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Bifurcation for Nonlinear Elliptic Boundary Value Problems II

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Abstract. This paper is a continuation of the previous paper [Ta] where we studied local static bifurcation theory for a class of *degenerate* boundary value problems for semilinear second-order elliptic differential operators which includes as particular cases the Dirichlet and Neumann problems. This paper is devoted to *global* static bifurcation theory.

Introduction and results.

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, $N\geq 2$, with C^∞ boundary ∂D; its closure $\bar{D}=D\cup\partial D$ is an n-dimensional, compact C^∞ manifold with boundary. We let

$$Au(x)=-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N}a^{ij}(x)\frac{\partial u}{\partial x_j}(x)\right)+c(x)u(x)$$

be a second-order, *elliptic* differential operator with real C^∞ coefficients on $\bar{D}$ such that:

1) $a^{ij}(x)=a^{ji}(x)$, $x\in\bar{D}$, $1\leq i,j\leq N$, and there exists a constant $a_0>0$ such that

$$\sum_{i,j=1}^{N}a^{ij}(x)\xi_i\xi_j\geq a_0|\xi|^2,\qquad x\in\bar{D},\ \xi\in\mathbf{R}^N.$$

2) $c(x)\geq 0$ on $\bar{D}$.

We consider the following linear elliptic boundary value problem: Given function f defined in D, find a function u in D such that

$$(*)\qquad\begin{cases}Au=f & \text{in}\quad D\,,\\[4pt] Bu=a\dfrac{\partial u}{\partial \mathbf{v}}+bu=0 & \text{on}\quad \partial D\,.\end{cases}$$

Here:

1) $a\in C^\infty(\partial D)$ and $a\geq 0$ on ∂D.

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2) $b \in C^\infty(\partial D)$ and $b \geq 0$ on ∂D.

3) $\partial/\partial \boldsymbol{\nu}$ is the conormal derivative associated with the operator A: $\partial/\partial \boldsymbol{\nu} = \sum_{i,j=1}^N a^{ij} n_j \partial/\partial x_i$, where $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ is the unit exterior normal to the boundary ∂D.

It is worth pointing out here that problem (*) is nondegenerate (or coercive) if and only if either $a > 0$ on ∂D or $a \equiv 0$ and $b > 0$ on ∂D. In particular, if $a \equiv 1$ and $b \equiv 0$ on ∂D (resp. $a \equiv 0$ and $b \equiv 1$ on ∂D), then the boundary condition L is the so-called Neumann (resp. Dirichlet) condition.

First we study problem (*) in the framework of L^2 spaces. To do so, we associate with problem (*) an unbounded linear operator $\mathscr{A}$ from the Hilbert space $L^2(D)$ into itself as follows:

(a) The domain of definition $D(\mathscr{A})$ of $\mathscr{A}$ is the space

$$D(\mathscr{A}) = \{u \in H^2(D)\,;\ Bu = 0 \text{ on } \partial D\}\,.$$

(b) $\mathscr{A}u = Au$, $u \in D(\mathscr{A})$.

Our starting point is the following (cf. [Ta, Theorem 1]):

THEOREM 0. *Assume that the following hypotheses* (H1) *and* (H2) *are satisfied*:

(H1) $b(x') > 0$ *on* $M = \{x' \in \partial D\,;\ a(x') = 0\}$.

(H2) $c(x) > 0$ *in* D.

Then the operator $\mathscr{A}$ *is a nonnegative, selfadjoint operator in the space* $L^2(D)$. *Moreover, the spectrum of* $\mathscr{A}$ *is discrete and the eigenvalues of* $\mathscr{A}$ *have finite multiplicities. In particular, the first eigenvalue* λ_1 *of* $\mathscr{A}$ *is positive and* simple, *and the associated eigenfunction* φ_1 *is* positive *everywhere in* D.

