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Laguerre Character Sums

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1. Introduction.

Evans [1] defined character sum analogues of orthogonal polynomials and studied the properties of Hermite character sums. Greene [2] conducted a similar study of character sum analogues for hypergeometric series. Sawabe [7] considered the properties of Legendre character sums and showed the relation between Legendre character sums and hypergeometric character sums.

In this paper, we consider the properties of Laguerre character sums as defined by Evans [1]. Greene [3] showed that hypergeometric functions over finite fields arise as matrix elements of the principal series representations for $SL(2, q)$. We show that Laguerre character sums arise as matrix elements of a certain representation for a certain group, and derive two more properties of Laguerre character sums by making use of this result.

Throughout this paper we will use the following notations. Let q be a positive integral power of an odd prime p. The finite field of q elements is denoted by $\mathbf{F}_q$. The multiplicative group of $\mathbf{F}_q$ is denoted by $\mathbf{F}_q^\times$. The sets of the multiplicative and additive characters on $\mathbf{F}_q$ are denoted by $\hat{\mathbf{F}}_q^\times$ and $\hat{\mathbf{F}}_q^+$ respectively. The capital letters A, B, C, M, N, $\cdots$ and χ will denote multiplicative characters on $\mathbf{F}_q$. The trivial character will be denoted by 1 or ε, and the quadratic character by ϕ. All multiplicative characters on $\mathbf{F}_q$ are defined to be zero at the zero element of $\mathbf{F}_q$. Define $\bar{N}$ by $N\bar{N}=1$. The letters λ, μ are reserved for additive characters on $\mathbf{F}_q$. For $x \in \mathbf{F}_q$, $\lambda_0(x)$ or ζ^x is defined by $\exp\left(\frac{2\pi i}{p}\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x)\right)$, where $\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x)$ denotes the trace of x from $\mathbf{F}_q$ to $\mathbf{F}_p$. λ_0 is a non-trivial additive character on $\mathbf{F}_q$. For $w \in \mathbf{F}_q^\times$ and $\lambda \in \hat{\mathbf{F}}_q^+$, ${}^{\omega}\lambda$ denotes an additive character on $\mathbf{F}_q$ defined by ${}^{\omega}\lambda(a)=\lambda(wa)$ for $a \in \mathbf{F}_q$. For any non-trivial additive character μ on $\mathbf{F}_q$, there exists an unique element w in $\mathbf{F}_q^\times$ such that $\mu={}^{\omega}\lambda_0$. We define a function δ on elements of $\mathbf{F}_q$ by

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$$\delta(x)=\begin{cases}1\,, & \text{if } \ x=0\,,\\ 0\,, & \text{if } \ x\neq 0\,,\end{cases}$$

and on multiplicative characters by

$$\delta(A)=\begin{cases}1\,, & \text{if } \ A=1\,,\\ 0\,, & \text{if } \ A\neq 1\,.\end{cases}$$

We also use the notation $\delta_{A,B}$ in place of $\delta(\bar{A}B)$. Write $\sum_x$ to denote the sum over all $x\in\mathbf{F}_q$ and $\sum_N$ to denote the sum over all $N\in\hat{\mathbf{F}}_q^\times$. The Gauss sum for $N\in\hat{\mathbf{F}}_q^+$ is defined by

$$G(N)=\sum_x N(x)\zeta^x\,,$$

and the Jacobi sum for M, $N\in\hat{\mathbf{F}}_q^\times$ is defined by

$$J(M,N)=\sum_x M(x)N(1-x)\,.$$

We will make use of elementary formulas with respect to Gauss and Jacobi sums repeatedly (See [5]).

This paper is organized as follows. In section 2, we describe the properties derived from the definition of Laguerre character sums. In section 3, we define a group G and a representation T of G. We show that Laguerre character sums arise as matrix elements of T (Theorem 3.1). From this result and the representation theory of G, we derive two more formulas on Laguerre character sums (Theorem 3.4, 3.6).

