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# Symplectic Geometry and Deformation of Infinite Dimensional Cycles Associated to Cauchy-Riemann Operators

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## 1. Introduction.

Let $M$ be a Riemann surface, and $P$ be a principal bundle over $M$ with structure group $U(n)$. For every connection $A$ on $P$ we can associate $\bar{\partial}_A$-operator as follows. We denote by $\Omega_C^1(M, \mathrm{ad}(P))$ the complex vector space of 1-forms over $M$ valued on $\mathrm{ad}(P)$. In parallel with the decomposition of this vector space:

$$\Omega_C^1(M, \mathrm{ad}(P)) = \Omega^{1,0}(M, \mathrm{ad}(P)) \oplus \Omega^{0,1}(M, \mathrm{ad}(P)),$$

the covariant derivative $d_A$ decomposes into $\partial_A$ and $\bar{\partial}_A$:

$$d_A = \partial_A + \bar{\partial}_A,$$
$$\partial_A : \Omega_C^0(M, \mathrm{ad}(P)) \to \Omega^{1,0}(M, \mathrm{ad}(P)),$$
$$\bar{\partial}_A : \Omega_C^0(M, \mathrm{ad}(P)) \to \Omega^{0,1}(M, \mathrm{ad}(P)).$$

Hence the Cauchy-Riemann operator $\bar{\partial}_A$ is associated to each connection $A$. We have therefore a family of Fredholm operators $\{\bar{\partial}_A\}_{A \in \mathscr{A}}$ parametrize by $\mathscr{A}$:

$$\mathscr{A} \ni A \mapsto \bar{\partial}_A.$$

Let $\mathscr{G}$ denote the gauge transformation group. Due to the gauge invariance of $\{\bar{\partial}_A\}_{A \in \mathscr{A}}$, if we take as a parameter space the quotient space $\mathscr{A}/\mathscr{G}$ instead of $\mathscr{A}$, then we see that the above association can be reduced to $\mathscr{A}/\mathscr{G}$. Thus we get a new family of operators

$$\mathscr{A}/\mathscr{G} \ni [A] \mapsto \bar{\partial}_{[A]}.$$

This defines an element of $k$-theory over $\mathscr{A}/\mathscr{G}$ (see [2], [5], [10]).

These correspondences among operators $\{\bar{\partial}_A\}$, $\{\bar{\partial}_{[A]}\}$ and parameter spaces $\mathscr{A}$, $\mathscr{A}/\mathscr{G}$ are easily generalized to a general situation, i.e., a correspondence with Fredholm

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operators and infinite dimensional parameter spaces. Let $X$ be an infinite dimensional paracompact space, and let $\mathscr{F}=\{\mathscr{F}_x\}$, $x\in X$ be a set of Fredholm operators acting on a fixed Hilbert space $E$:

$$X\ni x\mapsto\mathscr{F}_x, \qquad \mathscr{F}_x: E\to E.$$

The space $X$ will be called the parameter space of this family. In the case of Cauchy-Riemann operators above, the parameter space is

$$X=\mathscr{A}/\mathscr{G}=\{[A]\},$$

the family of operators is a collection $\{\mathscr{F}_x\}$ of

$$\mathscr{F}_x=\mathscr{F}_{[A]}=\bar{\partial}_{[A]}.$$

This general situation is studied in [10], where an infinite dimensional cycle theory is exhibited. We will recall some results obtained in that paper. If we are given a family of Fredholm operators:

$$X\ni x\mapsto\mathscr{F}_x,$$

then we obtain some series of infinite dimensional cycles in $X$, which are classified into three types. These three types of cycles are denoted, respectively, by $\chi_{p,q}(\mathscr{F})$, $\kappa^{r,s}_{p,q}(\mathrm{Ker}(\mathscr{F}))$ and $\psi^{r,s}_{p,q}(\mathrm{Ind}(\mathscr{F}))$.

