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Some New Surfaces of General Type

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Introduction.

In this paper we shall give two series of rare examples of algebraic surfaces of *general type*. One is a series of surfaces with *positive topological indices* and another with *the geometric genus* $p_g=0$.

In Part I, we construct surfaces with positive topological indices. In [V] (1966), A. Van de Ven pointed out that there are not many known examples of algebraic surfaces of general type with positive topological indices. By the index theorem of Hirzebruch, the topological index $\tau(S)$ of a surface S is equal to $(K_S^2-2e(S))/3$ where K_S is the canonical line bundle and $e(S)$ the topological Euler number. Hence the positivity of $\tau(S)$ is equivalent to $K_S^2>2e(S)$. At that time, the only known examples were the ones due to F. Hirzebruch [H], which are the compact quotients of the 2-dimensional unit ball and which therefore satisfy $K_S^2=3e(S)$. Following Van de Ven's remark, K. Kodaira [K] (1967) was the first to construct a series of examples as branched coverings of the product of two algebraic curves. Kodaira's examples satisfy $3e(s)>K_S^2>2e(S)$ and have many interesting properties. Afterwards some new examples were discovered, for instance, by Mostow-Siu (1979) and Y. Miyaoka (1980). But even now the known examples are rare. In Part I, we construct new examples of such surfaces as branched coverings of the so-called *elliptic modular surfaces* which are investigated in detail in T. Shioda [S] (1972). Our construction depends heavily on some results in [S], which we shall recall in §1. We remark that our examples are discovered independently also by R. Livné.

In Part II, we construct surfaces with $p_g=0$. If S is a minimal surface of general type with $p_g(S)=0$, then we know generally that $q(S)=0$, $K_S^2=1, 2, \cdots, 9$. Till 1974, the known and *verified* example was only the classical one due to L. Godeaux, on which we refer to Y. Miyaoka [M] (1976). Afterwards some classically known examples were verified and furthermore some new examples were discovered by, for instance, R. Barlow, A. Beauville, Y. Miyaoka, D. Mumford, C. A. M. Peters, M. Reid and I. Shavel and

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so on. But even now the known examples are rare. In Part II, we construct new examples of such surfaces as the quotients of the hypersurfaces of the product of three elliptic curves. Our construction is very elementary and has a close relation with a classical example due to P. Burniat, a geometer of the classical Italian school (see [I_2]).

The main part of this work was done during the author's stay at Bonn University from September 1979 to August 1980. The author would like to express his sincere thanks to Professor F. Hirzebruch, who provided him the opportunity to stay in Bonn, as well as warm encouragement and many invaluable suggestions.

PART I Surfaces of General Type with Positive Topological Indices

§1. Elliptic modular surfaces (due to T. Shioda).

In this section, we shall recall some results in T. Shioda [S]. From now on, we assume that $N \geqq 3$. Let $\Gamma(N)$ be the principal congruence subgroup of level N, namely,

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z}) \;\middle|\; a, d \equiv 1,\ b, c \equiv 0 \text{ modulo } N \right\}.$$

Let $\mu(N)=\frac{1}{2}N^3\prod_{p|N, p:\text{prime}}(1-p^{-2})$. Then $\boldsymbol{H}/\Gamma(N)$ has $t=\mu(N)/N$ cusps. Let

$$\Delta(N)=(\boldsymbol{H}/\Gamma(N)) \cup \{t \text{ cusps}\}.$$

Then

$$g(\Delta(N))=1+\frac{(N-6)\mu(N)}{12N},$$

$$2g(\Delta(N))-2=\frac{(N-6)\mu(N)}{6N}.$$

Let $\Phi: B(N) \to \Delta(N)$ be the elliptic modular surface attached to $\Gamma(N)$, which is called *the elliptic modular surface of level N*. Then

$$K_{B(N)}=\Phi^*(\mathfrak{k}-\mathfrak{f})$$

where

$$\mathfrak{k}=\text{the canonical line bundle of } \Delta(N),$$

$$\mathfrak{f}=\text{a line bundle on } \Delta(N) \text{ with } \deg\mathfrak{f}=-(p_g(B(N))-q(B(N))+1).$$

We know the following:

$$q(B(N))=g(\Delta(N))=1+\frac{(N-6)\mu(N)}{12N},$$

$$K^2_{B(N)}=0,$$

$$e(B(N))=\text{the Euler number of } B(N)=Nt=\mu(N)\,,$$

$$p_g(B(N))-q(B(N))+1=\frac{K^2_{B(N)}+e(B(N))}{12}=\frac{\mu(N)}{12}\,,$$

$$p_g(B(N))=\frac{(N-3)\mu(N)}{6N}\,,$$

$$\deg\mathfrak{t}=-\frac{\mu(N)}{12}\,,\qquad \deg\mathfrak{f}=2g-2=\frac{(N-6)\mu(N)}{6N}\,,$$

$$\deg(\mathfrak{t}-\mathfrak{f})=\frac{N-4}{4N}\,\mu(N)\,.$$

On the fibres of Φ, we know

$$\Phi^{-1}(v)=\begin{cases}\text{a non-singular elliptic curve} & \text{if } \ v\neq\text{cusp}\,,\\ \sum_{i=0}^{N-1}\Theta_{v,i} & \text{if } \ v=\text{cusp}\end{cases}$$

where $\Theta_{v,i}$ is a non-singular rational curve with $\Theta^2_{v,i}=-2$ and with the configuration as in Fig. 1. $B(N)$ has exactly N^2 sections

$$\Gamma(i,j)\,,\qquad i,j=0,\cdots,N-1$$

where $\Gamma(0,0)=$ the zero-section and as in Fig. 2

$$\Gamma(i,j)\cdot\Theta_{\infty,k}=\begin{cases}1 & \text{if } \ i=k\\ 0 & \text{if } \ i\neq k\,.\end{cases}$$

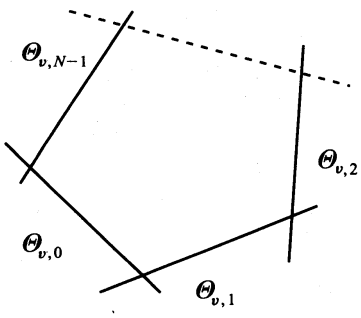


FIGURE 1

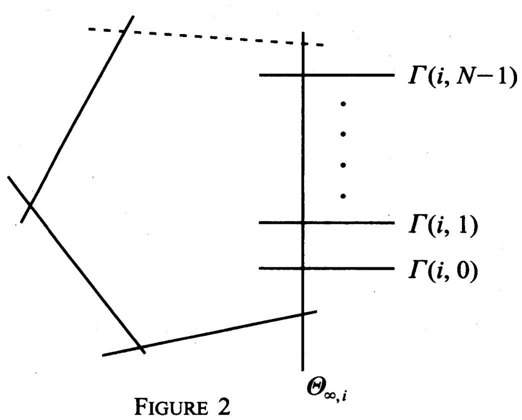


FIGURE 2

$\Gamma(i,j)$'s are mutually disjoint and

$$\{\Gamma(i,j)\cap F\}=\{N\text{-division points on } F\}$$

where F is a general fibre of $\Phi\colon B(N)\to\Delta(N)$. We know that

$$K_{B(N)}\cdot\Gamma(i,j)=\deg(\mathfrak{k}-\mathfrak{f})=\frac{N-4}{4N}\mu(N)\,,$$

$$g(\Gamma(i,j))=g(\Delta(N))=1+\frac{(N-6)\cdot\mu(N)}{12N}\,,$$

$$\Gamma(i,j)^2=\deg\mathfrak{f}=-\frac{\mu(N)}{12}\,.$$

Let

$$\Gamma=\sum_{i,j}\Gamma(i,j)\,.$$

Then Γ is a non-singular (reducible) curve on $B(N)$,

$$\Gamma\cap F=\{N\text{-division points on } F\}\sim N^2[0_F]$$

where $\sim$ is the linear equivalence relation and

$$\Gamma\cap\Theta_{v,i}=\{N\text{-division points on } \boldsymbol{C}^*=\boldsymbol{P}^1-\{0,1\}\}$$

where $\Theta_{v,i}=\boldsymbol{P}^1$, $\Theta_{v,i}\cap\Theta_{v,i-1}=0$, $\Theta_{v,i}\cap\Theta_{v,i+1}=\infty$.

