TOKYO J. MATH.
VOL. 19, NO. 2, 1996

Totally Real Submanifolds in a Quaternion Projective Space*

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(Communicated by T. Nagano)

Abstract. In this paper, we will obtain some new intrinsic rigidity theorems of compact totally real minimal submanifolds in a quaternion projective space. So the corresponding results due to B. Y. Chen and C. S. Houh as well as Y. B. Shen are improved.

1. Introduction.

A quaternion Kaehler manifold is defined as a $4m$-dimensional Riemannian manifold whose holonomy group is contained in $Sp(m)\cdot Sp(1)$ with the additional condition for $m=1$ that it is a self-dual Einstein space. A quaternion projective space $QP^m_{(c)}$ is a quaternion Kaehler manifold with constant quaternion sectional curvature $c>0$. A complex projective space $CP^m_{(c)}$ with constant holomorphic sectional curvature c can be isometrically imbedded in $QP^m_{(c)}$ as a totally geodesic submanifold. Let M be an n-dimensional Riemannian submanifolds and $\mathscr{F}: M\to QP^m_{(c)}$ an isometric immersion of M into $QP^m_{(c)}$. If each tangent 2-subspace of M is mapped by $\mathscr{F}$ into a totally real plane of $QP^m_{(c)}$, then M is called a totally real submanifold of $QP^m_{(c)}$. In [1], B. Y. Chen and C. S. Houh studied the fundamental properties of totally real submanifolds in $QP^m_{(c)}$ and obtained some intrinsic rigidity theorems on scalar curvature and sectional curvature. In [2], Y. B. Shen obtained some intrinsic rigidity theorems and improved the theorems in [1].

Let ρ and $\|\sigma\|^2$ denote the scalar curvature and the length square of the second fundamental form of M respectively, and K_c and Q denote the function which assigns to each point of M the infimum of the sectional curvature and Ricci curvature at that point respectively. Shen obtained the following:

THEOREM A [2, Th. 2.1]. *Let M be an n-dimensional compact totally real minimal submanifold in the quaternion projective space $QP^n_{(c)}$. If $\|\sigma\|^2\le(n+1)(3n+2)c/4(5n+2)$,*

Received February 17, 1995
Revised June 27, 1995
* Project supported by NSFS.

or equivalently $\rho \geq (n-2)(5n^2+4n+1)c/4(5n+2)$, *then either* (i) *M is totally geodesic in* $QP^n_{(c)}$ *or* (ii) $n=2$ *and M is a flat surface in* $QP^2_{(c)}$ *with the parallel second fundamental form and a parallel normal subbundle of fiber dimension* 4.

THEOREM B [2, Th. 3.1]. *Let M be an n-dimensional compact totally real minimal submanifold in* $QP^n_{(c)}$. *If* $K_c \geq (n-2)c/8n$, *then either* (i) *M is totally geodesic or* (ii) $n=2$ *and M is a flat surface in* $QP^2_{(c)}$ *with the parallel second fundamental form and a parallel normal subbundle of fiber dimension* 4.

THEOREM C [2, Th. 3.2]. *Let M be an n-dimensional compact totally real minimal submanifold in* $QP^n_{(c)}$ *with* $n \geq 4$. *If* $Q \geq (n-2-n^{-1})c/4$, *then either* (i) *M is totally geodesic in* $QP^n_{(c)}$ *or* (ii) $n=4$ *and M is a locally symmetric Einstein space which is not of constant curvature.*

In this paper, we shall further study the intrinsic rigidity of compact totally real minimal submanifolds in $QP^n_{(c)}$. We obtain the following:

THEOREM 1. *Let M be an n-dimensional compact totally real minimal submanifold in* $QP^n_{(c)}$. *If* $\|\sigma\|^2 \leq (n+1)c/6$, *or equivalently* $\rho \geq (3n^2-5n-2)c/12$, *then either* (i) *M is totally geodesic in* $QP^n_{(c)}$ *or* (ii) $n=2$ *and M is a flat surface in* $QP^2_{(c)}$ *with the parallel second fundamental form and a parallel normal subbundle of fiber dimension* 4.

THEOREM 2. *Let M be an n-dimensional compact totally real minimal submanifold in* $QP^n_{(c)}$. *If* $K_c \geq (2n-3)c/20n$, *then M is totally geodesic in* $QP^n_{(c)}$.