The first type of cycles $\chi_{p,q}(\mathscr{F})$ are defined by

$$\chi^*_{q,p}(\mathscr{F})=\chi_{p,q}(\mathscr{F}^*)=\{x\in X: \dim(\mathscr{F}^*_x)\geq p\},$$

where $p-q=k$, and $k$ is the numerical index of $\mathscr{F}$ (cf. U. Koshorke [5]).

The second type of cycles $\kappa^{r,s}_{p,q}(\mathrm{Ker}(\mathscr{F}))$, called *solution-cycles*, represents a global structure of the family of kernels of $\mathscr{F}$. These cycles are defined by conditions based on some degeneracy of kernels $\mathscr{F}_x$ with respect to a filtration of the trivial bundle $E\otimes X\to X$. Precisely speaking these cycles are defined as follows:

$$\kappa^{r,s}_{p,q}(\mathrm{Ker}(\mathscr{F}))=\{x\in\chi^*_{p,q}(\mathscr{F})\ ;\ \dim(\ker(\mathscr{F}_x)\cap E^{\infty-s})\geq\dim(\ker(\mathscr{F}_x))-s+r\},$$

where $\{E_i\}$ and $\{E^{\infty-i}\}$ $(i=1,2,\ldots)$ are filtrations of the bundle $E\otimes X\to X$, such that $E_i$ and $E^{\infty-i}$ are subbundles of $E\otimes X\to X$ for any $i$ with

$$E_1\subset E_2\subset\cdots,$$

$$E^{\infty-1}\supset E^{\infty-2}\supset\cdots,$$

$$E_i\oplus E^{\infty-i}=E\otimes X.$$

The global cohomological meanings of these solution cycles in the parameter space $X$ (i.e., the dual cohomology classes of these cycles in $H^*(X)$) were calculated in [10], and is given by a polynomial of the characteristic classes of the bundle $K$ which consists of kernels of $\mathscr{F}_x$, $x\in X$. The polynomial is the Hankel determinant of $C_*(K)$:

$$(-1)^{pq}\begin{vmatrix} C_{p-s+r}(K) & \cdots & C_{p-s+2r-l}(K) \\ \vdots & \ddots & \vdots \\ C_{p-s+1}(K) & \cdots & C_{p-s+r}(K) \end{vmatrix}.$$

The third type of cycles $\psi^{r,s}_{p,q}(\mathrm{Ind}(\mathscr{F}))$, called *index-cycles*, were constructed to estimate solution-cycles. The solution cycles $\kappa^{r,s}_{p,q}(\mathrm{Ker}(\mathscr{F}))$ are interesting in their nature, but they lack invariance (in the sense of global topological meaning). Therefore we need other cycles with good invariance relating to solution cycles. The index cycles were invented to meet this need. The index cycles represent the invariant part of the solution cycles. In this context we have:

$$|\kappa|^{r,s}_{p,q}(\mathrm{Ker}(\mathscr{F}))\supset|\psi|^{r,s}_{p,q}(\mathrm{Ind}(\mathscr{F}))\,,$$

here$|*|$ denotes a carrier of a cycle $*$. These index cycles are invariant in the sense that their cohomological meanings (i.e., the dual cohomology classes) are determined by the index (as a family) of $\mathscr{F}$. We see that the cohomology classes of $\psi^{r,s}_{p,q}(\mathrm{Ind}(\mathscr{F}))$ are polynomials of $C_*(\mathrm{Ind}(\mathscr{F}))$ (the characteristic classes of the index (as a family) of $\mathscr{F}$):

