2\|_t . \tag{2.10}$$

REMARK 2.2. McKean [1] treated (2.9) when $\alpha(x)=1/x$. His method can be applied to the above general case without any change.

PROPOSITION 2.2. *There exists a unique nonnegative solution of the SDE*

$$Z(t)=\int_0^t \sigma(Z(u))dB(u)+\int_0^t (\alpha+b)(Z(u))du\,, \qquad t\geqq 0\,. \tag{2.11}$$

PROOF. We construct a solution of (2.11) by the iteration method. Let $T>0$ be any finite fixed time and define a sequence of equations by

$$\begin{cases} Z_n(t)=W_n(t)+\int_0^t \alpha(Z_n(u))du\,, \\ Z_n(t)>0\,, \quad t\in[0,\,T]\,, \quad n=0,1,\cdots\,, \end{cases}$$

where $W_0(t)\equiv 0$ and

$$W_n(t)=\int_0^t \sigma(Z_{n-1}(u))dB(u)+\int_0^t b(Z_{n-1}(u))du\,, \quad n\geqq 1\,, \quad t\in[0,\,T]\,.$$

By Lemma 2.2, it is easy to see that

$$\begin{aligned} E\{\|Z_n-Z_{n-1}\|_t^2\} &\leqq 8E\left[\left\|\int_0^\cdot \{\sigma(Z_{n-1}(u))-\sigma(Z_{n-2}(u))\}dB(u)\right\|_t^2\right] \\ &\quad +8E\left[\left\|\int_0^\cdot \{b(Z_{n-1}(u))-b(Z_{n-2}(u))\}du\right\|_t^2\right] \\ &\leqq 8K^2(4+T)E\left\{\int_0^t \|Z_{n-1}-Z_{n-2}\|_u^2 du\right\}\,, \qquad t\in[0,\,T]\,. \end{aligned}$$

On the other hand, by using Lemma 2.2 again, we have

$$E\{\|Z_1-Z_0\|_t^2\}\leqq\{32\|\sigma(Z_0)\|_T^2+8T\|b(Z_0)\|_T^2\}t\,, \qquad t\in[0,\,T]\,.$$

Therefore, by a routine argument we see that $Z_n(t)$ converges uniformly in $t\in[0,\,T]$ as n tends to ∞. If we denote the limit by Z, it satisfies the equation

$$\begin{cases} Z(t)=\int_0^t \sigma(Z(u))dB(u)+\int_0^t (\alpha+b)(Z(u))du\,, \\ Z(t)\geqq 0\,, \qquad t\in[0,\,T]\,. \end{cases}$$

The proof of the uniqueness of the solution is routine and so it is omitted. □

§ 3. Proof of Theorem 1.1.

Let $(Y(t),\,L(t))$ be the solution of (1.3) and put

$$\hat{W}(t)=\int_0^t \sigma s'(Y(u)+L(u))dB(u) ,$$
$$\hat{Y}(t)=s(Y(t)+L(t))-s(L(t)) ,$$
$$\hat{L}(t)=s(L(t)) .$$

Before proving Theorem 1.1, we prepare two lemmas.

LEMMA 3.1. *The equality*

$$\hat{Y}(t)=\hat{W}(t)+\hat{L}(t) , \qquad t\geqq 0 , \tag{3.1}$$

holds and is a Skorohod equation.

PROOF. Since $\mathscr{L}s=0$, Itô's formula implies

$$s(Y(t)+L(t))=\int_0^t \sigma s'(Y(u)+L(u))dB(u)+2\int_0^t s'(Y(u)+L(u))dL(u) . \tag{3.2}$$

Noting that $L(t)=\int_0^t \mathbf{1}_{\{0\}}(Y(u))dL(u)$, we have

$$\begin{aligned}\int_0^t s'(Y(u)+L(u))dL(u)&=\int_0^t s'(L(u))dL(u)\\ &=s(L(t))=\hat{L}(t) .\end{aligned} \tag{3.3}$$