THEOREM 3. *Let M be an n-dimensional compact totally real minimal submanifold in* $QP^n_{(c)}$. *If* $Q \geq (2n-3)c/10$, *then M is totally geodesic in* $QP^n_{(c)}$.

REMARK. (i) Obviously $(n+1)c/6 > (n+1)(3n+2)c/4(5n+2)$ $(n>2)$. So Theorem 1 improves Theorem A when $n \geq 3$.

(ii) $(2n-3)c/20n < (n-2)c/8n$ $(n>4)$. So Theorem 2 improves Theorem B when $n \geq 5$.

(iii) $(2n-3)c/10 < (n-2-n^{-1})c/4$ $(n>5)$. So Theorem 3 improves Theorem C when $n \geq 6$. Moreover, it contains the results of $n=2, 3$ as well.

2. Basic formulas.

We give here a quick review of basic formulas about totally real submanifolds in a quaternion Kaehler manifold. For details see [1] or [2].

Let $QP^n_{(c)}$ $(n \geq 2)$ be a $4n$-dimensional quaternion projective space with constant quaternion sectional curvature $c>0$, whose almost quaternion structures I, J and K satisfy $IJ=K$, $JK=I$, $KI=J$, $I^2=J^2=K^2=-1$. Let M be an n-dimensional totally real submanifold in $QP^n_{(c)}$. Choose a field of orthonormal frames $e_1, \cdots, e_n$, $e_{I(1)}=Ie_1, \cdots, e_{I(n)}=Ie_n$, $e_{J(1)}=Je_1, \cdots, e_{J(n)}=Je_n$, $e_{K(1)}=Ke_1, \cdots, e_{K(n)}=Ke_n$ in $QP^n_{(c)}$ so that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. With respect to this frame field,

I, J, K have the following forms:

$$I=\begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & -E \\ 0 & 0 & E & 0 \end{pmatrix}, \qquad J=\begin{pmatrix} 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \end{pmatrix},$$

$$K=\begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & -E & 0 \\ 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix},$$

where E stands for the identity $(n\times n)$-matrix. The range of indices is as follows: $A, B, C, \cdots = 1, \cdots, n, I(1), \cdots, I(n), J(1), \cdots, J(n), K(1), \cdots, K(n)$; $i, j, k \cdots = 1, \cdots, n$; $\alpha, \beta, \gamma, \cdots = I(1), \cdots, I(n), J(1), \cdots, J(n), K(1), \cdots, K(n)$; $\varphi = I, J$ or K.

Let ω^A and ω^A_B be the dual frame field and the connection forms with respect to the frame field chosen above. Then the structure equations of $QP^n_{(c)}$ are

$$\begin{aligned} d\omega^A &= -\sum \omega^A_B \wedge \omega^B, \qquad \omega^A_B + \omega^B_A = 0, \\ d\omega^A_B &= -\sum \omega^A_C \wedge \omega^C_B + \frac{1}{2}\sum \bar{R}_{ABCD}\omega^C \wedge \omega^D, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \bar{R}_{ABCD} = \frac{c}{4}(&\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + I_{AC}I_{BD} + I_{AD}I_{BC} + 2I_{AB}I_{CD} + J_{AC}J_{BD} \\ &- J_{AD}J_{BC} + 2J_{AB}J_{CD} + K_{AC}K_{BD} - K_{AD}K_{BC} + 2K_{AB}K_{CD}). \end{aligned} \tag{2.2}$$

Restricting to M, we have

$$\begin{aligned} &\omega^\alpha = 0, \quad \omega^\alpha_i = \sum h^\alpha_{ij}\omega^j, \quad h^\alpha_{ij} = h^\alpha_{ji}, \\ &h^{\varphi(i)}_{jk} = h^{\varphi(j)}_{ki} = h^{\varphi(k)}_{ij}. \end{aligned} \tag{2.3}$$

The second fundamental form σ of M in $QP^n_{(c)}$ is defined as $\sigma = \sum h^\alpha_{ij}\omega^i \otimes \omega^j \otimes e_\alpha$, whose length square is $\|\sigma\|^2 = \sum_{i,j,\alpha}(h^\alpha_{ij})^2$. The Gauss-Codazzi-Ricci equations of M in $QP^n_{(c)}$ are

$$R_{ijkl} = \frac{c}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.4}$$

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{2.5}$$

$$\begin{aligned} R_{\alpha\beta kl} = \frac{c}{4}(&I_{\alpha k}I_{\beta l} - I_{\alpha l}I_{\beta k} + J_{\alpha k}J_{\beta l} - J_{\alpha l}J_{\beta k} \\ &+ K_{\alpha k}K_{\beta l} - K_{\alpha l}K_{\beta k}) + \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}). \end{aligned} \tag{2.6}$$