LEMMA OF T. SHIODA. *Let F be a general fibre of Φ: $B(N)\to\Delta(N)$ and let D be a divisor on $B(N)$ such that $D|F\sim 0$. Then*

$$D\approx(D\cdot\Gamma(0,0))\cdot F+\sum_{v:\mathrm{cusp}}(\Theta_{v,1},\cdots,\Theta_{v,N-1})\cdot A_N^{-1}\begin{pmatrix}D\cdot\Theta_{v,1}\\ \vdots\\ D\cdot\Theta_{v,N-1}\end{pmatrix}$$

where $\approx$ is the algebraic equivalence relation and

$$A_N=[\Theta_{v,i}\cdot\Theta_{v,j}]_{1\le i,j\le N-1}=\begin{pmatrix}-2&1&0&\cdots&&&0\\ 1&-2&1&0&\cdots&&0\\ 0&1&-2&1&0&\cdots&0\\ \vdots&&&\ddots&&&\vdots\\ \vdots&&&&\ddots&&\vdots\\ \vdots&&&&&\ddots&0\\ 0&\cdots&&0&1&-2&1\\ 0&\cdots&&\cdots&0&1&-2\end{pmatrix}\Bigg\updownarrow N-1$$

and the components of $A_N^{-1}\begin{pmatrix}D\cdot\Theta_{v,1}\\ \vdots\\ D\cdot\Theta_{v,N-1}\end{pmatrix}$ *are integers.*

PROOF. Take h general fibres $F_1, F_2, \cdots, F_h$ where $F_i\neq F_j$ $(i\neq j)$. Then

$$\to H^1\left(B(N), \mathcal{O}\left(D-\sum_{i=1}^{h} F_i\right)\right) \to H^1(B(N), \mathcal{O}(D))$$

$$\to H^1(F_1, \mathcal{O}) \oplus H^1(F_2, \mathcal{O}) \oplus \cdots \oplus H^1(F_h, \mathcal{O}) \to H^2\left(B(N), \mathcal{O}\left(D-\sum_{i=1}^{h} F_i\right)\right) \to$$

where $H^1(F_i, \mathcal{O}) \cong \boldsymbol{C}$. Hence, if h is sufficiently large,

$$H^2\left(B(N), \mathcal{O}\left(D-\sum_{i=1}^{h} F_i\right)\right) \neq 0 .$$

Since $H^2(B(N), \mathcal{O}(D-\sum_{i=1}^{h} F_i)) \cong H^0(B(N), \mathcal{O}(K+\sum_{i=1}^{h} F_i - D))$, there exists an effective divisor $D' \in |K+\sum_{i=1}^{h} F_i - D|$. Namely

$$K+\sum_{i=1}^{h} F_i - D \sim D' .$$

For any fibre F of $\Phi: B(N) \to \Delta(N)$,

$$D' \cdot F = K \cdot F + \sum_{i=1}^{h} F_i \cdot F - D \cdot F = 0 .$$

Hence $D' = \sum_\alpha m_\alpha D_\alpha$ where D_α's are irreducible curves contained in the fibres of Φ. Since

$$K \approx \frac{N-4}{4N} \mu(N) \cdot F , \quad F_i \approx F , \quad \sum_{i=0}^{N-1} \Theta_{v,i} \approx F ,$$

we obtain

$$D \approx pF + \sum_{i=1, v:\mathrm{cusp}}^{N-1} q_{v,i} \Theta_{v,i}$$

for some $p, q_{v,i} \in \boldsymbol{Z}$. Since $\Gamma(0, 0) \cdot \Theta_{v,i} = 1$ if $i=0$ and $\Gamma(0, 0) \cdot \Theta_{v,i} = 0$ if $i=1, \cdots, N-1$, we obtain

$$D \cdot \Gamma(0, 0) = pF \cdot \Gamma(0, 0) + \sum_{i=1}^{N-1} q_{v,i} \Theta_{v,i} \cdot \Gamma(0, 0) = p .$$

Since $F \cdot \Theta_{v,j} = 0$,

$$D \cdot \Theta_{v,j} = \sum_{i=1}^{N-1} q_{v,i} \Theta_{v,i} \cdot \Theta_{v,j}$$

for $j=1, 2, \cdots, N-1$. Let $A_N = [\Theta_{v,i} \cdot \Theta_{v,j}]_{1 \le i,j \le N-1}$. Then A_N is non-singular and

$$\begin{pmatrix} q_{v,1} \\ \cdot \\ \cdot \\ q_{v,N-1} \end{pmatrix} = A_N^{-1} \begin{pmatrix} D \cdot \Theta_{v,1} \\ \cdot \\ \cdot \\ D \cdot \Theta_{v,N-1} \end{pmatrix} .$$

Thus

$$D \approx (D\cdot\Gamma(0,0))F+\sum_{v:\text{cusp}}(\Theta_{v,1},\cdots,\Theta_{v,N-1})A_N^{-1}\begin{pmatrix} D\cdot\Theta_{v,1}\\ \vdots \\ D\cdot\Theta_{v,N-1}\end{pmatrix}. \qquad \text{Q.E.D.}$$

LEMMA 1-1. *Let* $A_N^{-1}=[x_{jk}]$. *Then*

(i) $$x_{jk}=\begin{cases}\dfrac{-j(N-k)}{N}, & j\leqq k\\[2mm] \dfrac{-k(N-j)}{N}, & j>k.\end{cases}$$

(ii) $$\sum_{k=1}^{N-1}x_{jk}=\frac{-j(N-j)}{2}=\begin{cases}-mj+\dfrac{j(j+1)}{2}-\dfrac{j}{2} & \textit{if}\ \ N=2m,\\[2mm] -mj+\dfrac{j(j-1)}{2} & \textit{if}\ \ N=2m+1.\end{cases}$$

LEMMA 1-2. $\mu(N)/12$ *is divisible by* N *if* $N\geqq 5$.

PROOF. Let $\sigma(N)=N^2\prod_{p|N,p:\text{prime}}(1-p^{-2})$. It is sufficient to prove that $24\mid\sigma(N)$.

(i) Assume $p\geqq 5$ and p is prime. Then (a) $p=3h+1$, $h=2m$, $m\geqq 1$, or (b) $p=3h+2$, $h=2m+1$, $m\geqq 0$.

(a) $\sigma(p)=p^2(1-p^{-2})=(p+1)(p-1)=12(3m+1)m$. Since $3m+1$ or m is even, $24\mid\sigma(p)$.

(b) $\sigma(p)=(p+1)(p-1)=12(m+1)(3m+2)$. Since $m+1$ or $3m+2$ is even, $24\mid\sigma(p)$.

(ii) Assume $N=p_1^{h_1}\cdots p_r^{h_r}$, $p_i\neq p_j$ $(i\neq j)$, p_i : prime, $h_i\geqq 1$ where some $p_i\neq 2, 3$. Then $p_i\geqq 5$ and, hence by (i), $24\mid\sigma(p_i)$. Since

$$\sigma(N)=\sigma(p_1^{h_1})\cdots\sigma(p_r^{h_r}),$$
$$\sigma(p_i^{h_i})=p_i^{2h_i}(1-p_i^{-2})=p_i^{2h_i-2}\cdot\sigma(p_i),$$

we get $24\mid\sigma(N)$.

(iii) Finally we assume $N=2^{h_1}\cdot 3^{h_2}\geqq 5$.

(a) In case $h_1, h_2\geqq 1$,

$$\begin{aligned}\sigma(N)&=\sigma(2^{h_1})\cdot\sigma(3^{h_2})=2^{2h_1}(1-2^{-2})3^{2h_2}(1-3^{-2})\\ &=2^{2h_1-2}\cdot 3^{2h_2-2}(2^2-1)(3^2-1)=24\cdot 2^{2h_1-2}\cdot 3^{2h_2-2}.\end{aligned}$$

Hence $24\mid\sigma(N)$.

(b) In case $h_1=0$ and $h_2\geqq 2$,

$$\sigma(N)=\sigma(3^{h_2})=3^{2h_2-2}(3^2-1)=3^{2h_2-2}\cdot 8=24\cdot 3^{2h_2-3},$$

where $2h_2-3>0$. Hence $24\mid\sigma(N)$.

(c) In case $h_1 \geqq 3$ and $h_2 = 0$,

$$\sigma(N) = \sigma(2^{h_1}) = 2^{2h_1-2}(2^2-1) = 2^{2h_1-2}\cdot 3 = 24\cdot 2^{2h_1-5}$$

where $2h_1 - 5 > 0$. Hence $24 \mid \sigma(N)$. Q.E.D.