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2. Properties of Laguerre character sums.

Evans [1] defined the *Laguerre character sum* $L_N^A(x)$ as follows.

DEFINITION 2.1 (Laguerre character sum). *For any* A, $N\in\hat{\mathbf{F}}_q^\times$, $x\in\mathbf{F}_q$,

$$L_N^A(x):=\frac{1}{q}\sum_u \bar{N}(u)AN(1+u)\zeta^{xu}\,.$$

This definition is motivated by the integral representation of Laguerre functions $L_n^a(x)$.

cf. (Laguerre function) [4, p. 77 (4.17.4)] $L_n^a(x)=\dfrac{1}{2\pi i}\displaystyle\int_C u^{-n}(1+u)^{a+n}e^{-xu}\frac{du}{u}$.

The analogue of the n-th derivative with respect to x was defined by Evans [1, (2.8)] as follows.

DEFINITION 2.2 (N-th derivative). *An operator D^N on the set of complex valued functions F on $\mathbf{F}_q$ is defined by*

$$D^N F(x) := \frac{1}{G(\bar{N})} \sum_t \bar{N}(t) F(x-t) \qquad \text{for} \quad x \in \mathbf{F}_q .$$

Laguerre character sums also have an analogue of the Rodrigues formula as Hermite and Legendre character sums (See [1, 7]).

THEOREM 2.1. *If $x \neq 0$, then*

$$L_N^A(x) = \frac{G(\bar{N})\bar{A}(x)\zeta^{-x}}{qN(-1)} D^N(AN(x)\zeta^x) .$$

In particular when $N \neq 1$,

$$L_N^A(x) = \frac{\bar{A}(x)\zeta^{-x}}{G(N)} D^N(AN(x)\zeta^x) .$$

cf. (Rodrigues formula) [6, p. 204 (3)] $L_n^{(\alpha)}(x) = \dfrac{x^{-\alpha}e^x}{n!}\left(\dfrac{d}{dx}\right)^n (x^{\alpha+n}e^{-x})$.

PROOF. By the definition of D^N,

$$D^N(AN(x)\zeta^x) = \frac{\zeta^x}{G(\bar{N})} \sum_t \bar{N}(t) AN(x-t)\zeta^{-t} .$$

Since $x \neq 0$, replace t by $-xu$. Then $D^N(AN(x)\zeta^x)$ is

$$\begin{aligned} \frac{\zeta^x}{G(\bar{N})} \sum_u \bar{N}(-xu) AN(x(1+u))\zeta^{xu} &= \frac{N(-1)A(x)\zeta^x}{G(\bar{N})} \sum_u \bar{N}(u) AN(1+u)\zeta^{xu} \\ &= \frac{qN(-1)}{G(\bar{N})\bar{A}(x)\zeta^{-x}} L_N^A(x) , \end{aligned}$$

and the result follows. If $N \neq 1$, then $G(\bar{N})G(N) = qN(-1)$. So

$$L_N^A(x) = \frac{\bar{A}(x)\zeta^{-x}}{G(N)} D^N(AN(x)\zeta^x) . \qquad \square$$

The analogue for m-th derivative of Laguerre polynomials is as follows.

THEOREM 2.2. $D^M L_N^A(x) = M(-1) L_{N\bar{M}}^{AM}(x)$.

cf. [6, p. 203 (11)] $DL_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x)$.

PROOF. By the definitions of D^M and L_N^A,

$$D^M L_N^A(x) = \frac{1}{G(\bar{M})} \sum_t \bar{M}(t) \frac{1}{q} \sum_u \bar{N}(u) AN(1+u) \zeta^{(x-t)u}$$
$$= \frac{1}{q} \sum_{u \neq 0} \bar{N}(u) AN(1+u) \zeta^{xu} \frac{1}{G(\bar{M})} \sum_t \bar{M}(t) \zeta^{-tu}.$$

Replace $-tu$ by t, and by the definition of Gauss sums,

$$D^M L_N^A(x) = \frac{1}{q} \sum_u \bar{N}(u) AN(1+u) \zeta^{xu} M(-u)$$
$$= M(-1) L_{NM}^{AM}(x). \qquad \square$$

Laguerre character sums satisfy the next formula, which is similar to [1, Th.4.3].