Thus, from (3.2) and (3.3) we obtain (3.1). It is obvious that $\hat{Y}$ is nonnegative and continuous, and $\hat{L}$ is nondecreasing and continuous. Since $s(x)$ is strictly increasing, $Y(t)=0$ if and only if $\hat{Y}(t)=0$ and so we have

$$\begin{aligned}\hat{L}(t)&=\int_0^t s'(L(u))dL(u)\\ &=\int_0^t \mathbf{1}_{\{0\}}(Y(u))s'(L(u))dL(u)\\ &=\int_0^t \mathbf{1}_{\{0\}}(\hat{Y}(u))s'(L(u))dL(u)\\ &=\int_0^t \mathbf{1}_{\{0\}}(\hat{Y}(u))d\hat{L}(u) .\end{aligned}$$

Thus, (3.1) is a Skorohod equation. This completes the proof of the lemma. □

Next, let ψ be the increasing process defined by

$$\psi(t)=\int_0^t |\sigma s'(Y(u)+L(u))|^2 du$$

and $\phi(t)$ its inverse function $\psi^{-1}(t)$. Put $\bar{W}(t)=\hat{W}(\phi(t))$, $\bar{L}(t)=\hat{L}(\phi(t))$, $\bar{Y}(t)=\hat{Y}(\phi(t))$ and $\bar{Z}(t)=\bar{Y}(t)+\bar{L}(t)$. It is easy to see that $\phi(t)$ is an $(\mathscr{F}_t)$-stopping time and $\bar{W}(t)$ is a one-dimensional $(\mathscr{F}_{\phi(t)})$-Brownian motion and

$$\phi(t)=\int_0^t|\sigma s'(s^{-1}(\bar{Z}(u)))|^{-2}du .$$

LEMMA 3.2. *$(\bar{Y}, \bar{L})$ is the unique solution of the Skorohod equation:*

$$\bar{Y}(t)=\bar{W}(t)+\bar{L}(t) , \qquad t\geqq 0 . \tag{3.4}$$

PROOF. Lemma 3.1 implies

$$\begin{cases} \hat{Y}(t)=\hat{W}(t)-\min\limits_{0\leqq u\leqq t}\hat{W}(u) , \\ \hat{L}(t)=-\min\limits_{0\leqq u\leqq t}\hat{W}(u) . \end{cases}$$

Thus, since ϕ is strictly increasing, we have

$$\begin{cases} \bar{Y}(t)=\bar{W}(t)-\min\limits_{0\leqq u\leqq t}\bar{W}(u) , \\ \bar{L}(t)=-\min\limits_{0\leqq u\leqq t}\bar{W}(u) . \end{cases}$$

This completes the proof of Lemma 3.2. □

PROOF OF THEOREM 1.1. Since $\bar{W}$ is a Brownian motion and $(\bar{Y}, \bar{L})$ is the unique solution of the Skorohod equation (3.4), by Pitman's theorem $\bar{Z}$ is a Bessel process with index 3 and is the unique nonnegative solution of the SDE

$$\bar{Z}(t)=\bar{B}(t)+\int_0^t\frac{1}{\bar{Z}(u)}du , \qquad t\geqq 0 , \tag{3.5}$$

where $\bar{B}\equiv\{\bar{B}(t): t\geqq 0\}$ is a one-dimensional Brownian motion starting from 0. We denote the proper reference family of $\bar{B}$ by $(\bar{\mathscr{F}}_t)$. Since $\psi(t)$ is the inverse function of $\phi(t)=\int_0^t|\sigma s'(s^{-1}(\bar{Z}(u)))|^{-2}du$, $\psi(t)$ is an $(\bar{\mathscr{F}}_t)$-stopping time. Noting that $Z(t)=s^{-1}(\bar{Z}(\psi(t)))$, $\{(s^{-1}(x)\}'=1/s'(s^{-1}(x))$ and $\{s^{-1}(x)\}''=-s''(s^{-1}(x))/s'(s^{-1}(x))^3$, by Itô's formula we have