Now, as an application of Theorem 0, we consider global static bifurcation problems for the following nonlinear elliptic boundary value problem:

$$(**)\qquad \begin{cases} Au - \lambda u + h(u) = 0 & \text{in} \quad D\,, \\ Bu = a\dfrac{\partial u}{\partial \boldsymbol{\nu}} + bu = 0 & \text{on} \quad \partial D\,. \end{cases}$$

Here λ is a real parameter and $h(t)$ is a real-valued function on $\mathbf{R}$, not depending explicitly on x.

A solution $u \in C^2(\bar{D})$ of problem (**) is said to be *nontrivial* if it does not identically equal zero on $\bar{D}$. We call a nontrivial solution u of problem (**) a *positive solution* (resp. *negative solution*) if $u(x) \geq 0$ (resp. $u(x) \leq 0$) on $\bar{D}$.

By using the bifurcation theory from a simple eigenvalue due to Crandall and Rabinowitz [CR], we can prove that there exist precisely two nontrivial branches of solutions of problem (**) bifurcating at the point $(\lambda_1, 0)$ where λ_1 is the first eigenvalue of $\mathscr{A}$ (cf. [Ta, Theorem 3]). The forthcoming two theorems characterize them *globally*.

The first theorem is a generalization of Szulkin [Sz, Theorem 1.3] to the degenerate case:

THEOREM 1. *Let λ_1 be the first eigenvalue of $\mathscr{A}$, and let h be a function of class C^1 on $\mathbf{R}$ such that $h(0)=0$ and $h'(0)=0$. Assume that the derivative h' is strictly decreasing for $t<0$ and strictly increasing for $t>0$, and that there exist constants $k_->0$ and $k_+>0$ such that*

$$\lim_{t\to-\infty} h'(t)=k_-, \qquad \lim_{t\to+\infty} h'(t)=k_+ .$$

Then the point $(\lambda_1,0)$ is a bifurcation point of problem (**). More precisely, the set of nontrivial solutions of problem (**) consists of two C^1 curves Γ_- and Γ_+ parametrized respectively by λ as follows (cf. Figure 1):

$$\Gamma_-=\{(\lambda,u_-(\lambda))\in\mathbf{R}\times C(\bar{D})\,;\ \lambda_1\le\lambda<\lambda_1+k_-\},$$
$$\Gamma_+=\{(\lambda,u_+(\lambda))\in\mathbf{R}\times C(\bar{D})\,;\ \lambda_1\le\lambda<\lambda_1+k_+\}.$$

The branch Γ_- is negative and the branch Γ_+ is positive except at $(\lambda_1,0)$, and the uniform norms $\| u_-(\lambda) \|$ and $\| u_+(\lambda) \|$ tend to ∞ as $\lambda\to\lambda_1+k_-$ and as $\lambda\to\lambda_1+k_+$, respectively. Furthermore, problem (**) has no other positive or negative solutions for all $\lambda\ge\lambda_1$.

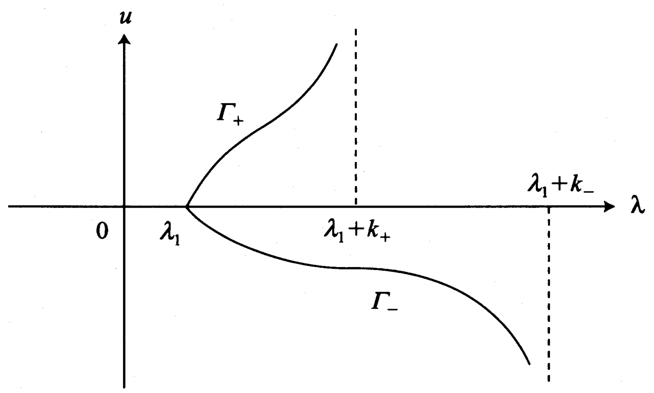

FIGURE 1

EXAMPLE 1. For Theorem 1, we give an example of the function $h(t)$:

$$h(t)=\begin{cases} k_+(t+1/(2t)-4/3) & \text{for } t>1, \\ (k_+/6)t^3 & \text{for } 0\le t\le 1, \\ (k_-/6)t^3 & \text{for } -1\le t\le 0, \\ k_-(t+1/(2t)+4/3) & \text{for } t<-1. \end{cases}$$