THEOREM 2.3. *If $MN \neq 1$, then*

$$D^M(A(x)\zeta^x L_N^A(x)) = \frac{G(\bar{N}) M(-1) A\bar{M}(x) \zeta^x}{G(\overline{NM})} L_{NM}^{A\bar{M}}(x).$$

PROOF. The result is clear for $x=0$, so assume $x \neq 0$. By Theorem 2.1,

$$D^M(A(x)\zeta^x L_N^A(x)) = \frac{G(\bar{N})}{qN(-1)} D^M D^N(AN(x)\zeta^x).$$

Since $MN \neq 1$, $D^M D^N = D^{MN}$ by [1, (2.11)]. By the definition of D^{MN}, the right-hand side of the above equation is

$$\frac{G(\bar{N})\zeta^x}{qN(-1)G(\overline{MN})} \sum_t \overline{MN}(t) AN(x-t) \zeta^{-t}.$$

Replace $-t$ by xu. Then

$$D^M(A(x)\zeta^x L_N^A(x)) = \frac{G(\bar{N})\zeta^x}{qN(-1)G(\overline{MN})} \sum_u \overline{MN}(-xu) AN(x(1+u)) \zeta^{xu}$$
$$= \frac{G(\bar{N})\zeta^x}{N(-1)G(\overline{MN})} \overline{MN}(-x) AN(x) \frac{1}{q} \sum_u \overline{MN}(u) AN(1+u) \zeta^{xu}$$
$$= \frac{G(\bar{N}) M(-1) A\bar{M}(x) \zeta^x}{G(\overline{NM})} L_{NM}^{A\bar{M}}(x). \qquad \square$$

Laguerre character sums with $N=1$ can be calculated as follows.

THEOREM 2.4. *If $x \neq 0$, then*

$$L_1^A(x) = -\frac{1}{q} + \frac{G(A)\bar{A}(x)\zeta^{-x}}{q}.$$

PROOF. Since $x \neq 0$, by Theorem 2.1

$$L_1^A(x) = \frac{G(1)\bar{A}(x)\zeta^{-x}}{q} D^1(A(x)\zeta^x).$$

$G(1) = -1$, and by [1, (2.9)], $L_1^A(x)$ equals

$$-\frac{\bar{A}(x)\zeta^{-x}}{q}\left(A(x)\zeta^x - \sum_t A(t)\zeta^t\right) = -\frac{1}{q} + \frac{G(A)\bar{A}(x)\zeta^{-x}}{q}. \qquad \square$$

We obtain the next theorem immediately from the definition of Laguerre character sums.

THEOREM 2.5. $L_N^A(-x) = A(-1)\zeta^x L_{\overline{AN}}^A(x)$.

PROOF. By the definition of L_N^A,

$$L_N^A(-x) = \frac{1}{q}\sum_u \bar{N}(u)AN(1+u)\zeta^{-xu}.$$

Replace $-u$ by $1+u$. Then

$$\begin{aligned} L_N^A(-x) &= \frac{1}{q}\sum_u \bar{N}(-1-u)AN(-u)\zeta^{x(1+u)} \\ &= A(-1)\zeta^x \frac{1}{q}\sum_u AN(u)\bar{N}(1+u)\zeta^{xu} \\ &= A(-1)\zeta^x L_{\overline{AN}}^A(x). \end{aligned} \qquad \square$$

When $x = 0$, we can describe Laguerre character sums in terms of binomial coefficients defined by Greene as follows [2, Def. 2.4].