$$\begin{aligned} Z(t)&=\int_0^{\psi(t)}\frac{1}{s'(s^{-1}(\bar{Z}(u)))}d\bar{B}(u) \\ &\quad+\int_0^{\psi(t)}\left\{\frac{1}{s'(s^{-1}(\bar{Z}(u)))\bar{Z}(u)}-\frac{s''(s^{-1}(\bar{Z}(u)))}{2s'(s^{-1}(\bar{Z}(u)))^3}\right\}du \\ &=\int_0^t\sigma(Z(u))d\hat{B}(u)+\int_0^t\tilde{b}(Z(u))du , \end{aligned} \tag{3.6}$$

where $\hat{B}$ is the $(\bar{\mathscr{F}}_{\psi(t)})$-Brownian motion defined by

$$\hat{B}(t)=\int_0^{\psi(t)}\frac{1}{\sigma s'(s^{-1}(\bar{Z}(u)))}d\bar{B}(u) .$$

It is easy to see that we can take $\delta>0$ so that $\alpha(x)=\sigma^2 s'(x\wedge\delta)/s(x\wedge\delta)$ satisfies the conditions (C.1)$\sim$(C.3) in §2. Then $\hat{b}_r(x)=\hat{b}(x\wedge r)$ is a Lipschitz continuous function for each r $(>\delta)$, where $\hat{b}(x)=\tilde{b}(x)-\alpha(x)$. If we put

$$\tau^{(r)}=\inf\{t>0 : Z(t)=r\} ,$$

from (3.6) Z satisfies the SDE

$$Z(t)=\int_0^t\sigma(Z(u))d\hat{B}(u)+\int_0^t(\alpha+\hat{b}_r)(Z(u))du , \tag{3.7}$$

for $0\leqq t\leqq\tau^{(r)}$. Since (3.7) has a unique nonnegative solution for each r $(>\delta)$ by Proposition 2.2 and $\tau^{(r)}\to\infty$ a.s. as $r\to\infty$, we see that Z is the unique nonnegative solution of the SDE

$$Z(t)=\int_0^t\sigma(Z(u))d\hat{B}(u)+\int_0^t\tilde{b}(Z(u))du . \tag{3.8}$$

It is then an easy matter to check that Z is a diffusion process starting from 0 with generator $\tilde{\mathscr{L}}$. The proof of Theorem 1.1 is finished. □

§4. Remarks.

In our original proof of Theorem 1.1 we employed a limit procedure by establishing first a discrete time version of Pitman's theorem for birth and death chains. The possibility of the present proof given in §3 was suggested by Prof. S. Watanabe, and also by Prof. H. Tanaka including some details. Although our first proof is longer than the present one, it will be worth while giving a key part of the first one, namely, a theorem of Pitman type for birth and death chains.

For each $x\in\boldsymbol{Z}$, let $p(x)$, $q(x)$ be given positive numbers with $p(x)+q(x)=1$ and put $r(x)=q(x)/p(x)$,

$$a(x, u)=\begin{cases}1 , & u\in[0, p(x)] ,\\ -1 , & u\in(p(x), 1] ,\end{cases}$$

for $u\in[0, 1]$. Next, let $U(i)$, $i=1, 2, \cdots$ be a sequence of i.i.d. random variables with the uniform distribution on [0, 1]. Then

$$x(k)=\sum_{i=1}^{k} a(x(i-1),U(i)) \;, \qquad k\in \boldsymbol{Z}_+\equiv\{0,1,2,\cdots\} \;, \tag{4.1}$$

is a birth and death chain with transition function

$$p(x,y)=\begin{cases} p(x) \;, & y=x+1 \;, \\ q(x) \;, & y=x-1 \;, \\ 0 \;, & \text{otherwise} \;. \end{cases}$$