$$(-1)^{pq+(p+r)(q-s+r)}\begin{vmatrix} (-1)^pC_p(\mathrm{Ind}(\mathscr{F})) & \cdots & (-1)^{p+q+1}C_{p+q+1}(\mathrm{Ind}(\mathscr{F})) \\ \vdots & \ddots & \vdots \\ (-1)^{p+q+1}C_{p-q+1}(\mathrm{Ind}(\mathscr{F})) & \cdots & (-1)^pC_p(\mathrm{Ind}(\mathscr{F})) \end{vmatrix}$$
$$\times\begin{vmatrix} (-1)^{p+r}C_{p+r}(\mathrm{Ind}(\mathscr{F})) & \cdots & (-1)^{p+q+s+1}C_{p+q+2r-s-1}(\mathrm{Ind}(\mathscr{F})) \\ \vdots & \ddots & \vdots \\ (-1)^{p+q+s+1}C_{p-q+s+1}(\mathrm{Ind}(\mathscr{F})) & \cdots & (-1)^{p+r}C_{p+r}(\mathrm{Ind}(\mathscr{F})) \end{vmatrix}.$$

The construction of these index cycles was carried out in [10], using the geometry of infinite dimensional Grassmannian manifolds. The models of index cycles are subvarieties in the intersections of Schubert varieties of infinite dimensional Grassmannian manifolds. (See [10] for the precise proof of the existence of index cycles.)

For a family of coupled Dirac operators over $S^n$, we calculated the value of these cohomology classes, and deduced non-triviality of these cycles (see [10] for details).

The present paper exhibits one of the applications of this general theory of infinite dimensional cycles to a variational problem, i.e., the Yang-Mills Gauge Field theory over Riemann surfaces. An application to other variational problem of this theory was found with respect to the problem of characteristic orbits (the orbits of Reeb field), concerning to Weinstein Conjecture in [9], where the theory was used to encounter the failure of the Palais-Smale condition.

The aim of the present paper is to find an *invariant* infinite dimensional cycle *via* the gradient flow of the Yang-Mills action over Riemann surfaces, other than stable (or unstable) manifolds, as an application of the above general theory of infinite dimensional cycles. In this paper, we attempt to discover an invariant cycle which is cobordant, in the space $\mathscr{A}/\mathscr{G}$, to a (non-invariant) cycle derived from operators $\{\bar{\partial}_{[A]}\}_{[A]\in\mathscr{A}/\mathscr{G}}$.

We exploit, for this purpose, a symplectic structure over the connection space $\mathscr{A}$. In the process of the proof of our main theorem, we discover a symplectic geometrical relationship between the Yang-Mills energy function and the infinite dimensional cycles associated to $\{\bar{\partial}_{[A]}\}$. The author believes that this insight goes back to T. Frankel [3].

Let us explain this role of symplectic geometry, since this is one of key points in our work. T. Frankel [3] studied the fixed points of a 1-parameter group of holomorphic isometries on a Kähler manifold. He used the equation

$$JX = \operatorname{grad}\phi\ ,$$

where $J$ is the complex structure, $X$ is the vector field caused by a 1-parameter holomorphic transformation and $\phi$ is an integral of 1-form $i(X)\omega$. From this equation we know that the fixed set of the 1-parameter group coincides with the critical set of $\phi$. Thus the study of fixed points was deduced from Morse theory of $\phi$. We go conversely, and start with a Morse function (a Yang-Mills functional in our situation). Instead of the above equation, we show in §3

$$\operatorname{grad}_A(L) = -*X_A(L)\ ,$$

where $L$ is a Yang-Mills functional, $X$ is a Hamiltonian vector field with respect to a symplectic structure on the connection space, and $*$ is the Hodge $*$-operator.

By means of this symplectic geometrical relation, we reduce the invariance for the gradient field to the invariance for $*$-operator and the Hamiltonian vector field. Thus we can find the invariant cycles (*via* the Yang-Mills gradient flow) in the same cobordism class as the given cycle (derived from the operators $\{\bar{\partial}_{[A]}\}_{[A]\in\mathscr{A}/\mathscr{G}}$). This relation is, in its essence, quite similar to the correspondence between the Birkhoff decomposition and unstable (or stable) manifolds in loop groups.