REMARK 1. This is not true if $N \leqq 4$.

$$N=4: \quad \mu(4)=24, \quad \mu(4)/12=2,$$

$$N=3: \quad \mu(3)=12, \quad \mu(3)/12=1.$$

LEMMA 1-3.

$$\Gamma \approx N^2\cdot\Gamma(0,0) - (1-N^2)\frac{\mu(N)}{12}F - \sum_{v:\mathrm{cusp},\, j=1,\cdots,N-1} \frac{j(N-j)}{2} N\cdot\Theta_{v,j}.$$

PROOF. Let $D = \Gamma - N^2\cdot\Gamma(0,0)$. Then $D \mid F = 0$. By the Lemma of Shioda

$$D \approx (D\cdot\Gamma(0,0))F + \sum_{v:\mathrm{cusp}} (\Theta_{v,1}, \cdots, \Theta_{v,N-1}) A_N^{-1} \begin{pmatrix} D\cdot\Theta_{v,1} \\ \vdots \\ D\cdot\Theta_{v,N-1} \end{pmatrix}$$

where

$$D\cdot\Gamma(0,0) = (1-N^2)\cdot\Gamma(0,0)^2 = -(1-N^2)\cdot\mu(N)/12,$$

$$D\cdot\Theta_{v,i} = \Gamma\cdot\Theta_{v,i} = N \qquad \text{for} \quad i=1,\cdots,N-1.$$

Thus Lemma 1-1 implies Lemma 1-3. Q.E.D.

Lemmas 1-1, 1-2, 1-3 and the above Remark 1 imply

PROPOSITION 1-1. *Assume* $N \geqq 4$. *Then*

$$\Gamma = \sum_{i,j=0}^{N-1} \Gamma(i,j) \text{ is divisible } \begin{cases} \text{by } N & \text{if } N \text{ is odd}, \\ \text{by } N/2 & \text{if } N \text{ is even}. \end{cases}$$

REMARK 2. In case $N=3$,

$$\mu(3) = 12,$$

$$\Gamma(i,j)^2 = -\mu(3)/12 = -1,$$

$$g(\Gamma(i,j)) = 1 + (3-6)\cdot\mu(3)/(12\cdot 3) = 0.$$

Hence Γ is not divisible by 3 and $\Gamma(i,j)$'s are exceptional curves of the first kind.

REMARK 3. If N is even, Γ is *not* divisible by N. We refer to our previous paper [I_1] for a proof.

REMARK 4. In §1 of [S], Shioda remarked that *the Néron-Severi group* $NS(B(N))$

is torsion-free. This fact can be proved as follows:

PROOF OF REMARK 4. Let D be a divisor on $B(N)$ such that $D\neq 0$ and $nD\approx 0$ for some positive integer n. Then

$$\sum_{\nu=0}^{2}(-1)^{\nu}\dim H^{\nu}(B(N),\mathcal{O}(D))=p_g-q+1=\frac{\mu(N)}{12}\geqq 1$$

where $H^0(B(N),\mathcal{O}(D))=0$ and $H^2(B(N),\mathcal{O}(D))\cong H^0(B(N),\mathcal{O}(K_{B(N)}-D))$. Hence there exists an effective divisor $D'\in|K_{B(N)}-D|$. Since D' is effective and $D'\cdot F=K_{B(N)}\cdot F-D\cdot F=0$, we obtain that $D'|F\sim 0$. By the Lemma of Shioda and by the fact that $D'\cdot\Theta_{v,i}=0$, we get

$$D'\approx(D'\cdot\Gamma(0,0))F\,.$$

Since $K_{B(N)}=\Phi^*(\mathfrak{k}-\mathfrak{f})$, we obtain that $D\approx hF$ for some integer h. Since $nD\approx 0$, $0=D\cdot\Gamma(0,0)=h\cdot F\cdot\Gamma(0,0)=h$. Thus $h=0$ and $D\approx 0\cdot F=0$. Q.E.D.

§2. The example $A(N, n)$.

From now on, we assume $N\geqq 4$. By Proposition 1-1, $\Gamma=\sum_{i,j=0}^{N-1}\Gamma(i,j)$ is divisible by N if N is odd, and by $N/2$ if N is even. Let n be an integer such that $n\geqq 2$ and

$$n\,|\,N \qquad \text{if } N \text{ is odd}\,,$$
$$n\,|\,(N/2) \qquad \text{if } N \text{ is even.}$$

Then $[\Gamma]=nL$ for some line bundle $L\in H^1(B(N),\mathcal{O}^*)$. Hence we can construct, in the bundle space of L, an n-fold branched covering

$$\varphi\colon A(N,n)\to B(N)$$

along a non-singular branch locus $\Gamma(\subset B(N))$.

Let

K_S = the canonical line bundle of a compact complex surface S (the canonical divisor of S is also denoted by K_s),

$e(X)$ = the topological Euler number of a space X,

$\tau(S)$ = the topological index of $S=(K_S^2-2e(S))/3$,

$p_g(S)$ = the geometric genus of S, $q(S)$ = the irregularity of S.

We have the following classically known

LEMMA 2-1. *Let $\varphi\colon A\to B$ be an n-fold branched covering along a non-singular branch locus $\Gamma(\subset B)$. Then*

(i) $K_A = \varphi^* K_B + (n-1)\Gamma^*$

where $n\Gamma^* = \varphi^*\Gamma$ *and* $\Gamma^* = \varphi^{-1}(\Gamma)$. *Hence*

$$K_A^2 = nK_B^2 + 2(n-1)K_B \cdot \Gamma + \frac{(n-1)^2}{n}\Gamma^2 .$$

(ii) $e(A) = ne(B) - (n-1)e(\Gamma)$.

(iii) $p_g(S) - q(S) + 1 = \dfrac{K_S^2 + e(S)}{12}$ *(Noether's formula)* .

In the following, we shall calculate some numerical invariants of $A(N, n)$.

PROPOSITION 2-1.

$$K_{A(N,n)}^2 = \frac{N(n-1)\{(5n+1)N - 24n\}}{12n} \cdot \mu(N) ,$$

$$e(A(N, n)) = \frac{6n + (n-1)N(N-6)}{6} \cdot \mu(N) .$$

PROOF. By Lemma 2-1,

$$K_{A(N,n)}^2 = nK_{B(N)}^2 + 2(n-1)K_{B(N)} \cdot \Gamma + \frac{(n-1)^2}{n}\Gamma^2$$

where

$$K_{B(N)}^2 = 0 ,$$

$$K_{B(N)} \cdot \Gamma = \sum_{i,j} K_{B(N)} \cdot \Gamma(i, j) = N^2 \cdot \frac{N-4}{4N}\mu(N) = \frac{N(N-4)}{4}\mu(N) ,$$

$$\Gamma^2 = \sum_{i,j} \Gamma(i, j)^2 = N^2 \frac{-\mu(N)}{12} = -\frac{N^2}{12}\mu(N) .$$

Hence

$$K_{A(N,n)}^2 = 2(n-1) \cdot \frac{N(N-4)}{4} \cdot \mu(N) - \frac{(n-1)^2}{n}\frac{N^2}{12}\mu(N)$$

$$= \frac{N(n-1)\{(5n+1)N - 24n\}}{12n} \cdot \mu(N) .$$

By Lemma 2-1,

$$e(A(N, n)) = ne(B(N)) - (n-1)e(\Gamma)$$

where

$$e(B(N))=\mu(N)\,,$$

$$e(\Gamma)=\sum_{i,j}e(\Gamma(i,j))=N^2\cdot e(\Delta(N))=N^2(2-2g(\Delta(N)))$$

$$=N^2\cdot 2\left(-\frac{N-6}{12N}\cdot\mu(N)\right)=-\frac{N(N-6)}{6}\cdot\mu(N)\,.$$

Hence

$$e(A(N,n))=n\cdot\mu(N)+(n-1)\frac{N(N-6)}{6}\cdot\mu(N)$$

$$=\frac{6n+(n-1)N(N-6)}{6}\mu(N)\,. \qquad \text{Q.E.D.}$$

PROPOSITION 2-2. *Assume* $N\geqq 5$. *Then*

(i) $3e(A(N,n))\geqq K^2_{A(N,n)}\geqq 2e(A(N,n))$,

(ii) $3e(A(N,n))=K^2_{A(N,n)}$ *if and only if* $(N,n)=(7,7),(8,4),(9,3),(12,2)$,

(iii) $K^2_{A(N,n)}=2e(A(N,n))$ *if and only if* $(N,n)=(5,5)$.