The second theorem asserts that if the function h is bounded, then the bifurcation curves "turn back" towards λ_1. More precisely, we have the following generalization of [Sz, Theorem 5.2] to the degenerate case:

THEOREM 2. *Let λ_1, λ_2 be the first and second eigenvalues of $\mathscr{A}$, respectively, and let h be a function of class C^1 on $\mathbf{R}$ such that $h(0)=0$ and $h'(0)=0$. Assume that h is* *bounded and that there exists a constant* $k>0$ *such that*

$$0 \leq h'(t) \leq k < \lambda_2 - \lambda_1 \qquad \textit{for all } t \in \mathbf{R}\,.$$

Then the set of nontrivial solutions of problem (**), *bifurcating at* $(\lambda_1, 0)$, *consists of two* C^1 *branches* Γ_1 *and* Γ_2. *The branches* Γ_1 *and* Γ_2 *may be parametrized respectively by* s *as follows* (*cf. Figure* 2):

$$\Gamma_1 = \{(\lambda^1(s), u^1(s)) \in \mathbf{R} \times C(\bar{D})\,;\ 0 \leq s < \infty\}\,,$$
$$\Gamma_2 = \{(\lambda^2(s), u^2(s)) \in \mathbf{R} \times C(\bar{D})\,;\ 0 \leq s < \infty\}\,.$$

Here $(\lambda^i(0), u^i(0)) = (\lambda_1, 0)$ *and* $\lambda^i(s) \to \lambda_1$ *as* $s \to \infty$ $(i = 1, 2)$.

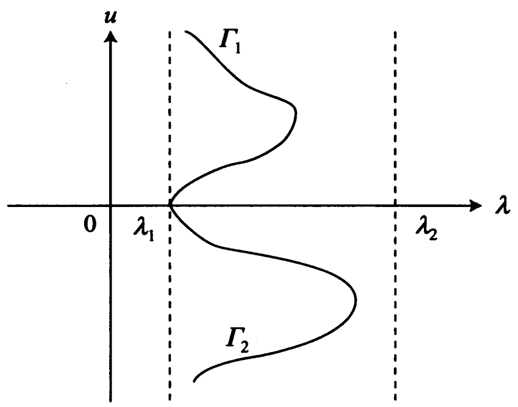

FIGURE 2

EXAMPLE 2. For Theorem 2, we give an example of the function $h(t)$:

$$h(t) = \begin{cases} k(-1/(2t) + 2/3) & \text{for } t > 1\,, \\ (k/6)t^3 & \text{for } -1 \leq t \leq 1\,, \\ k(-1/(2t) - 2/3) & \text{for } t < -1\,. \end{cases}$$

The rest of this paper is organized as follows. In Section 1 we give an existence and uniqueness theorem for problem (*) in the framework of Sobolev spaces of L^p style (Theorem 1.1) which will play an essential role in the proof of Theorem 1. In Section 2 we study problem (**) and prove Theorems 1 and 2. Problem (**) is reduced to the study of an operator equation for the resolvent K of problem (*) (equation (2.1)). This equation is solved by using the theory of positive mappings in ordered Banach spaces (cf. [Am2], [Da]), just as in [Sz]. The essential step in the proof is Proposition 2.2 where the compactness and strong positivity of K are proved.

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1. Existence and uniqueness theorem for problem (*).

We study problem (*) in the framework of Sobolev spaces of L_p style. If k is a nonnegative integer and $1<p<\infty$, we define the Sobolev space

$H^{k,p}(D)$ = the space of (equivalence classes of) functions $u \in L^p(D)$ whose derivatives $D^\alpha u$, $|\alpha| \leq k$, in the sense of distributions are in $L^p(D)$.