The symplectic structure which we will use in order to find a relation between the cycles associated to $\{\bar{\partial}_{[A]}\}$, $A\in\mathscr{A}/\mathscr{G}$ and the Yang-Mills energy function over Riemann surfaces, comes from the Kähler metric on the space of Cauchy-Riemann operators. There is a standard Kähler metric on the space of the Cauchy-Riemann operators on $P$ (cf., D. G. Quillen [12]). We note that the imaginary part of this Kähler metric defines a symplectic structure on $\mathscr{A}$.

Main Theorem will be presented in §2, the proof of which will be given in §3.

## 2. Invariant cycles and $\bar{\partial}$-operators.

There are various ways of defining the symplectic structure over the connection space which are essentially the same. Though the symplectic structure we will adopt here may be interpreted as the imaginary part of the Kähler metric, or can be defined using $\zeta$-function corresponding to the determinant bundle over $\mathscr{A}$, we will give an alternative, straightforward definition for the later purpose, using Hodge $*$-operator.

Recall first that the connection space $\mathscr{A}$ is an affine space with model $\Omega^1(M, \mathrm{ad}(P))$. We denote its natural inner product by $B(x, y)$, $x, y \in \Omega^1(M, \mathrm{ad}(P))$. Let $*$ denote the Hodge $*$-operator over $\Omega^1(M, \mathrm{ad}(P))$. We set, for any $x, y \in T_A(\mathscr{A})(\simeq \Omega^1(M, \mathrm{ad}(P)))$,

$$S(x, y) = -B(x, *(y)) .$$

The form $S(x, y)$ defines a two-form on the connection space $\mathscr{A}$, and it is easily checked to be non-degenerate. Therefore this gives rise to a symplectic structure on $\mathscr{A}$. (Our symplectic structure $S(x, y)$ is equivalent to the imaginary part of Quillen's metric which was determined by the curvature derived from $\zeta'(0)$ on the determinant bundle).

We will consider a Hamiltonian vector field over $\mathscr{A}$ corresponding to a functional over $\mathscr{A}$, using the above symplectic structure. Let $L$ be the Yang-Mills functional over $\mathscr{A}$;

$$L(A) = \int_M \|F(A)\|^2 dv , \qquad A \in \mathscr{A} .$$

The Hamiltonian vector field which is associated to the above Yang-Mills functional with respect to the symplectic structure $S(x, y)$ will be denoted by $X(L)$:

$$X(L) : \mathscr{A} \ni A \mapsto X_A(L) .$$

We will apply our infinite dimensional cycle theory to the family of Fredholm operators $\{\bar{\partial}_A\}_{A \in \mathscr{A}}$ (parametrized by the space $\mathscr{A}$),

$$\bar{\partial}_A : \Omega^0_{\boldsymbol{C}}(M, \mathrm{ad}(P)) \to \Omega^{0,1}(M, \mathrm{ad}(P)) , \qquad A \in \mathscr{A} .$$

In order to state our Main Theorem we need a notion of *complementary filtrations*. Let $\bar{E} = \Omega^0(M, \mathrm{ad}(P)) \otimes \mathscr{A}$ be a trivial vector bundle over $\mathscr{A}$ with fibre $\Omega^0(M, \mathrm{ad}(P))$. Take two filtrations of $\bar{E}$,

$$\bar{E}_1 \subset \bar{E}_2 \subset \cdots ,$$
$$\bar{E}^{\infty-1} \supset \bar{E}^{\infty-2} \supset \cdots ,$$

which are consisting of subbundles of $\bar{E}$ and satisfy, for any $n$,

$$\bar{E} = \bar{E}_n \oplus \bar{E}^{\infty-n} .$$

We call this set of filtrations a *complementary filtration* of $\bar{E}$.

We notice that we can choose a particular filtration of the bundle $\bar{E}$ which is invariant with respect to the Hamiltonian vector field $X(L)$. This fact follows from the fact that $X(L)$ is tangent to the orbits of gauge transformation group. The integrability of fields which are tangent to each orbit of the gauge transformation group comes from Freed-Uhlenbeck [4]. Then we have:

LEMMA. *There exists a complementary filtration of* $\bar{E} = \Omega^0(M, \mathrm{ad}(P)) \otimes \mathscr{A}$ *which* *is invariant by the action of the Hamiltonian vector field* $X(L)$.