Consider the equation

$$y(k)=\sum_{i=1}^{k} a(y(i-1)+l(i-1),U(i))+l(k) \;, \qquad k\in \boldsymbol{Z}_+ \;, \tag{4.2}$$

under conditions

$$y(k)\geqq 0 \;, \qquad k\in \boldsymbol{Z}_+ \;, \tag{4.3}$$

$$\begin{aligned} &l \text{ is nonincreasing, } l(0)=0 \text{ and} \\ &\quad l(k+1)>l(k) \text{ only when } y(k+1)=0 \;. \end{aligned} \tag{4.4}$$

The equation (4.2) together with (4.3) and (4.4) is considered as a discrete time version of Skorohod's equation. The existence and uniqueness of the solution of (4.2) is clear, indeed, if we define (y, l) by $(y(0), l(0))=(0, 0)$ and

$$\begin{cases} y(k+1)=\{y(k)+a(y(k)+l(k),U(k+1))\}\vee 0 \;, \\ l(k+1)=l(k)+\mathbf{1}_{\{0\}}(y(k))\mathbf{1}_{\{-1\}}(a(y(k)+l(k),U(k+1))) \;, \end{cases} \tag{4.5}$$

$k\in \boldsymbol{Z}_+$, inductively, (y, l) is the unique solution of (3.2). Put $z(k)=y(k)+l(k)$, $k\in \boldsymbol{Z}_+$. Then we have the following theorem.

THEOREM 4.1. *$\{z(k), k\in \boldsymbol{Z}_+\}$ is a Markov chain on $\boldsymbol{Z}_+$ with transition function*

$$p^H(x,y)=h(x)^{-1}p(x,y)h(y) \;, \qquad x,y\in \boldsymbol{Z}_+ \;,$$

where

$$h(x)=\begin{cases} 1 \;, & x=0 \;, \\ 1+\sum_{j=1}^{x}\prod_{i=0}^{j-1} r(i) \;, & x=1,2,3,\cdots \;. \end{cases}$$

SKETCH OF PROOF. By (4.5) we see that $\{(z(k), l(k)), k\in \boldsymbol{Z}_+\}$ is a Markov chain on $\{(x_1, x_2)\in \boldsymbol{Z}_+\times \boldsymbol{Z}_+ : x_1\geqq x_2\}$ with transition function

$$\hat{p}((x_1, x_2), (y_1, y_2)) = \begin{cases} p(x_1, y_1) & \text{if } y_2 = x_2 , \\ p(x_1, y_1)r(x_1) & \text{if } x_1 = x_2 \text{ and} \\ & \quad (y_1, y_2) = (x_1+1, x_2+1) , \\ 0 & \text{otherwise} . \end{cases} \tag{4.6}$$

Thus, we can calculate the probability of events $\{z(k)=a_k,\ 0\leqq k\leqq m\}$, $m \in \boldsymbol{Z}_+$, $a_k \in \boldsymbol{Z}_+$, $k=0, 1, \cdots, m$, and consequently obtain

$$P(z(k)=a_k,\ 0\leqq k\leqq m)=\prod_{k=1}^{m} p^H(a_{k-1}, a_k) . \qquad \square$$

ACKNOWLEDGEMENT. The authors would like to express their thanks to Professor H. Tanaka for helping them with valuable suggestion and constant encouragement.

References

[1] H. P. McKEAN, The Bessel motion and a singular integral equation, Mem. Coll. Sci. Univ. Kyoto Ser. A, **33** (1960), 317–322.

[2] J. W. PITMAN, One-dimensional Brownian motion and the three-dimensional Bessel process, Adv. in Appl. Probab., **7** (1975), 511–526.

[3] L. C. G. ROGERS and J. W. PITMAN, Markov functions, Ann. Probab., **9** (1981), 573–582.

[4] D. WILLIAMS, Path decomposition and continuity of local time for one-dimensional diffusions, I, Proc. London Math. Soc. (3), **28** (1974), 738–768.

Present Address:

YASUMASA SAISHO
DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING, KUMAMOTO UNIVERSITY
KUMAMOTO 860, JAPAN

HIDEKI TANEMURA
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHIBA UNIVERSITY
CHIBA 260, JAPAN