Then we can obtain the following existence and uniqueness theorem for problem (*) (cf. [Um, Theorem 1]):

THEOREM 1.1. *If hypotheses* (H1) *and* (H2) *are satisfied, then the mapping*

$$A: H_B^{k,p}(D) \longrightarrow H^{k-2,p}(D)$$

is an algebraic and topological isomorphism *for all integer* $k \geq 2$. *Here*

$$H_B^{k,p}(D) = \{u \in H^{k,p}(D)\,;\ Bu=0 \text{ on } \partial D\}\,.$$

2. Proof of Theorems 1 and 2.

2.1. Reduction to an operator equation. By Theorem 1.1, we can introduce a continuous linear operator

$$K: H^{k-2,p}(D) \longrightarrow H_B^{k,p}(D)$$

as follows: For any $v \in H^{k-2,p}(D)$, the function $u=Kv \in H^{k,p}(D)$ is the unique solution of the problem

$$\begin{cases} Au=v & \text{in} \quad D\,, \\ Bu=0 & \text{on} \quad \partial D\,. \end{cases}$$

Then we find that problem (**) is equivalent to the following operator equation:

$$\lambda K u - K(h(u)) = u \qquad \text{in} \quad C(\bar{D})\,. \tag{2.1}$$

Indeed, it suffices to note that the operator K can be uniquely extended to an operator K: $C(\bar{D}) \to C^1(\bar{D})$, and also an operator K: $C^1(\bar{D}) \to C^2(\bar{D})$, since we have, by Sobolev's imbedding theorem,

$$C^k(\bar{D}) \subset H^{k,p}(D) \subset C^{k-N/p}(\bar{D})$$

if $p>N$. Here we remark that, by the Ascoli-Arzelà theorem, the operators K: $C(\bar{D}) \to C^1(\bar{D})$ and K: $C^1(\bar{D}) \to C^2(\bar{D})$ are compact.

2.2. Theory of positive mappings in ordered Banach spaces. We make use of the theory of positive operators in ordered Banach spaces to find nontrivial solutions of equation (2.1) (cf. [Am2]).

A Banach space X is called an *ordered Banach space* if it is an ordered set. For an ordered Banach space X having the ordering $\leq$, the set $Q=\{x\in X;\ x\geq 0\}$ is called the *positive cone* in X.

For functions u and v in $C(\bar{D})$, we write $u\leq v$ if $u(x)\leq v(x)$ for all $x\in\bar{D}$. Then the space $C(\bar{D})$ becomes an ordered Banach space with the ordering $\leq$. Moreover, if we let $P=\{u\in C(\bar{D});\ u\geq 0\}$, then the set P is the positive cone in $C(\bar{D})$.

Now we introduce an ordered Banach space which is associated with the operator K: $C(\bar{D})\to C^1(\bar{D})$. To do so, we need the following:

LEMMA 2.1. *Assume that hypotheses* (H1) *and* (H2) *are satisfied. If* $v\in C^1(\bar{D})$ *and if* $v\geq 0$ *but* $v\not\equiv 0$ *on* $\bar{D}$, *then the function* $u=Kv\in C^2(\bar{D})$ *satisfies the following conditions*:

(1) $u(x')=0$ *on* $M=\{x'\in\partial D;\ a(x')=0\}$.

(2) $u(x)>0$ *on* $\bar{D}\backslash M$.

(3) *For the conormal derivative* $\partial u/\partial\boldsymbol{\nu}$ *of* u, *we have* $(\partial u/\partial\boldsymbol{\nu})(x')<0$ *on* M.

Furthermore, the operator K: $C(\bar{D})\to C(\bar{D})$ *is* positive, *that is*, $K(P)\subset P$.