DEFINITION 2.3 (Binomial coefficients). *For any $A, N \in \hat{\mathbf{F}}_q^\times$,*

$$\binom{A}{N} := \frac{N(-1)}{q} J(A, \bar{N}).$$

THEOREM 2.6. $L_N^A(0) = \binom{AN}{N}$.

PROOF. By the definition of L_N^A,

$$L_N^A(0) = \frac{1}{q}\sum_u \bar{N}(u)AN(1+u).$$

Replace u by $-u$. Then

$$\begin{aligned} L_N^A(0) &= \frac{1}{q} \sum_u \bar{N}(-u) AN(1-u) \\ &= \frac{N(-1)}{q} \sum_u \bar{N}(u) AN(1-u) \\ &= \frac{N(-1)}{q} J(AN, \bar{N}) = \binom{AN}{N}. \end{aligned} \qquad \square$$

The next theorem shows that Laguerre character sums have the generating function.

THEOREM 2.7. *If $t \neq 0, 1$, then*

$$\bar{A}(1-t)\zeta^{\frac{xt}{1-t}} = \frac{q}{q-1} \sum_N L_N^A(x) N(t).$$

cf. (generating function) [6, p. 202 (4)] $\displaystyle \frac{1}{(1-t)^{1+\alpha}} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n.$

PROOF. By the definition of L_N^A,

$$\begin{aligned} \frac{q}{q-1} \sum_N L_N^A(x) N(t) &= \frac{1}{q-1} \sum_N \sum_u \bar{N}(u) AN(1+u) \zeta^{xu} N(t) \\ &= \frac{1}{q-1} \sum_{u \neq 0} A(1+u) \zeta^{xu} \sum_N N\left(\frac{(1+u)t}{u}\right). \end{aligned}$$

If $u = t/(1-t)$, then $\sum_N N((1+u)t/u) = q-1$, and if $u \neq t/(1-t)$, then $\sum_N N((1+u)t/u) = 0$. Therefore

$$\frac{q}{q-1} \sum_N L_N^A(x) N(t) = A\left(1 + \frac{t}{1-t}\right) \zeta^{\frac{xt}{1-t}} = \bar{A}(1-t) \zeta^{\frac{xt}{1-t}}. \qquad \square$$

COROLLARY 2.8. *If $a \neq 0, -1$, then*

$$\zeta^{ax} = \frac{q\bar{A}(1+a)}{q-1} \sum_N N\left(\frac{a}{1+a}\right) L_N^A(x).$$

cf. [4, p. 90 (4.24.3)] $\displaystyle e^{-ax} = (a+1)^{-\alpha-1} \sum_{n=0}^{\infty} \left(\frac{a}{a+1}\right)^n L_n^{\alpha}(x).$

PROOF. By Theorem 2.7 with $t = a/(1+a)$,

$$\bar{A}\left(\frac{1}{1+a}\right) \zeta^{ax} = \frac{q}{q-1} \sum_N L_N^A(x) N\left(\frac{a}{1+a}\right),$$

and the result follows. $\square$

Greene defined the analogue of confluent hypergeometric functions as follows, motivated by the integral representation [3, (5.20)].

DEFINITION 2.4 (Confluent hypergeometric character sum).
For any A, $B \in \hat{\mathbf{F}}_q^\times$, $x \in \mathbf{F}_q$,

$$ {}_1F_1\left(\begin{matrix} A \\ B \end{matrix} \middle| x\right) := \varepsilon(x)AB(-1)\sum_y A(y)\bar{A}B(1-y)\zeta^{-xy}. $$

cf. [6, p. 124 (9)] $\quad {}_1F_1(a; b; z) = \dfrac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \displaystyle\int_0^1 e^{zt} t^{a-1}(1-t)^{b-a-1}dt.$

Laguerre functions can be described by confluent hypergeometric functions. The next theorem shows the relation between the Laguerre character sums and the confluent hypergeometric character sums.