The proof of this lemma will be given in the next section in the process of the proof of our Main Theorem. In this lemma, the action caused by $X(L)$ is extended to the bundle $\bar{E}$ in a trivial way.

Hereafter we will take as a family of operators, $\mathscr{D}=\{\bar{\partial}_{[A]}\}$, $[A]\in\mathscr{A}/\mathscr{G}$, and simply write the solution cycle and the index cycle associated to this family as $\kappa_0$, and $\psi_0$ respectively, instead of writing $\kappa_{p,q}^{r,s}(\mathscr{D})$ or $\psi_{p,q}^{r,s}(\mathscr{D})$ (fixing arbitrary non-negative integers $p, q, r$ and $s$).

REMARK. We will see in the proof of Main Theorem that the cycles $\chi_{p,q}$ and $\kappa_0$ are already invariant *via* the gradient flow. On the contrary the cycle $\psi_0$ is not invariant. The problem is to *deform* the cycle $\psi_0$ and to find an invariant cycle $\psi$ among the cobordism class of $\psi_0$ in $\mathscr{A}/\mathscr{G}$.

We will get the following solution of our problem, the proof of which will be given in the next section.

MAIN THEOREM. *Let* $\mathscr{D}=\{\bar{\partial}_{[A]}\}$ $([A]\in\mathscr{A}/\mathscr{G})$ *be the family of operators of* $\bar{\partial}_{[A]}$, *and let* $\kappa_0$ *and* $\psi_0$ *be a solution-cycle and an index-cycle of* $\mathscr{D}$ *respectively. Then we have*:

(a) *The cycle* $\kappa_0$ *is invariant under the gradient vector field of the Yang-Mills functional* $L$.

(b) *There exists a cycle* $\psi$ *among the cobordism class of the cycle* $\psi_0$ *which is invariant under the gradient vector field of the functional* $L$.

## 3. Proof of Main Theorem.

The proof of Main Theorem is based on a relation between the energy function and cycles. We can describe this relation precisely using the symplectic structure defined in the previous section, $S(x, y)$, $x, y\in T_A\mathscr{A}$ $(\simeq\Omega^1(M, \mathrm{ad}(P))$ which was defined on the connection space $\mathscr{A}$. We begin by investigating a relation among the Hamiltonian vector fields and tangent spaces of orbits of the gauge transformation group $\mathscr{G}$.

As in the previous section we will denote by

$$X(L):\quad \mathscr{A}\ni A\mapsto X_A(L)\,,$$

the Hamiltonian vector field which corresponds to the Yang-Mills functional $L$ with respect to the symplectic structure $S(x, y)$. We will show later that the Hamiltonian vector field $X(L)$ is tangent to $\mathscr{G}$-orbit (here $\mathscr{G}$ is the gauge transformation group). If we restrict the Hamiltonian vector field $X(L)$ over each $\mathscr{G}$-orbit, then $X(L)$ can be regarded as a vector field on $\mathscr{G}$-orbit itself. It follows that $X(L)$ is integrable on each orbit (cf. Freed and Uhlenbeck [4]).

Let us construct a gauge-invariant filtration in Lemma in the previous section. Let

$\mathscr{A}\to\mathscr{A}/X(L)$ be the fibration of $A$ *via* orbits of $X(L)$. (This fibration makes sense because $X(L)$ is integrable). Because $\mathscr{A}$ is contractible, there is a lift of $\mathscr{A}/X(L)$ into $\mathscr{A}$. (This follows, for example, from the fact that $H^2(\mathscr{A}/X(L), Z)=0$ in case $X(L)\neq 0$.) Take a filtration $\{\bar{E}_n\}$, $\{\bar{E}^{\infty-n}\}$ $(n=1, 2, \dots)$ of $\bar{E}=\Omega^0(M, \mathrm{ad}(P))\otimes\mathscr{A}$ (restricted to the lift) at each point $A\in\mathscr{A}$ which is included in the image of the lift. Denote the projections of this filtration by $\pi_i: \bar{E}\mapsto\bar{E}_i$. Extend them to the whole $\mathscr{A}$ by