Suppose M is minimal in $QP^n_{(c)}$, i.e. $\mathrm{tr}\,\sigma=0$. From (2.4),

$$R_{ij}=\frac{c}{4}(n-1)\delta_{ij}-\sum_{\alpha,k}h^\alpha_{ik}h^\alpha_{kj}\,, \tag{2.7}$$

$$\rho=\frac{c}{4}n(n-1)-\|\sigma\|^2\,, \tag{2.8}$$

where R_{ij} is the Ricci tensor of M. If H^α and Δ denote the $(n\times n)$-matrix (h^α_{ij}) and the Laplacian on M respectively, the following formula can be found in [1] or [2]: for any real number a,

$$\frac{1}{2}\Delta(\|\sigma\|^2)=\|\nabla\sigma\|^2+(1+a)\sum_{\alpha,i,j,k,l}h^\alpha_{ij}(h^\alpha_{kl}R_{lijk}+h^\alpha_{il}R_{lkjk})+\frac{1}{2}(1-a)$$
$$\cdot\sum_{\alpha,\beta}\mathrm{tr}(H^\alpha H^\beta-H^\beta H^\alpha)^2+a\sum_{\alpha,\beta}(\mathrm{tr}\,H^\alpha H^\beta)^2-\frac{c}{4}(na-1)\|\sigma\|^2\,. \tag{2.9}$$

3. Proof of main theorems.

Firstly, from [3] we have

LEMMA 3.1 [3, Th.1]. *Let A_1 $A_2,\cdots,A_p$ be symmetric $(n\times n)$-matrices $(p\geq 2)$. Denote $S_\alpha=\mathrm{tr}\,A^2_\alpha$, $S=\sum_\alpha S_\alpha$. Then*

$$\sum_{\alpha,\beta}\mathrm{tr}(A_\alpha A_\beta-A_\beta A_\alpha)^2-\sum_{\alpha,\beta}(\mathrm{tr}\,A_\alpha A_\beta)^2\geq-\frac{3}{2}S^2 \tag{3.1}$$

and the equality holds if and only if one of the following conditions holds: (1) $A_1=A_2=\cdots=A_p=0$; (2) *only two of the matrices $A_1, A_2,\cdots,A_p$ are different from zero. Moreover, assuming $A_1\neq 0$, $A_2\neq 0$, $A_3=\cdots=A_p=0$, then $S_1=S_2$ and there exists an orthogonal $(n\times n)$-matrix T such that*

$$TA_1T^t=\sqrt{\frac{S_1}{2}}\left(\begin{array}{cc|c}1&0&0\\0&-1&\\\hline \multicolumn{2}{c|}{0}&0\end{array}\right),\qquad TA_2T^t=\sqrt{\frac{S_2}{2}}\left(\begin{array}{cc|c}0&1&0\\1&0&\\\hline \multicolumn{2}{c|}{0}&0\end{array}\right),$$

where T^t denotes the transposed matrix of T.

We can also establish the following lemmas.

LEMMA 3.2. *Let M be an n-dimensional totally real minimal submanifold in $QP^n_{(c)}$. Then*

$$\sum_{\alpha,\beta}\mathrm{tr}(H^\alpha H^\beta-H^\beta H^\alpha)^2\geq[4Q-(n-1)c]\|\sigma\|^2+\frac{4}{n}\sum_\alpha[\mathrm{tr}(H^\alpha)^2]^2\,. \tag{3.2}$$

PROOF. For a fixed α, let λ^α_i be the eigenvalues of the matrix H^α and choose $\{e_i\}$ (tangential part of the frame fields) so that H^α is diagonal and $h^\alpha_{ii}=\lambda^\alpha_i$. Then we see from (2.7) that

$$\sum_{\substack{i\\ \beta\neq\alpha}} (h^{\beta}_{ik})^2 \leq \frac{c}{4}(n-1)-Q-(\lambda^{\alpha}_k)^2 \qquad \text{for each} \quad k\,.$$