PROOF. By Proposition 2-1

$$3e(A(N,n))-K^2_{A(N,n)}=\frac{\mu(N)}{12n}\{(n-1)N-6n\}^2\geqq 0\,.$$

The equality holds if and only if $N=6n/(n-1)$. Since $n\geqq 2$, this is equivalent to $(N,n)=(7,7),(8,4),(9,3),(12,2)$. By Proposition 2-1

$$K^2_{A(N,n)}-2e(A(N,n))=\frac{\mu(N)}{12n}\{(n^2-1)N^2-24n^2\}\,.$$

If $N=5$ (and hence $n=5$), then

$$K^2_{A(N,n)}-2e(A(N,n))=\frac{\mu(5)}{12\cdot 5}\{(5^2-1)\cdot 5^2-24\cdot 5^2\}=0\,.$$

If $N\geqq 6$, then, since $n\geqq 2$,

$$K^2_{A(N,n)}-2e(A(N,n))\geqq\frac{\mu(N)}{12n}\{(n^2-1)36-24n^2\}$$

$$=\frac{\mu(N)}{12n}(12n^2-36)=\frac{\mu(N)}{n}(n^2-3)>0\,. \qquad \text{Q.E.D.}$$

PROPOSITION 2-3. (i) *If* $N\geqq 6$, *then* $A(N,n)$ *is a minimal surface of general type* *with positive topological index.*

(ii) $A(5, 5)$ *is a surface of general type with* $K^2_{A(5,5)}=200$ *and* $e(A(5, 5))=100$. *Let*

$$\Gamma^*(i,j)=\varphi^{-1}(\Gamma(i,j)) \qquad (\text{hence } 5\Gamma^*(i,j)=\varphi^*\Gamma(i,j)) .$$

Then $\Gamma^*(i,j)$*'s are exceptional curves of the first kind. Let* A_0 *be the surface obtained by blowing down* $\Gamma^*(i,j)$*'s. Then* A_0 *is a minimal surface of general type with* $K^2_{A_0}=225$ *and* $e(A_0)=75$ (*and hence* $K^2_{A_0}=3e(A_0)$).

PROOF. By Lemma 2-1

$$K_{A(N,n)}=\varphi^*K_{B(N)}+(n-1)\Gamma^*=\varphi^*\Phi^*(\mathfrak{k}-\mathfrak{f})+(n-1)\sum_{i,j}\Gamma^*(i,j) ,$$

$$\dim|\mathfrak{f}|-\dim|\mathfrak{k}-\mathfrak{f}|=\deg\mathfrak{f}+1-g(\Delta(N))=\frac{3-N}{6N}\mu(N) .$$

Since $\deg\mathfrak{f}=-\mu(N)/12<0$, $\dim|\mathfrak{f}|=-1$. Hence

$$\dim|\mathfrak{k}-\mathfrak{f}|=\frac{N-3}{6N}\mu(N)-1>0 \qquad \text{if} \quad N\geqq 5 .$$

In particular, $p_g(A(N, n))>0$. If there exists an exceptional curve of the first kind on $A(N, n)$, then it is contained in the divisor $K_{A(N,n)}$ and, hence, is one of $\Gamma^*(i,j)$'s, while

$$g(\Gamma^*(i,j))=g(\Gamma(i,j))=1+\frac{(N-6)\mu(N)}{12N} ,$$

$$\Gamma^*(i,j)^2=\frac{\Gamma(i,j)^2}{n}=-\frac{\mu(N)}{12n} .$$

If $N\geqq 6$, then $g(\Gamma^*(i,j))\geqq 1$ and hence $A(N, n)$ is minimal. If $N=5$, then $\mu(5)=\frac{1}{2}5^3\cdot(1-5^{-2})=60$. Hence $g(\Gamma^*(i,j))=0$ and $\Gamma^*(i,j)^2=-1$, namely, $\Gamma^*(i,j)$'s are exceptional curves of the first kind. Since $\mu(5)=60$, $g(\Delta(5))=1+(5-6)\mu(5)/(12\cdot 5)=0$ and $\deg(\mathfrak{k}-\mathfrak{f})=((5-4)/(4\cdot 5))\mu(5)=(1/20)60=3$,

$$K_{A(5,5)}=3\varphi^*F+4\sum_{i,j=0}^{4}\Gamma^*(i,j)$$

where F is a general fibre of Φ: $B(5)\to\Delta(5)$ and $F\cdot\Gamma^*(i,j)=1$. Hence $K_{A_0}=3F_*$ where F_* is a non-singular curve with $g(F_*)=11$. In particular A_0 is minimal. Since $K^2_{A(5,5)}=200$ and $e(A(5, 5))=100$ by Proposition 2-1, A_0 is a minimal surface of general type with $K^2_{A_0}=225$ and $e(A_0)=75$. By Proposition 2-1,

$$K^2_{A(N,n)}=\frac{N(n-1)\{(5n+1)N-24n\}}{12n}\cdot\mu(N)$$

$$\geqq \frac{N(n-1)\{(5n+1)5-24n\}}{12n}\cdot\mu(N)$$

$$=\frac{N(n-1)(n+5)}{12n}\cdot\mu(N)>0$$

for $N\geqq 5$. Thus $A(N,n)$ is of general type. The topological index $\tau(A(N,n))=(K^2_{A(N,n)}-2e(A(N,n)))/3$ is positive if $N\geqq 6$ by Proposition 2-2. Q.E.D.

REMARK 1. On the geometric genus $p_g(A_0)$ and the irregularity $q(A_0)$ of A_0, we know

$$p_g(A_0)=34\,,\qquad q(A_0)=10\,.$$

As for the detailed calculations, we refer to [I_1].

REMARK 2. In case $N=4$ and $n=2$, $B(4)$ is a $K3$ surface and $\Gamma(i,j)^2=-2$, $g(\Gamma(i,j))=0$. Let $\Gamma^*(i,j)=\varphi^{-1}(\Gamma(i,j))$. Then

$$K_{A(4,2)}=\sum_{i,j}\Gamma^*(i,j)\,,$$

$$\Gamma^*(i,j)^2=-1\,,\qquad g(\Gamma^*(i,j))=0\,,$$

$$K^2_{A(4,2)}=-16\,,\qquad e(A(4,2))=16$$

by Lemma 2-1 and Proposition 2-1. Let $A_0(4,2)$ be the surface obtained by blowing down $\Gamma^*(i,j)$'s. Then

$$K_{A_0(4,2)}=0\,,\qquad e(A_0(4,2))=0\,.$$

This implies that $A_0(4,2)$ is an abelian surface and $B(4)$ is a Kummer surface.

REMARK 3. Fix $n\geqq 2$ and consider N's which are multiples of n. Then, by Proposition 2-1,

$$\lim_{N\to\infty}\frac{K^2_{A(N,n)}}{e(A(N,n))}=\frac{5n+1}{2n}\,.$$

Moreover $K^2_{A(N,n)}/e(A(N,n))>5/2$ if and only if $(N,n)\neq(5,5)$.

REMARK 4. (i) The canonical line bundle of $A(N,n)$, $N\geqq 6$, and of A_0 are *ample*.

(ii) $A(6,3)$ contains *elliptic curves* $\Gamma^*(i,j)$. Hence its universal covering space is *not* a bounded domain, while

$$\frac{K^2_{A(6,3)}}{e(A(6,3))}=\frac{8}{3}=2.66\cdots\,.$$

REMARK 5. There exist some other congruence relations between $\Gamma(i,j)$'s and

$\Theta_{v,i}$'s. For instance, in the case $N=2m$ (even), $\Gamma_e=\sum_{i,j:\text{even}}\Gamma(i,j)$ is divisible by m if m is odd and divisible by $m/2$ if m is even. Hence we can construct other branched coverings of $B(N)$ corresponding to them. We refer to [I_1] for details.

REMARK 6. E. Horikawa also gave another series of surfaces of general type with positive indices as branched coverings of the product of two algebraic curves. His construction is very simple but has a close relation with the moduli of algebraic curves. We refer also to [I_1] for details.

PART II Surfaces of General Type with $p_g=0$

§3. Hypersurfaces of the product of three elliptic curves.

We denote by θ_1 and θ_2 the usual theta functions, namely,

$$\theta_1(z)=2\left(\sum_{n=1}^{\infty}(-1)^{n-1}q^{((2n-1)/2)^2}\sin(2n-1)\pi z\right),$$

$$\theta_2(z)=2\left(\sum_{n=1}^{\infty}q^{((2n-1)/2)^2}\cos(2n-1)\pi z\right)$$

where $\tau\in \boldsymbol{C}$, $\mathrm{Im}\,\tau>0$ and $q=\exp(\pi\sqrt{-1}z)$. Then as is classically known we have

LEMMA 3-1.

$$\begin{aligned}
&\theta_1(z+1)=-\theta_1(z), &&\theta_2(z+1)=-\theta_2(z),\\
&\theta_1(z+\tau)=-\delta\theta_1(z), &&\theta_2(z+\tau)=\delta\theta_2(z),\\
&\theta_1(z+\tfrac{1}{2})=\theta_2(z), &&\theta_2(z+\tfrac{1}{2})=-\theta_1(z),\\
&\theta_1(-z)=-\theta_1(z), &&\theta_2(-z)=\theta_2(z),
\end{aligned}$$

where $\delta=\exp(\pi\sqrt{-1}(2z+\tau))$.