THEOREM 2.9. *If* $x \neq 0$, *then*

$$ L_N^A(x) = \frac{A(-1)}{q}\, {}_1F_1\left(\begin{matrix} \bar{N} \\ A \end{matrix} \middle| x\right). $$

cf. [6, p. 200 (1)] $\quad L_n^{(\alpha)}(x) = \dfrac{(1+\alpha)_n}{n!}\, {}_1F_1(-n; 1+\alpha; x).$

PROOF. Since $x \neq 0$, by the definition of ${}_1F_1$,

$$ {}_1F_1\left(\begin{matrix} \bar{N} \\ A \end{matrix} \middle| x\right) = \bar{N}A(-1)\sum_y \bar{N}(y)NA(1-y)\zeta^{-xy}. $$

Replace $-y$ by y, then

$$ {}_1F_1\left(\begin{matrix} \bar{N} \\ A \end{matrix} \middle| x\right) = A(-1)\sum_y \bar{N}(y)AN(1+y)\zeta^{xy} = A(-1)qL_N^A(x), $$

and the result follows. □

Corresponding to

[6, p. 209 (3)] $\quad L_n^{(\alpha+\beta+1)}(x+y) = \displaystyle\sum_{k=0}^{n} L_k^{(\alpha)}(x)L_{n-k}^{(\beta)}(y),$

we have

THEOREM 2.10. $L_N^{AB}(x+y) = q/(q-1)\sum_M L_M^A(x)L_{N\bar{M}}^B(y).$

PROOF. By the definition of L_N^A,

$$ \begin{aligned} &\frac{q}{q-1}\sum_M L_M^A(x)L_{N\bar{M}}^B(y) \\ &= \frac{q}{q-1}\sum_M \frac{1}{q}\sum_s \bar{M}(s)AM(1+s)\zeta^{xs}\frac{1}{q}\sum_t \bar{N}M(t)BN\bar{M}(1+t)\zeta^{yt} \\ &= \frac{1}{q(q-1)}\sum_{s,t\neq 0,-1} A(1+s)\bar{N}(t)BN(1+t)\zeta^{xs+yt}\sum_M M\left(\frac{t(1+s)}{s(1+t)}\right). \end{aligned} $$

If $s=t$, then $\sum_M M(t(1+s)/s(1+t))=q-1$, and if $s\neq t$, then $\sum_M M(t(1+s)/s(1+t))=0$. Therefore

$$\frac{q}{q-1}\sum_M L_M^A(x)L_{N\bar{M}}^B(y)=\frac{1}{q}\sum_t A(1+t)\bar{N}(t)BN(1+t)\zeta^{(x+y)t}=L_N^{AB}(x+y)\,. \qquad \square$$

We need the following lemma to prove Theorem 2.12.

LEMMA 2.11. *If $x\neq 0$, then*

$$L_{\bar{A}}^A(x)=\frac{1}{q}\bar{A}(x)G(A)-\frac{1}{q}A(-1)\zeta^{-x}\,.$$

PROOF.

$$L_{\bar{A}}^A(x)=\frac{1}{q}\sum_u A(u)\varepsilon(1+u)\zeta^{xu}=\frac{1}{q}\sum_{u\neq -1}A(u)\zeta^{xu}$$

$$=\frac{1}{q}\sum_u A(u)\zeta^{xu}-\frac{1}{q}A(-1)\zeta^{-x}\,.$$

Since $x\neq 0$, replace xu by u. Then

$$L_{\bar{A}}^A(x)=\frac{1}{q}\sum_u A\left(\frac{u}{x}\right)\zeta^u-\frac{1}{q}A(-1)\zeta^{-x}$$

$$=\frac{1}{q}\bar{A}(x)G(A)-\frac{1}{q}A(-1)\zeta^{-x}\,. \qquad \square$$

Corresponding to

[6, p. 209 (5)] $\quad L_n^{(\alpha)}(xy)=\sum_{k=0}^{\infty}\frac{(1+\alpha)_n(1-y)^{n-k}y^kL_k^{(\alpha)}(x)}{(n-k)!(1+\alpha)_k}$,

we have

THEOREM 2.12. *If $AN\neq 1$ and $xy\neq 0$, then*

$$L_N^A(xy)=\frac{q}{q-1}I+N(y)L_N^A(x)+\frac{1}{q}\bar{A}(xy)AN(1-y)G(A)\,,$$

where

$$I=\sum_M\frac{G(AN)}{G(AM)G(N\bar{M})}N\bar{M}(1-y)M(y)L_M^A(x)\,.$$

PROOF. If $y=1$, then both sides equal $L_N^A(x)$. So assume $y\neq 1$.