$$g^{-1}\circ\pi_i\circ g\,, \qquad g\in\mathscr{G}\,,$$

at an arbitrary point $A\in\mathscr{A}$ with $g(A)=A_0$, $[A]=[A_0]\in A/X(L)$. This gives a filtration which is invariant by the action caused by the Hamiltonian vector field $X(L)$. (This proves the lemma in the previous section). Hereafter we will fix this filtration.

We now calculate the Hamiltonian vector field $X(L)$ as follows. Take $X\in T_A\mathscr{A}\simeq\Omega^1(M, \mathrm{ad}(P))$. Then we have:

$$\begin{aligned} dL(X)\frac{d}{dt}\bigg|_{t=0}&\int_M \|*F(A+tX)\|^2 dv \\ &=\frac{d}{dt}\bigg|_{t=0}\int_M \|*\{F(A)+td_AX+t^2[X, X]\}\|^2 dv \\ &=(2d_AX, F)=2(X, d_A*F)=-2(X, *d_A*F)=2S(X, d_A*F)\,. \end{aligned}$$

This yields:

$$X_A(L)=d_A(*2F)\,.$$

Therefore we have:

$$X_A(L)\in\{d_A\alpha\in\Omega^1(M, \mathrm{ad}(P))\ ;\ \alpha\in\Omega^0(M, \mathrm{ad}(P))\}\,.$$

Consequently we see that the vector field $X(L)$ is tangent to the orbits of the gauge transformation group $\mathscr{G}$.

We proceed to show the invariance of infinite dimensional cycles. Recall that we have set,

$$\mathscr{D}=\{\bar{\partial}_{[A]}\}$$

$$\kappa_0=\kappa^{r,s}_{p,q}(\mathscr{D})\,, \qquad \psi_0=\psi^{r,s}_{p,q}(\mathscr{D})\,.$$

Since $\mathscr{D}$ is the quotient of $\{\bar{\partial}_A\}_{A\in\mathscr{A}}$, we will pull up everything upstairs (i.e., on $\mathscr{A}$), and set

$$\tilde{\mathscr{D}}=\{\bar{\partial}_A\}$$

$$\tilde{\kappa}_0=\kappa^{r,s}_{p,q}(\tilde{\mathscr{D}})\,, \qquad \tilde{\psi}_0=\psi^{r,s}_{p,q}(\tilde{\mathscr{D}})\,.$$

Because of the invariance of these cycles and filtrations *via* the gauge transformation group, it suffices to prove the assertions of Main Theorem for cycles $\tilde{\kappa}_0$ and $\tilde{\psi}_0$ instead of $\kappa_0$ and $\psi_0$. We will begin with the investigation of the cycle $\chi_{p,q}(\tilde{\mathscr{D}})$.

We will show that the cycle $\chi_{p,q}(\tilde{\mathscr{D}})$ for the family of operators $\tilde{\mathscr{D}}=\{\bar{\partial}_A\}_{(A\in\mathscr{A})}$ is invariant under the gradient flow of the Yang-Mills functional $L$.