In particular, $(\theta_1)^2$ and $(\theta_2)^2$ are sections of a line bundle $2[o]$ on the elliptic curve $E=\boldsymbol{C}/\langle 1,\tau\rangle$ with periods 1, τ where o is the origin of E.

Let $\wp(z)$ be the $\wp$-function, namely,

$$\wp(z)=\wp(z,\tau)=\frac{\theta_2(z)^2-\theta_1(z)^2}{\theta_2(z)^2+\theta_1(z)^2}.$$

Then we have the following also well-known

LEMMA 3-2.
(i) $$\wp(z)=\wp(z+1)=\wp(z+\tau),\qquad \wp(z+\tfrac{1}{2})=-\wp(z),$$

$$\wp(-z)=\wp(z),\qquad \wp(z+\tau/2)=a/\wp(z),$$

where $a=\wp(\tau/2)$ *can take any value* $\in\boldsymbol{C}-\{0,\pm 1\}$. *In particular,* $\wp(z)$ *is a meromorphic* *function on the elliptic curve E.*

(ii) $\wp(\frac{1}{2})=-1, \quad \wp(0)=1, \quad \wp(\tau/2)=a, \quad \wp((1+\tau)/2)=-a.$

(iii) $\dfrac{d\wp}{dz}(z)=0$ *if and only if* $z=0, \frac{1}{2}, \tau/2, (1+\tau)/2$.

In particular, $\wp: E \to \boldsymbol{P}^1$ *is a double covering ramified over* $\pm 1, \pm a \in \boldsymbol{P}^1$.

(iv) *Let* $b=\wp(\tau/4)$. *Then* $b^2=a$.

Now we take three elliptic curves $E_i=\boldsymbol{C}/\langle 1, \tau_i\rangle$, $i=1, 2, 3$. Let (z_1, z_2, z_3) be the coordinates on the product $E_1 \times E_2 \times E_3$ and

$$o_i = \text{the origin of } E_i, \qquad \wp_i(z_i)=\wp(z_i, \tau_i),$$

$$a_i=\wp_i(\tau_i/2), \quad b_i=\wp_i(\tau_i/4) \qquad (b_i^2=a_i)$$

for $i=1, 2, 3$. For any $c \in \boldsymbol{C}^*$, we define the subvariety X_c by

$$X_c=\{(z_1, z_2, z_3) \in E_1 \times E_2 \times E_3 \mid \wp_1(z_1) \cdot \wp_2(z_2) \cdot \wp_3(z_3)=c\}.$$

Let $\psi_i: E_1 \times E_2 \times E_3 \to E_i$ be the projection to the i-th factor. Then

$$[X_c]=\psi_1^* 2[o_1] \otimes \psi_2^* 2[o_2] \otimes \psi_3^* 2[o_3]$$

and $[X_c]$ is *ample* on $E_1 \times E_2 \times E_3$. By the theorem of Bertini, X_c is irreducible. By Lemma 3-2, the singular points of X_c are isolated and at most ordinary double points,

$$\{\text{singular points of } X_c\}=X_c \cap \{\text{2-division points on } E_1 \times E_2 \times E_3\},$$

and moreover, if (z_1, z_2, z_3) is a singular point on X_c, then

$$(z_1+\tfrac{1}{2}, z_2+\tfrac{1}{2}, z_3), \quad (z_1, z_2+\tfrac{1}{2}, z_3+\tfrac{1}{2}), \quad (z_1+\tfrac{1}{2}, z_2, z_3+\tfrac{1}{2})$$

are also singular points on X_c. Let

$$\mathscr{E}=\{\text{the values of } \wp_1(z_1) \cdot \wp_2(z_2) \cdot \wp_3(z_3) \text{ on the 2-division points}\}.$$

Then

$$\mathscr{E}=\{\pm 1, \pm a_i, \pm a_i \cdot a_j, \pm a_1 \cdot a_2 \cdot a_3, (i \neq j, i, j=1, 2, 3)\}.$$

Let n_c be the number of the singular points on X_c. Then by elementary calculations we obtain the following:

(0) The possible values of n_c are 0, 4, 8, 12 and 16. n_c really takes these values. For instance

(1) If $c \notin \mathscr{E}$, then $n_c=0$, namely, X_c is non-singular.

(2) If $c \in \mathscr{E}$ and E_i's are *general*, then $n_c=4$.

(3) If $c \in \mathscr{E}$ and

$$c=\pm a_i, \qquad a_i=\pm a_j \cdot a_k, \quad ((i, j, k) \text{ is a permutation of } (1, 2, 3))$$

or

$$c=\pm 1, \qquad a_i \cdot a_j = \pm 1 \quad \text{for some } i, j \ (i \neq j),$$

then $n_c = 8$.

(4) If $c \in \mathscr{E}$ and

$$c = a_i \quad \text{for some } i, \qquad a_j = \pm a_i \ (\neq \pm\sqrt{-1}) \quad (\text{for any } j \neq i),$$

then $n_c = 12$.

(5) If $c \in \mathscr{E}$ and

$$c = \pm 1 \quad \text{or} \quad \pm\sqrt{-1}, \qquad a_i = \pm\sqrt{-1} \quad (i = 1, 2, 3),$$

then $n_c = 16$.

Thus we obtain

PROPOSITION 3-1. *The subvariety X_c is irreducible and non-singular outside exactly n_c ordinary double points where*

$$n_c = 0, \quad 4, \quad 8, \quad 12 \quad \text{or} \quad 16.$$

Let $\iota : \tilde{X}_c \to X_c \subset E_1 \times E_2 \times E_3$ be the minimal resolution of X_c. We shall calculate some numerical invariants of $\tilde{X}_c$ in the following

PROPOSITION 3-2. (i) *$\tilde{X}_c$ is a minimal surface of general type with the numerical invariants*:

$$p_g(\tilde{X}_c) = 10, \quad q(\tilde{X}_c) = 3, \quad K^2_{\tilde{X}_c} = e(\tilde{X}_c) = 48.$$

(ii) *ι induces an isomorphism between the spaces of holomorphic 1-forms*

$$\iota^* : H^0(E_1 \times E_2 \times E_3, \Omega^1) \to H^0(\tilde{X}_c, \Omega^1).$$

PROOF. (i) Minimality is clear from the construction. Since $\tilde{X}_c$'s are homeomorphic to each other and the numerical invariants p_g, q, K^2 and e are homologically invariant, we may assume that $\tilde{X}_c = X_c$, namely, X_c is non-singular. By the adjunction formula,

$$K_{X_c} = (K_{E_1 \times E_2 \times E_3} + [X_c]) | X_c = [X_c] | X_c.$$

Since

$$\begin{aligned}[X_c] &= \psi_1^* 2[o_1] \otimes \psi_2^* 2[o_2] \otimes \psi_3^* 2[o_3] \\ &= 2[o_1 \times E_2 \times E_3 + E_1 \times o_2 \times E_3 + E_1 \times E_2 \times o_3]\end{aligned}$$

and

$$K^2_{X_c} = \begin{cases} [X_c]^2 & \text{on} \quad X_c \\ [X_c]^3 & \text{on} \quad E_1 \times E_2 \times E_3, \end{cases}$$

we obtain

$$K_{X_c}^2 = 8 \times 6 = 48 .$$

From the short exact sequence

$$0 \to \mathcal{O}_{E_1 \times E_2 \times E_3} \to \mathcal{O}_{E_1 \times E_2 \times E_3}([X_c]) \to \mathcal{O}_{X_c}(K_{X_c}) \to 0 ,$$

it follows

$$\begin{aligned} &0 \to H^0(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}) \to H^0(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}([X_c])) \\ &\to H^0(X_c, \mathcal{O}_{X_c}(K_{X_c})) \to H^1(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}) \\ &\to H^1(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}([X_c])) \to H^1(X_c, \mathcal{O}_{X_c}(K_{X_c})) \\ &\to H^2(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}) \to H^2(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}([X_c])) \to . \end{aligned}$$

Since $[X_c]$ is *ample* on $E_1 \times E_2 \times E_3$,

$$\dim H^i(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}([X_c])) = 0 \qquad \text{for} \quad i \geqq 1 .$$