$$I=-N(y)L_N^A(x)-\bar{A}(y)AN(1-y)L_{\bar{A}}^A(x)$$
$$+\sum_{M\neq N,\bar{A}}\frac{G(AN)}{G(AM)G(N\bar{M})}N\bar{M}(1-y)M(y)L_M^A(x)\,. \qquad (1)$$

When $M \neq N, \bar{A}$,

$$G(N\bar{M})G(\bar{N}M)=qNM(-1), \quad G(AM)G(\overline{AM})=qAM(-1).$$

Then the last term on the right-hand side of (1) is

$$G(AN)\sum_{M\neq N,\bar{A}}\frac{G(\overline{AM})G(\bar{N}M)}{qAM(-1)qNM(-1)}N\bar{M}(1-y)M(y)L_M^A(x)$$

$$=\frac{AN(-1)G(AN)}{q^2}\sum_{M\neq N,\bar{A}}G(\overline{AM})G(\bar{N}M)N\bar{M}(1-y)M(y)L_M^A(x)$$

$$=\frac{AN(-1)G(AN)}{q^2}\sum_M G(\overline{AM})G(\bar{N}M)N\bar{M}(1-y)M(y)L_M^A(x)$$

$$-\frac{AN(-1)G(AN)}{q^2}G(\overline{AN})G(1)\varepsilon(1-y)N(y)L_N^A(x)$$

$$-\frac{AN(-1)G(AN)}{q^2}G(1)G(\overline{AN})NA(1-y)\bar{A}(y)L_{\bar{A}}^A(x).$$

Since $AN\neq 1$, $G(AN)G(\overline{AN})=qAN(-1)$. Therefore

$$I=\frac{AN(-1)G(AN)}{q^2}\sum_M G(\overline{AM})G(\bar{N}M)N\bar{M}(1-y)M(y)L_M^A(x)$$

$$-\frac{q-1}{q}N(y)L_N^A(x)-\frac{q-1}{q}\bar{A}(y)AN(1-y)L_{\bar{A}}^A(x).$$

By Lemma 2.11,

$$I=\frac{AN(-1)G(AN)}{q^2}\sum_M G(\overline{AM})G(\bar{N}M)N\bar{M}(1-y)M(y)L_M^A(x) \tag{2}$$

$$-\frac{q-1}{q}N(y)L_N^A(x) \tag{3}$$

$$-\frac{q-1}{q^2}\bar{A}(xy)AN(1-y)G(A) \tag{4}$$

$$+\frac{q-1}{q^2}A(-1)\bar{A}(y)AN(1-y)\zeta^{-x}. \tag{5}$$

By the definitions of Gauss and Laguerre character sums,

$$(2)=\frac{AN(-1)G(AN)}{q^2}\sum_M\sum_s\overline{AM}(s)\zeta^s\sum_t\bar{N}M(t)\zeta^t N\bar{M}(1-y)M(y)$$

$$\times\frac{1}{q}\sum_u\bar{M}(u)AM(1+u)\zeta^{xu}$$

$$=\frac{AN(-1)G(AN)}{q^3}\sum_{\substack{s,t\neq 0\\ u\neq 0,-1}}\bar{A}(s)\bar{N}(t)N(1-y)A(1+u)\zeta^{s+t+xu}$$
$$\times\sum_M M\left(\frac{ty(1+u)}{su(1-y)}\right).$$