First we observe;

$$\mathrm{grad}_A(L)=-2*(d_A*F)=*(d_A*(-2F))=-*X_A(L)\,,$$

here $\mathrm{grad}_A(L)$ denotes the gradient vector field at $A\in\mathscr{A}$ of the functional $L$. Recall that the cycle $\chi_{p,q}(\tilde{\mathscr{D}})$ is defined only by the condition on the dimension of the kernels of $\bar{\partial}_A$. Therefore the cycle $\chi_{p,q}(\tilde{\mathscr{D}})$ is invariant by the gauge transformation group, and hence invariant under the Hamiltonian vector field $X(L)$. Moreover we notice that $\chi_{p,q}(\tilde{\mathscr{D}})$ is invariant under the Hodge $*$-operator (here, the action is considered on the tangent spaces). In view of these, we see that $\chi_{p,q}(\tilde{\mathscr{D}})$ is invariant with respect to the vector field $-*X_A(L)$. Therefore it follows, from the above equation, that the cycle $\chi_{p,q}(\tilde{\mathscr{D}})$ is invariant under the gradient vector field $\mathrm{grad}_A(L)$ (cf., T. Frankel [3]).

We proceed to prove the invariance of the kernel cycle $\tilde{\kappa}_0$. The problem here is the fact that the cycle $\tilde{\kappa}_0$ is not necessarily invariant under the action of the vector field $X(L)$. Recall that we have chosen the complementary filtration, $\{\bar{E}_n\}$, $\{\bar{E}^{\infty-n}\}$ $(n=1, 2, \ldots.)$ which are invariant by the action of the vector field $X(L)$ (the existence of which has been proven already). Therefore we can apply the same argument as the case of the cycle $\chi_{p,q}(\tilde{\mathscr{D}})$ to the case of the cycle $\tilde{\kappa}_0$. This proves the first assertion (a) of Main Theorem.

Let us now proceed to prove the existence of an invariant cycle $\tilde{\psi}$ (among the cobordism class of the cycle $\tilde{\psi}_0$) which is invariant under the gradient flow.

So far we have been discussing the family of Fredholm operators with parameter space $\mathscr{A}$,

$$\mathscr{A}\ni A\mapsto\bar{\partial}_A\in\mathbf{Fred}(\Omega^0(M, \mathrm{ad}(P)), \Omega^{0,1}(M, \mathrm{ad}(P)))\,.$$

In order to prove the part (b) of Main Theorem, we will deal with a deformation of another family of operators restricted over the subset $|\chi_{p,q}(\tilde{\mathscr{D}})|$ (the carrier of the cycle $\chi_{p,q}(\tilde{\mathscr{D}})$).

Let $\pi^{\infty-n}$ denote the projection of $\bar{E}=\Omega^0(M, \mathrm{ad}(P))\otimes\mathscr{A}$ into $\bar{E}^{\infty-n}$. First, let us consider a family of operators which associates, for each $\mathrm{A}\in|\chi_{p,q}(\tilde{\mathscr{D}})|$, the operator $\pi^{\infty-n}\circ\bar{\partial}_A$. We will denote this new family of operators $\{\pi^{\infty-n}\circ\bar{\partial}_A\}$, $A\in|\chi_{p,q}(\tilde{\mathscr{D}})|$ with parameter space $|\chi_{p,q}(\tilde{\mathscr{D}})|$ by $\mathscr{H}_\chi$:

$$\mathscr{H}_\chi:\quad|\chi_{p,q}(\tilde{\mathscr{D}})|\ni A\mapsto\pi^{\infty-n}\circ\bar{\partial}_A\,.$$

Secondly, we will consider the following continuous deformation of the family of operators $\mathscr{H}_\chi$. The invariant cycle $\tilde{\psi}$ will be given as an intersection of an infinite dimensional cycles derived from a deformation family of Fredholm operators $\mathscr{H}_\chi$.

For any infinite matrix $\varepsilon=(\varepsilon_j^r)$, $\varepsilon_j^r\in\boldsymbol{R}$, $r, j=1, 2, \ldots$, we will define a family of operators $\mathscr{H}_\chi^\varepsilon$ as a deformation of $\mathscr{H}_\chi$ as follows. Let $\bar{\pi}_j$ $(j=1, 2, \ldots.)$ denote the projection of $\bar{E}=\Omega^0(M, \mathrm{ad}(P))\otimes|\chi_{p,q}(\tilde{\mathscr{D}})|$ onto $\bar{E}_j/\bar{E}_{j-1}$, the complement space of $\bar{E}_{j-1}$ in $\bar{E}_j$:

$$\bar{\pi}_j: \quad \bar{E}=\Omega^0(M, \mathrm{ad}(P))\otimes|\chi_{p,q}(\tilde{\mathscr{D}})| \to \bar{E}_j/\bar{E}_{j-1}.$$