From the formula of Künneth, it follows

$$\begin{aligned} &\dim H^\nu(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}) \\ &\quad = \sum_{i+j+k=\nu} \dim H^i(E_1, \mathcal{O}_{E_1}) \cdot \dim H^j(E_2, \mathcal{O}_{E_2}) \cdot \dim H^k(E_3, \mathcal{O}_{E_3}) \\ &\quad = \begin{cases} 1 & \text{for} \quad \nu = 0, 3 \\ 3 & \text{for} \quad \nu = 1, 2 \\ 0 & \text{for} \quad \nu > 3 , \end{cases} \end{aligned}$$

$$\begin{aligned} &\dim H^0(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}([X_c])) \\ &\quad = \dim H^0(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}(\psi_1^* 2[o_1] \otimes \psi_2^* 2[o_2] \otimes \psi_3^* 2[o_3]) \\ &\quad = \prod_{i=1}^{3} \dim H^0(E_i, \mathcal{O}_{E_i}(\psi_i^* 2[o_i])) = 2 \cdot 2 \cdot 2 = 8 . \end{aligned}$$

Thus we obtain

$$\begin{aligned} p_g(X_c) &= \dim H^0(X_c, \mathcal{O}[K_{X_c}]) \\ &= \dim H^1(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}) \\ &\quad + \dim H^0(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}([X_c])) \\ &\quad - \dim H^0(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}) \\ &= 3 + 8 - 1 = 10 , \end{aligned}$$

$$\begin{aligned} q(X_c) &= \dim H^1(X_c, \mathcal{O}_{X_c}) = \dim H^1(X_c, \mathcal{O}[K_{X_c}]) \\ &= \dim H^2(E_1 \times E_2 \times E_3, \mathcal{O}_{E_1 \times E_2 \times E_3}) = 3 . \end{aligned}$$

By Noether's formula (Lemma 2-1)

$$48+e(X_c)=K_{\tilde{X}_c}^2+e(X_c)=12(p_g(X_c)-q(X_c)+1)=12(10-3+1)=96\,.$$

Hence $e(X_c)=48$. Since $p_g(X_c)>0$ and $K_{\tilde{X}_c}^2>0$, X_c is a surface of general type.

(ii) Since X_c is *not* linear, ι^*dz_1, ι^*dz_2 and ι^*dz_3 are linearly independent. Since dz_1, dz_2 and dz_3 form a system of basis of $H^0(E_1\times E_2\times E_3, \Omega^1)$, ι^* is injective, while

$$\dim H^0(E_1\times E_2\times E_3, \Omega^1)=3\,,$$

$$\dim H^0(\tilde{X}_c, \Omega^1)=\dim H^1(\tilde{X}_c, \mathcal{O}_{\tilde{X}_c})=q(\tilde{X}_c)=3\,.$$

Hence ι^* is an isomorphism. Q.E.D.

§4. The example Y_c.

We consider the following automorphisms of $E_1\times E_2\times E_3$:

$$g_1: (z_1, z_2, z_3)\to(-z_1+\tfrac{1}{2}, z_2+\tfrac{1}{2}, z_3)\,,$$

$$g_2: (z_1, z_2, z_3)\to(z_1, -z_2+\tfrac{1}{2}, z_3+\tfrac{1}{2})\,,$$

$$g_3: (z_1, z_2, z_3)\to(z_1+\tfrac{1}{2}, z_2, -z_3+\tfrac{1}{2})\,.$$

Let G be the group generated by g_1, g_2 and g_3:

$$G=\langle g_1, g_2, g_3\rangle\ (\cong(\mathbf{Z}/2\mathbf{Z})\oplus(\mathbf{Z}/2\mathbf{Z})\oplus(\mathbf{Z}/2\mathbf{Z}))\,.$$

Then, by Lemma 3-2, X_c is *invariant* under the action of G. Thus G operates on X_c and, hence, naturally on the minimal resolution $\tilde{X}_c$ of X_c. Let Y_c be the quotient surface of $\tilde{X}_c$ by G: $Y_c=\tilde{X}_c/G$.

PROPOSITION 4-1. *Y_c is a non-singular minimal surface of general type with*

$$p_g(Y_c)=q(Y_c)=0\,,\quad K_{Y_c}^2=6-n_c/4\,,\quad e(Y_c)=6+n_c/4$$

where n_c = the number of singular points of X_c = 0, 4, 8, 12 *or* 16. *Hence*

$$K_{Y_c}^2=6, 5, 4, 3 \text{ or } 2\,,\qquad e(Y_c)=6, 7, 8, 9 \text{ or } 10$$

according as n_c = 0, 4, 8, 12 *or* 16.

OUTLINE OF THE PROOF. (For details on the proof, we refer to our forthcoming paper [I_2].) Let

$$g_0=g_1\circ g_2\circ g_3: (z_1, z_2, z_3)\to(-z_1, -z_2, -z_3)$$

$$\bar{G}=G/\langle g_0\rangle\,,\qquad \bar{Y}_c=\tilde{X}_c/\langle g_0\rangle\,.$$

Then, as is clear, g_0 is the only one element of G which has fixed points on the ambient space $E_1\times E_2\times E_3$. Hence

$$\bar{G}\cong(Z/2Z)\oplus(Z/2Z)\,,\qquad Y_c=\bar{Y}_c/\bar{G}$$

where $\bar{G}$ has no fixed points on $\bar{Y}_c$. Let

$$\varphi_1:\tilde{X}_c\to\bar{Y}_c\quad\text{and}\quad\varphi_2:\bar{Y}_c\to Y_c$$

be the projections. Since

$$\{\text{fixed points of } g_0 \text{ on } E_1\times E_2\times E_3\}=\{\text{2-division points on } E_1\times E_2\times E_3\}\,,$$

we obtain

$$\begin{aligned}&\{\text{fixed points of } g_0 \text{ on } X_c\}\\&\quad=\{\text{2-division points on } E_1\times E_2\times E_3\}\cap X_c\\&\quad=\{\text{singular points on } X_c\}\,.\end{aligned}$$

Let $s_1, s_2, \cdots, s_{n_c}$ be all of the singular points on X_c and

$$C_i=\iota^{-1}(s_i)\qquad i=1,2,\cdots,n_c$$

where $\iota:\tilde{X}_c\to X_c$ is the minimal resolution of all singular points on X_c. Then C_i's are non-singular rational curves with

$$(C_i)^2=-2\,,\qquad C_i\cdot C_j=0\quad(i\neq j)\,.$$

By considering g_0 in local coordinates, we obtain

$$\{\text{fixed points of } g_0 \text{ on } \tilde{X}_c\}=\bigcup_{i=1}^{n_c}C_i\,.$$

Therefore $\bar{Y}_c$ is non-singular, the projection

$$\varphi_2:\bar{Y}_c\to Y_c$$

is an *unbranched* 4-fold covering surface and the projection

$$\varphi_1:\tilde{X}_c\to\bar{Y}_c$$

is a *branched* double covering surface along a non-singular branch locus $\Gamma=\bigcup_{i=1}^{n_c}\bar{C}_i$ $(\subset\bar{Y}_c)$ where $\bar{C}_i=\varphi_1(C_i)$'s are non-singular rational curves with $(\bar{C}_i)^2=-4$ and $K_{\bar{Y}_c}\cdot\bar{C}_i=2$.

By Lemma 2-1,

$$K^2_{\tilde{X}_c}=2K^2_{\bar{Y}_c}+2(2-1)K_{\bar{Y}_c}\cdot\Gamma+\frac{(2-1)^2}{2}\Gamma^2\,,$$

$$e(\tilde{X}_c)=2e(\bar{Y}_c)-(2-1)e(\Gamma)$$

where $K^2_{\tilde{X}_c}=e(\tilde{X}_c)=48$, $K_{\bar{Y}_c}\cdot\Gamma=\sum_{i=1}^{n_c}K_{\bar{Y}_c}\cdot\bar{C}_i=2n_c$, $\Gamma^2=\sum_{i=1}^{n_c}\bar{C}_i^2=-4n_c$ and $e(\Gamma)=\sum_{i=1}^{n_c}e(\bar{C}_i)=2n_c$. Therefore

$$48 = K^2_{\tilde{X}_c} = 2K^2_{\bar{Y}_c} + 2 \cdot 1 \cdot 2n_c + \tfrac{1}{2}(-4n_c) = 2K^2_{\bar{Y}_c} + 2n_c\,,$$
$$48 = e(\tilde{X}_c) = 2e(\bar{Y}_c) - 2n_c\,.$$

Hence

$$K^2_{\bar{Y}_c} = 24 - n_c\,, \qquad e(\bar{Y}_c) = 24 + n_c\,.$$

Since $\varphi_2 : \bar{Y}_c \to Y_c$ is an unbranched and 4-fold covering, we arrive at

$$K^2_{Y_c} = \frac{K^2_{\bar{Y}_c}}{4} = 6 - \frac{n_c}{4}\,, \qquad e(Y_c) = \frac{e(\bar{Y}_c)}{4} = 6 + \frac{n_c}{4}\,.$$