Then

$$-\sum_{\beta}\mathrm{tr}(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})^2=\sum_{\substack{i,k\\ \beta\neq\alpha}}(h^{\beta}_{ik})^2(\lambda^{\alpha}_i-\lambda^{\alpha}_k)^2\leq 4\sum_{\substack{i,k\\ \beta\neq\alpha}}(h^{\beta}_{ik})^2(\lambda^{\alpha}_k)^2$$

$$\leq 4\sum_k\left(\frac{c}{4}(n-1)-Q-(\lambda^{\alpha}_k)^2\right)(\lambda^{\alpha}_k)^2=[(n-1)c-4Q]\sum_k(\lambda^{\alpha}_k)^2-4\sum_k(\lambda^{\alpha}_k)^4$$

$$\leq[(n-1)c-4Q]\sum_k(\lambda^{\alpha}_k)^2-\frac{4}{n}\left(\sum_k(\lambda^{\alpha}_k)^2\right)^2.$$

Taking the sum over α, we get (3.2).

LEMMA 3.3. *Let M be an n-dimensional totally real submanifold in $QP^n_{(c)}$. Then*

$$\sum_{\alpha,\beta}\mathrm{tr}[(H^{\alpha})^2(H^{\beta})^2]=\sum_{\alpha,\beta}(\mathrm{tr}\,H^{\alpha}H^{\beta})^2\,. \tag{3.3}$$

PROOF. From (2.3), we get

$$\sum_{\alpha,\beta}\mathrm{tr}[(H^{\alpha})^2(H^{\beta})^2]=\sum_{\substack{\alpha,\beta\\ l,m,n,k}}h^{\alpha}_{kl}h^{\alpha}_{lm}h^{\beta}_{mn}h^{\beta}_{nk}=\sum_{\substack{\varphi,i,j\\ l,m,n,k}}h^{\varphi(i)}_{kl}h^{\varphi(i)}_{lm}h^{\varphi(j)}_{mn}h^{\varphi(j)}_{nk}$$

$$=\sum_{\substack{\varphi,k,m\\ i,j,l,n}}h^{\varphi(k)}_{il}h^{\varphi(m)}_{li}h^{\varphi(m)}_{jn}h^{\varphi(k)}_{nj}=\sum_{\alpha,\beta}(\mathrm{tr}\,H^{\alpha}H^{\beta})^2\,.$$

LEMMA 3.4. *Let M be an n-dimensional totally real minimal submanifold in $QP^n_{(c)}$. Then*

$$-\sum_{\alpha,\beta}(\mathrm{tr}\,H^{\alpha}H^{\beta})^2\geq\|\sigma\|^2\left(Q-\frac{c}{4}(n-1)\right). \tag{3.4}$$

PROOF. Firstly we note that the following formula holds:

$$\sum_{\alpha,\beta}\mathrm{tr}(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})^2=-2\sum_{\alpha,\beta}\{\mathrm{tr}[(H^{\alpha})^2(H^{\beta})^2]-\mathrm{tr}(H^{\alpha}H^{\beta})^2\}\,. \tag{3.5}$$

Putting $a=0$ and $a=1$ in (2.9), we get two equalities. Taking the difference of the two, and then using (3.5), we have

$$-\sum_{\alpha,\beta}(\mathrm{tr}\,H^{\alpha}H^{\beta})^2=\sum_{\alpha,i,j,k,l}h^{\alpha}_{ij}(h^{\alpha}_{kl}R_{lijk}+h^{\alpha}_{li}R_{lkjk})-\frac{1}{2}\sum_{\alpha,\beta}\mathrm{tr}(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})^2-\frac{n}{4}c\|\sigma\|^2$$

$$=\sum_{\alpha,i,j,k,l}h^{\alpha}_{ij}(h^{\alpha}_{kl}R_{lijk}+h^{\alpha}_{li}R_{lkjk})+\sum_{\alpha,\beta}\{\mathrm{tr}[(H^{\alpha})^2(H^{\beta})^2]-\mathrm{tr}(H^{\alpha}H^{\beta})^2\}-\frac{n}{4}c\|\sigma\|^2\,.$$