If $s=ty(1+u)/u(1-y)$, then $\sum_M M(ty(1+u)/su(1-y))=q-1$, and if $s\neq ty(1+u)/u(1-y)$, then $\sum_M M(ty(1+u)/su(1-y))=0$. Then (2) equals

$$\frac{(q-1)AN(-1)G(AN)}{q^3}\sum_{\substack{t\neq 0\\ u\neq 0,-1}}\bar{A}\left(\frac{ty(1+u)}{u(1-y)}\right)\bar{N}(t)N(1-y)A(1+u)\zeta^{\frac{ty(1+u)}{u(1-y)}+t+xu}$$
$$=\frac{(q-1)AN(-1)G(AN)}{q^3}\bar{A}(y)AN(1-y)\sum_{u\neq 0,-1}A(u)\zeta^{xu}\sum_{t\neq 0}\overline{AN}(t)\zeta^{\frac{u+y}{u(1-y)}t}.$$

Now,

$$\sum_{u\neq 0,-1}A(u)\zeta^{xu}\sum_{t\neq 0}\overline{AN}(t)\zeta^{\frac{u+y}{u(1-y)}t}$$
$$=\sum_{u\neq 0,-1,-y}A(u)\zeta^{xu}\sum_{t\neq 0}\overline{AN}(t)\zeta^{\frac{u+y}{u(1-y)}t}+A(-y)\zeta^{-xy}\sum_{t\neq 0}\overline{AN}(t).$$

Since $AN\neq 1$, the sum in the second term is zero. Replace $(u+y)t/u(1-y)$ by t. Then the right-hand side of the above equation becomes

$$\sum_{u\neq 0,-1,-y}A(u)\zeta^{xu}\sum_{t\neq 0}\overline{AN}\left(\frac{u(1-y)}{u+y}t\right)\zeta^{t}=G(\overline{AN})\overline{AN}(1-y)\sum_{u\neq -1}\bar{N}(u)AN(u+y)\zeta^{xu}.$$

Therefore,

$$(2)=\frac{q-1}{q^2}\bar{A}(y)\sum_{u\neq -1}\bar{N}(u)AN(u+y)\zeta^{xu}$$
$$=\frac{q-1}{q^2}\bar{A}(y)\sum_{u}\bar{N}(u)AN(u+y)\zeta^{xu}$$
$$-\frac{q-1}{q^2}\bar{A}(y)N(-1)AN(-1+y)\zeta^{-x}$$
$$=\frac{q-1}{q^2}\bar{A}(y)\sum_{u}\bar{N}(u)AN(u+y)\zeta^{xu} \tag{6}$$
$$-\frac{q-1}{q^2}A(-1)\bar{A}(y)AN(1-y)\zeta^{-x}. \tag{7}$$

Replace u by uy in (6), then

$$(6)=\frac{q-1}{q^2}\bar{A}(y)\sum_u \bar{N}(uy)AN(y(1+u))\zeta^{uxy}$$
$$=\frac{q-1}{q^2}\sum_u \bar{N}(u)AN(1+u)\zeta^{uxy}=\frac{q-1}{q}L_N^A(xy).$$

Therefore,

$$I=(6)+(7)+(3)+(4)+(5)$$
$$=(6)+(3)+(4)$$
$$=\frac{q-1}{q}L_N^A(xy)-\frac{q-1}{q}N(y)L_N^A(x)-\frac{q-1}{q^2}\bar{A}(xy)AN(1-y)G(A),$$

and the result follows. □

Corresponding to

[4, p. 96, 20] $L_p^\beta(x)=\sum_{n=0}^{p}\frac{\Gamma(\beta-\alpha+p-n)}{(p-n)!\Gamma(\beta-\alpha)}L_n^\alpha(x)$,