From (ii) of Proposition 3-2, it follows that

$$H^0(Y_c, \Omega^1) \cong H^0(\tilde{X}_c, \Omega^1)^G \cong H^0(E_1 \times E_2 \times E_3, \Omega^1)^G$$

where $H^0(\ ,\ )^G$ is the subspace of the elements invariant under the action of G. Since dz_1, dz_2 and dz_3 form a system of basis of holomorphic 1-forms on $E_1 \times E_2 \times E_3$ and the group G contains

$$g_0 : (z_1, z_2, z_3) \to (-z_1, -z_2, -z_3)\,,$$

we obtain

$$H^0(Y_c, \Omega^1) \cong H^0(E_1 \times E_2 \times E_3, \Omega^1)^G = 0\,.$$

Hence

$$q(Y_c) = \dim H^1(Y_c, \mathcal{O}_{Y_c}) = \dim H^0(Y_c, \Omega^1) = 0\,.$$

By the aboves and Noether's formula (Lemma 2-1), we obtain

$$p_g(Y_c) - 0 + 1 = p_g(Y_c) - q(Y_c) + 1 = \frac{K^2_{Y_c} + e(Y_c)}{12}$$
$$= \frac{6 - n_c/4 + 6 + n_c/4}{12} = 1\,,$$
$$p_g(Y_c) = 0\,.$$

By considering more precisely the zeros of the 2-canonical forms $(\iota^* dz_i \wedge \iota^* dz_j)^2$ $(i \neq j)$ on $\bar{Y}_c$, we can express explicitly the 2-canonical divisor $2K_{Y_c}$:

$$2K_{Y_c} = 2(E_1 + E_2 + F_1 + F_2)$$

where E_1 and E_2 are non-singular elliptic curves with $E_i^2 = -1$ and F_1 and F_2 are as follows according as the values of n_c:

(Case 0) In case $n_c = 0$

F_1 and F_2 are non-singular curves with $g(F_i) = 2$ and $F_i^2 = 0$.

(Case 1) In case $n_c=4$

F_1 is a non-singular curve with $g(F_1)=2$ and $F_1^2=0$,

F_2 is a non-singular elliptic curve with $F_2^2=-1$.

(Case 2) In case $n_c=8$

(2-1) F_1 and F_2 are non-singular elliptic curves with $F_i^2=-1$, or

(2-2) F_1 is a non-singular curve with $g(F_1)=2$ and $F_1^2=0$,

F_2 is a non-singular rational curve with $F_2^2=-2$.

(Case 3) In case $n_c=12$

F_1 is a non-singular elliptic curve with $F_1^2=-1$,

F_2 is a non-singular rational curve with $F_2^2=-2$.

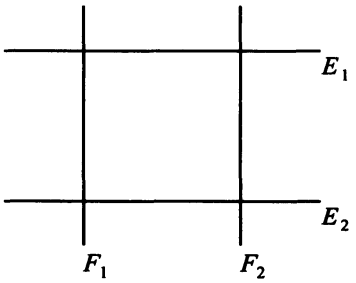
(Case 4) In case $n_c=16$

F_1 and F_2 are non-singular rational curves with $F_i^2=-2$.

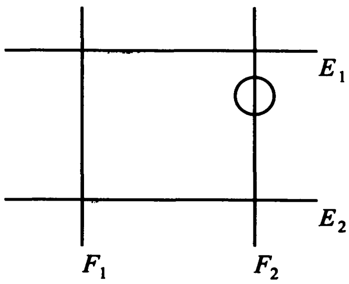
Hence $2K_{Y_c}$ is effective and contains no exceptional curves of the first kind. In particular Y_c is a minimal surface of general type. END OF THE OUTLINE OF THE PROOF.

In the following, we shall give some remarks on our examples. For details on these remarks, we refer to [I$_2$].

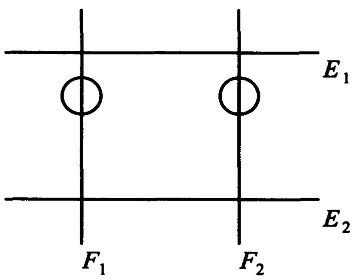
REMARK 1. The configuration of $2K_{Y_c}$ is as follows:



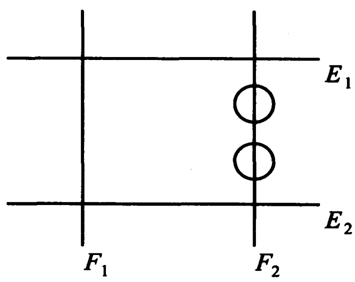
(Case 0)



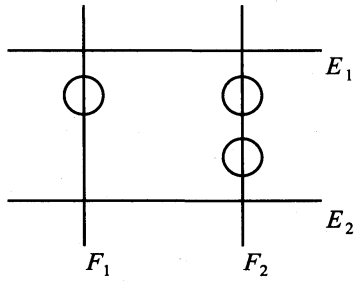
(Case 1)



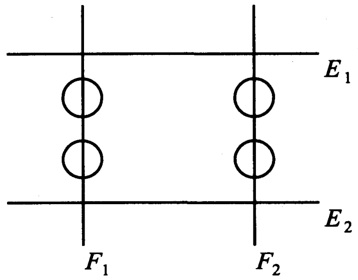
(Case 2-1)



(Case 2-2)



(Case 3)



(Case 4)

O's are non-singular rational curves with $(\cdot)^2=-4$.

REMARK 2. The fundamental group $\pi_1(Y_c)$ and the homology group $H_1(Y_c, \boldsymbol{Z})$ of Y_c are as follows. Let $(\boldsymbol{Z}/2\boldsymbol{Z})^m$ be the direct sum of m copies of $\boldsymbol{Z}/2\boldsymbol{Z}$ and let

$$H=((\boldsymbol{Z}/2\boldsymbol{Z})*(\boldsymbol{Z}/2\boldsymbol{Z}))/\langle(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1})^2\rangle$$

where $*$ denotes the free product of two groups $\boldsymbol{Z}/2\boldsymbol{Z}=\langle\gamma_1\rangle$ and $\boldsymbol{Z}/2\boldsymbol{Z}=\langle\gamma_2\rangle$. Then we have the following table:

n_c	$K_{Y_c}^2$	$\pi_1(Y_c)$	$H_1(Y_c, \boldsymbol{Z})$
0	6	$0\to\boldsymbol{Z}^6\to\pi_1(Y_c)\to G\to 0$	$(\boldsymbol{Z}/2\boldsymbol{Z})^6$
4	5	$H\times(\boldsymbol{Z}/2\boldsymbol{Z})^3$	$(\boldsymbol{Z}/2\boldsymbol{Z})^5$
8	4	$H\times(\boldsymbol{Z}/2\boldsymbol{Z})^2$	$(\boldsymbol{Z}/2\boldsymbol{Z})^4$
12	3	$H\times(\boldsymbol{Z}/2\boldsymbol{Z})$	$(\boldsymbol{Z}/2\boldsymbol{Z})^3$
16	2	H	$(\boldsymbol{Z}/2\boldsymbol{Z})^2$

REMARK 3. In case $\tau_1=\tau_2=\tau_3$ (namely $E_1=E_2=E_3$), $E_1\times E_2\times E_3$ admits the cyclic permutation of the components of the coordinates as an automorphism:

$$\sigma:(z_1, z_2, z_3)\to(z_3, z_1, z_2)\,.$$

By the definition of X_c, the subvariety X_c is invariant under the action of σ. Hence σ induces an automorphism

$$\sigma: \tilde{X}_c\to\tilde{X}_c\,.$$

Since $\sigma^{-1}G\sigma=G$, σ induces an automorphism

$$\sigma:\ Y_c=\tilde{X}_c/G\to Y_c=\tilde{X}_c/G$$

such that $\sigma^3=1$. σ has isolated fixed points on Y_c which are described locally in the following two types:

type 1: $\sigma: (\zeta_1, \zeta_2) \to (\omega\zeta_1, \omega^2\zeta_2)$,

type 2: $\sigma: (\zeta_1, \zeta_2) \to (\omega\zeta_1, \omega\zeta_2)$

where $\omega^3 = 1$ and $\omega \neq 1$. The number of the fixed points is as follows:

n_c	number of type 1	number of type 2	total
0	3	0	3
4	2	2	4
8	1	4	5
12	3	0	3
16	2	2	4

Let $\rho: Z_c \to Y_c/\langle\sigma\rangle$ be the minimal resolution of $Y_c/\langle\sigma\rangle$. Then for each singular point $s \in Y_c/\langle\sigma\rangle$, $\rho^{-1}(s)$ is as follows:

type 1: $\rho^{-1}(s) = C_1 \cup C_2$

where C_1 and C_2 are non-singular rational curves with $C_1 \cdot C_2 = 1$ and $(C_1)^2 = (C_2)^2 = -2$.