For each fixed α, choose $\{e_i\}$ as in the proof of Lemma 3.2. Then we see $\sum_{i,j}\lambda_i^\alpha\lambda_j^\alpha=(\operatorname{tr}H^\alpha)^2=0$. Moreover, from Lemma 3.3, $\sum_{\alpha,\beta}\operatorname{tr}[(H^\alpha)^2(H^\beta)^2]=\sum_\alpha(\sum_{i,j,\beta}\lambda_i^\alpha\lambda_j^\alpha h_{ii}^\beta h_{jj}^\beta)$ (note that H_{ij}^β depend on α here). Then, by (2.4), the above becomes:

$$
\begin{aligned}
&=\sum_\alpha\left(\frac{1}{2}\sum_{i,j}(\lambda_i^\alpha-\lambda_j^\alpha)^2R_{ijij}+\sum_{i,j,\beta}\lambda_i^\alpha\lambda_j^\alpha h_{ii}^\beta h_{jj}^\beta-\sum_{i,j,\beta}\lambda_i^\alpha\lambda_j^\alpha h_{ij}^\beta h_{ij}^\beta\right)-\frac{n}{4}c\|\sigma\|^2\\
&=\sum_\alpha\left[\sum_{i,j}(\lambda_i^\alpha)^2R_{ijij}-\sum_{i,j}\lambda_i^\alpha\lambda_j^\alpha\left(R_{ijij}+\sum_\beta h_{ij}^\beta h_{ij}^\beta-\sum_\beta h_{ii}^\beta h_{jj}^\beta\right)\right]-\frac{n}{4}c\|\sigma\|^2\\
&=\sum_\alpha\sum_i(\lambda_i^\alpha)^2R_{ii}-\sum_{\alpha,i,j}\lambda_i^\alpha\lambda_j^\alpha\frac{c}{4}(\delta_{ii}\delta_{jj}-\delta_{ij}\delta_{ij})-\frac{n}{4}c\|\sigma\|^2\\
&\geq Q\sum_{\alpha,i}(\lambda_i^\alpha)^2-\frac{c}{4}\sum_{\alpha,i,j}\lambda_i^\alpha\lambda_j^\alpha+\frac{c}{4}\sum_{\alpha,i}(\lambda_i^\alpha)^2-\frac{n}{4}c\|\sigma\|^2\\
&=Q\|\sigma\|^2+\frac{c}{4}\|\sigma\|^2-\frac{n}{4}c\|\sigma\|^2=\left(Q-\frac{c}{4}(n-1)\right)\|\sigma\|^2 .
\end{aligned}
$$

LEMMA 3.5. *Let M be an n-dimensional totally real submanifold in $QP^n_{(c)}$. Then*

$$\sum_{\alpha,\beta}(\operatorname{tr}H^\alpha H^\beta)^2+\sum_{\alpha,\beta}\operatorname{tr}(H^\alpha H^\beta)^2\geq 0 . \tag{3.6}$$

PROOF. For a fixed α, choose $\{e_i\}$ again as in the proof of Lemma 3.2. Then

$$\operatorname{tr}[(H^\alpha)^2(H^\beta)^2]+\operatorname{tr}(H^\alpha H^\beta)^2=\frac{1}{2}\sum_{k,l}(h_{kl}^\beta)^2(\lambda_k^\alpha+\lambda_l^\alpha)^2\geq 0 .$$

Taking the sum over β, and then α, and using Lemma 3.3, we get (3.6).

It is also easy to see in [1, p. 194] that

LEMMA 3.6. *Let M be an n-dimensional totally real minimal submanifold in $QP^n_{(c)}$. Then*

$$\sum_{\alpha,i,j,k,l}h_{ij}^\alpha(h_{kl}^\alpha R_{lijk}+h_{il}^\alpha R_{lkjk})\geq nK_c\|\sigma\|^2 , \tag{3.7}$$

and the equality holds if and only if $R_{ijij}=K_c$ for all i, j $(i\neq j)$.

PROOF OF THEOREM 1. Putting $a=-1$ in (2.9), we get

$$\frac{1}{2}\Delta(\|\sigma\|^2)=\|\nabla\sigma\|^2+\sum_{\alpha,\beta}\operatorname{tr}(H^\alpha H^\beta-H^\beta H^\alpha)^2-\sum_{\alpha,\beta}(\operatorname{tr}H^\alpha H^\beta)^2+\frac{c}{4}(n+1)\|\sigma\|^2 . \tag{3.8}$$

If $\|\sigma\|^2\leq(n+1)c/6$, then from (3.8) and Lemma 3.1, we have

$$\frac{1}{2}\Delta(\|\sigma\|^2)\geq\|\nabla\sigma\|^2-\frac{3}{2}\|\sigma\|^4+\frac{c}{4}(n+1)\|\sigma\|^2$$

$$= \|\nabla\sigma\|^2 + \|\sigma\|^2\left(-\frac{3}{2}\|\sigma\|^2 + \frac{c}{4}(n+1)\right) \geq 0\,. \tag{3.9}$$

Since M is compact, by Hopf's lemma, $\Delta(\|\sigma\|^2)=0$ and all equalities in (3.9) hold. Thus we have either $\|\sigma\|^2=0$, i.e., M is totally geodesic, or $\|\sigma\|^2=(n+1)c/6$. Moreover, the equality in (3.1) holds, where A_α in (3.1) should be read as H^α.