PROOF. The lemma follows by using Theorem 1.1 and the maximum principle, just as in the proof of [Ta, Proposition 7.5]. Indeed, it suffices to note that the operator K is nothing but the resolvent of problem (*). □

If we let $e=K1$, it follows from Lemma 2.1 that the function $e\in C^2(\bar{D})$ satisfies:

$$\begin{cases} e(x')=0 & \text{on} \quad M, \\ e(x)>0 & \text{on} \quad \bar{D}\backslash M, \\ (\partial e/\partial\boldsymbol{\nu})(x')<0 & \text{on} \quad M. \end{cases}$$

Further we let

$$C_e(\bar{D})=\{u\in C(\bar{D});\ \text{there is a constant } c>0 \text{ such that } -ce\leq u\leq ce\}.$$

Then the space $C_e(\bar{D})$ is given a norm by the formula

$$\|u\|_e=\inf\{c>0;\ -ce\leq u\leq ce\}.$$

It is easy to verify that the space $C_e(\bar{D})$ is an ordered Banach space having the positive cone $P_e=C_e(\bar{D})\cap P$ with nonempty interior.

The next proposition, which is a generalization of [Am1, Lemma 5.3] to the degenerate case, is the essential step in the proof of Theorem 1:

PROPOSITION 2.2. *The operator* K *maps* $C(\bar{D})$ compactly *into* $C_e(\bar{D})$. *Moreover,* K *is* strongly positive, *that is, if* $v\in P$ *and* $v\not\equiv 0$ *on* $\bar{D}$, *then the function* Kv *is an interior point of* P_e.

PROOF. (i) First, by the positivity of K, we find that K maps $C(\bar{D})$ into $C_e(\bar{D})$. Indeed, since we have $-\|v\| \leq v(x) \leq \|v\|$ on $\bar{D}$ for all $v \in C(\bar{D})$, it follows that

$$-\|v\| K1(x) \leq Kv(x) \leq \|v\| K1(x) \qquad \text{on} \quad \bar{D}.$$

This proves that $-ce \leq Kv \leq ce$ with $c = \|v\|$.

(ii) Next we prove that $K: C(\bar{D}) \to C_e(\bar{D})$ is compact. To do so, we let

$$C_B^1(\bar{D}) = \{u \in C^1(\bar{D})\,;\ Bu = 0 \text{ on } \partial D\}.$$

Since K maps $C(\bar{D})$ compactly into $C_B^1(\bar{D})$, it suffices to show that the inclusion mapping $\iota: C_B^1(\bar{D}) \to C_e(\bar{D})$ is continuous.

(ii-a) We verify that ι maps $C_B^1(\bar{D})$ into $C_e(\bar{D})$.

Let u be an arbitrary function in $C_B^1(\bar{D})$. Since we have for some neighborhood ω of M in ∂D

$$\begin{cases} b > 0 & \text{in} \quad \omega, \\ \partial e/\partial \boldsymbol{v} < 0 & \text{in} \quad \omega, \end{cases}$$

it follows that

$$\frac{u}{e} = \frac{(-a/b)\partial u/\partial \boldsymbol{v}}{(-a/b)\partial e/\partial \boldsymbol{v}} = \frac{\partial u/\partial \boldsymbol{v}}{\partial e/\partial \boldsymbol{v}} \qquad \text{in} \quad \omega \setminus M.$$

Hence there exists a constant $c_1 > 0$ such that $|u(x')| \leq c_1 e(x')$ in ω. Thus, by using Taylor's formula, we can find a neighborhood W of ω in D and a constant $c_2 > 0$ such that $|u(x)| \leq c_2 e(x)$ in W.

On the other hand, since we have, for some constant $\alpha > 0$, $e(x) \geq \alpha$ on $\bar{D} \setminus W$, we can find a constant $c_3 > 0$ such that $|u(x)/e(x)| \leq c_3$ on $\bar{D} \setminus W$.

Therefore, there exists a constant $c > 0$ such that $-ce(x) \leq u(x) \leq ce(x)$ on $\bar{D}$. This proves that $u \in C_e(\bar{D})$.