For given $\varepsilon=(\varepsilon_j^r)$ and $A\in|\chi_{p,q}(\tilde{\mathscr{D}})|$, we can find an operator $\mathscr{H}^{\varepsilon}_{\chi,A}$ which satisfies the conditions:

$$\bar{\pi}_r[\mathscr{H}^{\varepsilon}_{\chi,A}]=\bar{\pi}_r[\pi^{\infty-n}\bar{\partial}_A]+\sum_{1\le j<\infty}\varepsilon_j^r\bar{\pi}_j[\bar{\partial}_A],$$

for $r=1, 2, \ldots$, if we take sufficiently small $\varepsilon_j^r$. This defines a family of operators with the parameter space $|\chi_{p,q}(\tilde{\mathscr{D}})|$ associated to matrix $\varepsilon=(\varepsilon_j^r)$. The matrix $\varepsilon$ is regarded as a parameter of deformation.

Let $\psi^{\varepsilon}_{\chi}$ denote the index cycle associated to the family of operators $\mathscr{H}^{\varepsilon}_{\chi}$. Since the Hamiltonian vector field $X_A(L)$ is integrable, there exists, for every $A_0\in|\chi_{p,q}(\tilde{\mathscr{D}})|$, the orbit of $X_A(L)$, denoted by $g_t(A_0)$, $t\in\boldsymbol{R}$. Then it is easy to see:

$$g_t^{-1}[\mathscr{H}^{\varepsilon}_{\chi,g_t(A_0)}]g_t=\mathscr{H}^{\varepsilon}_{\chi,A_0}.$$

Therefore we see that the index cycle $\psi^{\varepsilon}_{\chi}$, is invariant *via* the Hamiltonian vector field $X(\boldsymbol{L})$.

Following the approximation method in [6], we will show that there exists some $\varepsilon$ such that the corresponding cycle

$$\tilde{\psi}^{\varepsilon}=\psi^{r,s}_{p,q}(\mathscr{H}^{\varepsilon}_{\chi})$$

is invariant with respect to the gradient vector field of the Yang-Mills functional.

The approximation method given in [6] is aimed to deform holomorphic maps to satisfy a transversality condition with varieties in the Grassmannian manifold. We can extend this method to our symplectic category, because all the equations which determines $\mathscr{H}^{\varepsilon}_{\chi}$ are invariant with respect to Hodge $*$-operator. Moreover, we can see that the index cycle $\tilde{\psi}^{\varepsilon}$ corresponding to $\mathscr{H}^{\varepsilon}_{\chi}$ is $X(L)$-invariant for any $\varepsilon$. (In other words, all the invariance we need are being kept during the deformation). It follows that there exists a matrix $\varepsilon$ such that $\mathscr{H}^{\varepsilon}_{\chi}$ satisfies the transversality condition with Koschorke varieties [5] and such that the corresponding index cycle $\tilde{\psi}^{\varepsilon}_{\chi}$ is invariant with respect to the gradient vector field of the Yang-Mills functional. Note that $\tilde{\psi}^{\varepsilon}_{\chi}$ is the intersection of $\chi_{p,q}(\tilde{\mathscr{D}})$ and one of Koschorke varieties [5]. (See for details [6] and [10].)

We can also extend the complex analytic cobordism theory of [10] to symplectic category in a similar way as above. We see finally that the cycle $\tilde{\psi}=\tilde{\psi}^{\varepsilon}_{\chi}$ is cobordant to the index cycle $\tilde{\psi}_0$ in the connection space $\mathscr{A}$. Noticing that all the cycles constructed above are gauge-invariant, this completes the proof of Main Theorem.

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