Assume M is not totally geodesic. Then $\|\sigma\|^2=(n+1)c/6\neq 0$. By Lemma 3.1, we may assume without loss of generality that

$$H^{I(1)} = \frac{\sqrt{(n+1)c/6}}{2}\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \qquad H^{I(2)} = \frac{\sqrt{(n+1)c/6}}{2}\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)$$

and $H^\alpha=0$ for $\alpha\neq I(1)$, $I(2)$. Then, as in [4, p. 70], by simple explicit computations, we see that $n=2$. In that case, our theorem coincides with Theorem A since $(n+1)c/6=(n+1)(3n+2)c/4(5n+2)$ if $n=2$, and nothing is left to be proved.

Proof of Theorem 2. Put $a=2/3$ in (2.9). If $K_c\geq(2n-3)c/20n$, using firstly (3.5) and (3.7), and then Lemmas 3.3 and 3.5, we have

$$\begin{aligned}\frac{1}{2}\Delta(\|\sigma\|^2) &\geq \|\nabla\sigma\|^2 + \frac{5}{3}nK_c\|\sigma\|^2 - \frac{1}{3}\sum_{\alpha,\beta}\operatorname{tr}[(H^\alpha)^2(H^\beta)^2] \\ &\quad + \frac{1}{3}\sum_{\alpha,\beta}\operatorname{tr}(H^\alpha H^\beta)^2 + \frac{2}{3}\sum_{\alpha,\beta}(\operatorname{tr} H^\alpha H^\beta)^2 - \frac{c}{4}\left(\frac{2}{3}n-1\right)\|\sigma\|^2 \\ &\geq \|\nabla\sigma\|^2 + \frac{1}{12}\|\sigma\|^2[20nK_c-(2n-3)c] \geq 0\,. \end{aligned} \tag{3.10}$$

Then, by Hopf's lemma again, $\Delta(\|\sigma\|^2)=0$ and all equalities in (3.10) hold. Then we have either $\|\sigma\|^2=0$, i.e., M is totally geodesic, or $K_c=(2n-3)c/20n$. Moreover, the equality in (3.7) holds.

Assume M is not totally geodesic, i.e., $K_c=(2n-3)c/20n$. Then it follows from Lemma 3.6 that M has constant sectional curvature K_c which is positive for $n\geq 2$. On the other hand, according to [1, Th. 9], K_c cannot be positive, which is a contradiction. So this case cannot occur, completing the proof.

Proof of Theorem 3. Let $Q\geq(2n-3)c/10$. From (3.8), using Lemmas 3.2 and 3.4, we have

$$\begin{aligned}\frac{1}{2}\Delta(\|\sigma\|^2) &\geq \|\nabla\sigma\|^2 + [4Q-(n-1)c]\|\sigma\|^2 + \frac{4}{n}\sum_\alpha[\operatorname{tr}(H^\alpha)^2]^2 \\ &\quad + \|\sigma\|^2\left(Q - \frac{c}{4}(n-1)\right) + \frac{c}{4}(n+1)\|\sigma\|^2\end{aligned}$$

$$= \|\nabla\sigma\|^2 + \left(5Q - \frac{2n-3}{2}c\right)\|\sigma\|^2 + \frac{4}{n}\sum_\alpha [\mathrm{tr}(H^\alpha)^2]^2$$

$$\geq \|\nabla\sigma\|^2 + \frac{1}{2}\|\sigma\|^2[10Q-(2n-3)c] \geq 0\,. \tag{3.11}$$

Then, similarly as before, we have $\Delta(\|\sigma\|^2)=0$ and all equalities in (3.11) hold. In particular, we get $\sum_\alpha [\mathrm{tr}(H^\alpha)^2]^2=0$. Therefore, for every α, $\mathrm{tr}(H^\alpha)^2=0$. Hence $\|\sigma\|^2=\sum_\alpha \mathrm{tr}(H^\alpha)^2=0$, i.e., M is totally geodesic in $QP^n_{(c)}$. This completes the proof of Theorem 3.

Give thanks to Professor Wang Xin-min for his kind help. Thanks are also due to the referee for his kind advice and opinion to this paper.

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