(ii-b) Now assume that

$$\begin{cases} u_j \in C_B^1(\bar{D}), \\ u_j \longrightarrow u & \text{in } C_B^1(\bar{D}), \\ u_j \longrightarrow v & \text{in } \ C_e(\bar{D}). \end{cases}$$

Then there exists a sequence $\{c_j\}$, $c_j \to 0$, such that $\|u_j - v\| \leq c_j \|e\|$. This implies that $u_j \to v$ in $C(\bar{D})$. Hence we have $u = v$. By the closed graph theorem, it follows that the mapping ι is continuous.

(iii) It remains to prove the strong positivity of K.

(iii-a) We show that, for any $v \geq 0$ but $v \not\equiv 0$ on $\bar{D}$, there exist constants $\beta > 0$ and $\gamma > 0$ such that

$$\beta e(x) \leq Kv(x) \leq \gamma e(x) \qquad \text{on} \quad \bar{D}. \tag{2.2}$$

By the positivity of K, one may modify the function v in such a way that $v \in C^1(\bar{D})$.

Furthermore, since the functions $u=Kv$ and $e=K1$ vanish only on the set M, it suffices to prove that there exists a neighborhood W of M in D such that

$$\beta e(x)\leq u(x) \quad \text{in} \quad W. \tag{2.2'}$$

We recall that in a neighborhood ω of M in ∂D,

$$u=\left(-\frac{a}{b}\right)\frac{\partial u}{\partial \mathbf{v}}, \quad \frac{\partial u}{\partial \mathbf{v}}<0 \quad \text{in} \quad \omega,$$

$$e=\left(-\frac{a}{b}\right)\frac{\partial e}{\partial \mathbf{v}}, \quad \frac{\partial e}{\partial \mathbf{v}}<0 \quad \text{in} \quad \omega.$$

Thus we have for β sufficiently small

$$u(x')-\beta e(x')\geq 0, \quad \frac{\partial}{\partial \mathbf{v}}(u-\beta e)(x')<0 \quad \text{in} \quad \omega.$$

Therefore, by using Taylor's formula, we can find a neighborhood W of M in D such that

$$u(x)-\beta e(x)\geq 0 \quad \text{in} \quad W.$$

This proves estimate (2.2').

(iii-b) Finally we show that the function $u=Kv$ is an interior point of P_e. Take $\varepsilon=\beta/2$, where β is the same constant as in estimate (2.2). Then, for all functions $w\in C_e(\bar{D})$ satisfying $\| w-Kv \|_e<\varepsilon$, we have by estimate (2.2)

$$w\leq Kv+\varepsilon e\leq(\gamma+\varepsilon)e, \qquad w\geq Kv-\varepsilon e\geq \beta e/2.$$

This implies that $w\in P_e$, that is, the function Kv is an interior point of P_e.

The proof of Proposition 2.2 is now complete. □

Now let m be a function in $C(\bar{D})$ such that $m(x)>0$ on $\bar{D}$, and consider the following eigenvalue problem for the operator K:

$$K(mu)=\mu u \quad \text{in} \quad C(\bar{D}). \tag{2.3}$$

This problem has a countable number of positive eigenvalues, $\mu_j(m)$, which may accumulate only at 0. Hence they may be arranged in a decreasing sequence $\mu_1(m)\geq \mu_2(m)\geq\cdots$, where each eigenvalue is repeated according to its multiplicity.

In the proof of Theorem 1, we need the following two results about problem (2.3):

Proposition 2.3. *The largest eigenvalue $\mu_1(m)$ is* simple, *i.e., $\mu_1(m)>\mu_2(m)$, and has a* positive *eigenfunction. No other eigenvalues have positive eigenfunctions.*

Proof. Proposition 2.2 tells us that the operator K: $C(\bar{D})\to C_e(\bar{D})$ is strongly positive and compact. Hence the assertions follow from an application of [Am2, Theorem 3.2]. □

PROPOSITION 2.4. *If* $m_1(x) \le m_2(x)$ *for all* $x \in \bar{D}$, *then we have* $\mu_j(m_1) \le \mu_j(m_2)$ *for all* j. *If* $m_1(x) < m_2(x)$ for almost all $x \in \bar{D}$, *then we have* $\mu_j(m_1) < \mu_j(m_2)$ *for all* j.