type 2: $\rho^{-1}(s) = C$

where C is a non-singular rational curve with $(C)^2 = -3$. From these we can derive some results on Z_c: Z_c is a minimal surface with $p_g(Z_c) = q(Z_c) = 0$ and

n_c	$K_{Z_c}^2$	structure	$\pi_1(Z_c)$
0	2	general	$(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/2\mathbf{Z})$
4	1	general	$\mathbf{Z}/2\mathbf{Z}$
8	0	elliptic	0
12	1	general	$\mathbf{Z}/2\mathbf{Z}$
16	0	elliptic	0

REMARK 4. In case $c = b_1 \cdot b_2 \cdot b_3$ where $b_i = \wp_i(\tau_i/4)$, the possible values of n_c are 0 and 8 and, by Lemma 3-2, X_c is invariant under the action of

$$h: (z_1, z_2, z_3) \to (z_1 + \tau_1/2, z_2 + \tau_2/2, z_3 + \tau_3/2).$$

Since h and each element of G are commutative to each other, h induces an automorphism

$$h: Y_c = \tilde{X}_c/G \to Y_c = \tilde{X}_c/G$$

such that $h^2=1$. h has four isolated fixed points on Y_c. Let W be the minimal resolution of $Y_c/\langle h\rangle$. Then W is a minimal surface with $p_g(W)=q(W)=0$ and

n_c	K_W^2	structure	$\pi_1(W)$
0	3	general	$H\times(\boldsymbol{Z}/2\boldsymbol{Z})^3$
8	2	general	$H\times(\boldsymbol{Z}/2\boldsymbol{Z})^2$

REMARK 5. In case $c=b_1\cdot b_2\cdot b_3$ where $b_i=\wp_i(\tau_i/4)$ and, moreover, $b_1=b_2=b_3$. Then X_c is invariant under the action of σ and h. Let V be the minimal resolution of $X_c/\langle\sigma, h\rangle$. Then V is a minimal surface with $p_g(V)=q(V)=0$ and

n_c	K_V^2	structure	$\pi_1(V)$
0	1	general	$\boldsymbol{Z}/2\boldsymbol{Z}$
8	0	elliptic	0

REMARK 6. In a similar way, we can construct examples of surfaces S of general type with $p_g(S)=0$ and $K_S^2=7$ and 8.

[Surfaces with $K_S^2=7$] We take four elliptic curves $E_i=\boldsymbol{C}/\langle 1, \tau_i\rangle$, $i=1, 2, 3, 4$. Let (z_1, z_2, z_3, z_4) be the coordinates of the product $E_1\times E_2\times E_3\times E_4$ and

$$\wp_i(z_i)=\wp(z_i, \tau_i)\,,\quad a_i=\wp_i(\tau_i/2)\,,\quad b_i=\wp_i(\tau_i/4)\quad (b_i^2=a_i)$$

for $i=1, 2, 3, 4$. We define the subvariety X of $E_1\times E_2\times E_3\times E_4$ by

$$X=\{(z_1, z_2, z_3, z_4)\in E_1\times E_2\times E_3\times E_4 \mid \wp_1(z_1)\wp_2(z_2)\wp_3(z_3)=b_1\cdot b_2\cdot b_3,\ \wp_3(z_3)\wp_4(z_4)=b_3\cdot b_4\}\,.$$

We consider several conditions on a_i's:

(C0) $a_3a_4\neq 1$ and $a_3\neq a_4$,
(C1-0) $a_1\cdot a_2\cdot a_3=1$,
(C1-i) $a_i=a_j\cdot a_k$ where (i, j, k) is a permutation of $(1, 2, 3)$,
(C2-0) $a_1\cdot a_2\cdot a_4=1$,
(C2-i) $a_i=a_j\cdot a_k$ where (i, j, k) is a permutation of $(1, 2, 4)$.

Then we know

(1) X is irreducible if and only if a_i's satisfy (C0).
(2) Under the condition (C0):

(2-0) X is non-singular if and only if a_i's satisfy neither (C1)'s nor (C2)'s.

(2-1) If a_i's satisfy one of (C1)'s but none of (C2)'s, or, satisfy one of (C2)'s but none of (C1)'s, then X has 16 ordinary double points and is non-singular elsewhere.

(2-2) If a_i's satisfy one of (C1)'s and one of (C2)'s both, then X has 32 ordinary double points and non-singular elsewhere.

(2-3) Any two of (C1)'s cannot hold simultaneously and any two of (C2)'s also.

Now we assume the condition (C0). Let n be the number of the singular points on X. Then the subvariety X is irreducible and non-singular outside exactly n ordinary double points where $n=0$, 16 or 32. Let $\iota: \tilde{X}\to X$ be the minimal resolution of X. We consider the following automorphisms of $E_1\times E_2\times E_3\times E_4$:

$$g_1: (z_1, z_2, z_3, z_4)\to(-z_1+\tfrac{1}{2}, z_2+\tfrac{1}{2}, z_3, z_4),$$

$$g_2: (z_1, z_2, z_3, z_4)\to(z_1, -z_2+\tfrac{1}{2}, z_3+\tfrac{1}{2}, -z_4+\tfrac{1}{2}),$$

$$g_3: (z_1, z_2, z_3, z_4)\to(z_1+\tfrac{1}{2}, z_2, -z_3+\tfrac{1}{2}, -z_4+\tfrac{1}{2}),$$

$$g_4: (z_1, z_2, z_3, z_4)\to(z_1, z_2, -z_3, -z_4),$$

$$g_5: (z_1, z_2, z_3, z_4)\to(z_1+\tau_1/2, z_2+\tau_2/2, z_3+\tau_3/2, z_4+\tau_4/2).$$

Let G be the group generated by g_1, g_2, g_3, g_4 and g_5:

$$G=\langle g_1, g_2, g_3, g_4, g_5\rangle\ (\cong(\boldsymbol{Z}/2\boldsymbol{Z})^5).$$

Then X is invariant under the action of G. G operates on X and, hence, naturally on the minimal resolution $\tilde{X}$ of X. Let S be the quotient surface of $\tilde{X}$ by G. Then we have the following results:

S is a non-singular minimal surface of general type with

$$p_g(S)=q(S)=0, \quad K_S^2=7-n/16, \quad e(S)=5+n/16$$

where $n=$ the number of the singular points on $X=0$, 16 or 32. Hence

$$K_S^2=7, 6 \text{ or } 5, \qquad e(S)=5, 6 \text{ or } 7$$

according as $n=0$, 16 or 32.

[Surfaces with $K_S^2=8$] Under the same circumstances, we define the subvariety X of $E_1\times E_2\times E_3\times E_4$ by

$$X=\{(z_1, z_2, z_3, z_4)\in E_1\times E_2\times E_3\times E_4 \mid \wp_1(z_1)\wp_2(z_2)=b_1\cdot b_2,\ \wp_3(z_3)\wp_4(z_4)=b_3\cdot b_4\}.$$

We assume

$$a_1 \cdot a_2 \neq 1\,, \qquad a_1 \neq a_2\,,$$
$$a_3 \cdot a_4 \neq 1\,, \qquad a_3 \neq a_4\,.$$

Then X is a product of two irreducible and non-singular algebraic curves. We consider the following automorphisms of $E_1 \times E_2 \times E_3 \times E_4$:

$$g_1 : (z_1, z_2, z_3, z_4) \rightarrow (-z_1 + \tfrac{1}{2}, z_2 + \tfrac{1}{2}, -z_3, -z_4)\,,$$
$$g_2 : (z_1, z_2, z_3, z_4) \rightarrow (-z_1 + \tau_1/2, -z_2 + \tau_2/2, z_3 + \tau_3/2, z_4 + \tau_4/2)\,,$$
$$g_3 : (z_1, z_2, z_3, z_4) \rightarrow (z_1 + \tau_1/2, z_2 + \tau_2/2, -z_3 + \tfrac{1}{2}, -z_4 + \tfrac{1}{2})\,,$$
$$g_4 : (z_1, z_2, z_3, z_4) \rightarrow (-z_1 + \tfrac{1}{2}, -z_2 + \tfrac{1}{2}, -z_3 + \tau_3/2, z_4 + \tau_4/2)\,.$$

Let G be the group generated by g_1, g_2, g_3 and g_4:

$$G = \langle g_1, g_2, g_3, g_4 \rangle \ (\cong (\boldsymbol{Z}/2\boldsymbol{Z})^4).$$

Then X is invariant under the action of G. G has no fixed points on X. Let S be the quotient surface of X by G. Then we have the following results:

S is a non-singular minimal surface of general type with

$$p_g(S) = q(S) = 0\,, \quad K_S^2 = 8\,, \quad e(S) = 4\,.$$

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