PROOF. The proposition is an immediate consequence of the well-known minimax property of eigenvalues. □

2.3. Proof of Theorem 1. The proof of Theorem 1 is essentially the same as that of [Sz, Theorem 1.3]; so we only give a sketch of the proof.

(i) First, by [Ta, Theorem 3], we obtain that equation (2.1) (or problem (*)) has precisely two branches of nontrivial solutions emanating from the point $(\lambda_1, 0)$.

(ii) Secondly, by using Propositions 2.3 and 2.4, we find that the nontrivial solutions of equation (2.1) with $\lambda_1 < \lambda \le \lambda_2$ must necessarily be positive or negative.

(iii) In order to study *globally* the bifurcation solution curves, we need the following three lemmas:

LEMMA 2.5. *If* u *is a positive (or negative) solution of equation* (2.1) *with* $\lambda_1 < \lambda < \infty$, *then* u *is a regular point of the mapping* $G(\lambda, u): \mathbf{R} \times C(\bar{D}) \to C(\bar{D})$, *given by the formula*

$$G(\lambda, u) = u - \lambda K u + K(h(u)),$$

that is, the partial Fréchet derivative $G_u(\lambda, u)$ *at* u *is invertible.*

LEMMA 2.6. *Equation* (2.1) *has a unique positive solution for each* $\lambda_1 < \lambda < \lambda_1 + k_+$. *No positive solutions exist for* $\lambda \ge \lambda_1 + k_+$. *The uniform norm* $\| u_+(\lambda) \|$ *of the positive solution* $u_+(\lambda)$ *tends to* ∞ *as* $\lambda \to \lambda_1 + k_+$. *Similar assertions are valid for negative solutions* $u_-(\lambda)$, *with* k_+ *replaced by* k_-.

LEMMA 2.7. *There is a constant* $\delta > 0$ *such that equation* (2.1) *has no nontrivial solutions for* $\lambda_1 - \delta \le \lambda \le \lambda_1$.

Lemmas 2.5, 2.6 and 2.7 are proved just as in the proof of [Sz, Lemmas 2.1, 2.2 and 2.3], by using Propositions 2.3 and 2.4 and the theory of positive mappings in ordered Banach spaces.

(iii-a) By using Lemma 2.6, Lemma 2.5 and the implicit function theorem, we can prove that equation (2.1) has a unique positive solution $u_+(\lambda)$ for all $\lambda_1 < \lambda < \lambda_1 + k_+$, and that the branch Γ_+ of positive solutions emanating from $(\lambda_1, 0)$ is a C^1 curve given by the formula

$$\Gamma_+ = \{(\lambda, u) \in \mathbf{R} \times C(\bar{D}) \,;\; u = u_+(\lambda)\,,\; \lambda_1 \le \lambda < \lambda_1 + k_+\}\,.$$

The other branch Γ_- is obtained in a similar way.

(iii-b) Furthermore, it follows from an application of Lemma 2.6 that no other positive or negative solutions exist for $\lambda > \lambda_1$, and also $\|u_+(\lambda)\| \to \infty$ as $\lambda \to \lambda_1 + k_+$ and $\| u_-(\lambda) \| \to \infty$ as $\lambda \to \lambda_1 + k_-$.

(iv) Finally, Lemma 2.7 tells us that there are no nontrivial solutions at $\lambda = \lambda_1$.

The proof of Theorem 1 is complete. □

2.4. Proof of Theorem 2. The proof of Theorem 2 is carried out by using the global theory of positive mappings (cf. [Da]), just as in the proof of [Sz, Theorems 5.1 and 5.2]. □

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