we have

THEOREM 2.13. $$L_N^A(x)=\frac{q}{q-1}I+L_N^B(x),$$

where

$$I=\sum_M \frac{G(A\bar{B}N\bar{M})}{G(A\bar{B})G(N\bar{M})}L_M^B(x).$$

PROOF. $$I=\frac{1}{G(A\bar{B})}\sum_M \frac{G(A\bar{B}N\bar{M})}{G(N\bar{M})}L_M^B(x)$$
$$=\frac{1}{G(A\bar{B})}\left(\sum_{M\neq N}\frac{G(A\bar{B}N\bar{M})}{G(N\bar{M})}L_M^B(x)-G(A\bar{B})L_N^B(x)\right)$$
$$=\frac{1}{G(A\bar{B})}\left(\sum_{M\neq N}\frac{G(\bar{N}M)G(A\bar{B}N\bar{M})}{qNM(-1)}L_M^B(x)-G(A\bar{B})L_N^B(x)\right)$$
$$=\frac{1}{G(A\bar{B})}\left(\sum_M \frac{G(\bar{N}M)G(A\bar{B}N\bar{M})}{qNM(-1)}L_M^B(x)-\frac{q-1}{q}G(A\bar{B})L_N^B(x)\right).$$

By the definitions of Gauss and Laguerre character sums, the first term inside the parenthesis is

$$\begin{aligned}
&\sum_M \frac{G(\bar{N}M)G(A\bar{B}N\bar{M})}{qNM(-1)} L_M^B(x) \\
&= \frac{1}{q} N(-1) \sum_M M(-1) \sum_s \bar{N}M(s)\zeta^s \sum_t A\bar{B}N\bar{M}(t)\zeta^t \frac{1}{q} \sum_u \bar{M}(u)BM(1+u)\zeta^{xu} \\
&= \frac{1}{q^2} N(-1) \sum_{\substack{s,t\neq 0 \\ u\neq 0,-1}} \bar{N}(s)A\bar{B}N(t)B(1+u)\zeta^{s+t+xu} \sum_M M\left(\frac{-s(1+u)}{tu}\right) \\
&= \frac{q-1}{q^2} N(-1) \sum_{\substack{s\neq 0 \\ u\neq 0,-1}} \bar{N}(s)A\bar{B}N\left(\frac{-s(1+u)}{u}\right)B(1+u)\zeta^{s+\frac{-s(1+u)}{u}+xu} \\
&= \frac{q-1}{q^2} \sum_{\substack{s\neq 0 \\ u\neq 0,-1}} A\bar{B}(-s)\bar{A}B\bar{N}(u)AN(1+u)\zeta^{-\frac{s}{u}+xu} \\
&= \frac{q-1}{q^2} \sum_{u\neq 0} \bar{A}B\bar{N}(u)AN(1+u)\zeta^{xu} \sum_s A\bar{B}(-s)\zeta^{-\frac{s}{u}} \\
&= \frac{q-1}{q^2} G(A\bar{B}) \sum_{u\neq 0} \bar{N}(u)AN(1+u)\zeta^{xu} \\
&= \frac{q-1}{q} G(A\bar{B})L_N^A(x)\,,
\end{aligned}$$

and the result follows. □

3. Matrix elements of the representation T of G.

$$\text{Let } G = \left\{ \begin{pmatrix} 1 & a & b \\ & c & d \\ & & 1 \end{pmatrix} \middle|\; \begin{matrix} a, b, d \in \mathbf{F}_q \\ c \in \mathbf{F}_q^\times \end{matrix} \right\}.$$

G is a subgroup of $GL(3, q)$ and its order is $q^3(q-1)$. In this section, we show that Laguerre character sums arise as matrix elements of a certain representation T of the group G. From this result and the representation theory of G, we derive two more formulas on Laguerre character sums.

Let V be the vector space of all complex-valued functions on $\mathbf{F}_q$. We define an inner product on V by

$$\langle f, f' \rangle := \frac{1}{q-1} \sum_{z\in\mathbf{F}_q} f(z)\overline{f'(z)} \qquad \text{for } f, f' \in V.$$

An orthogonal normal basis for V with respect to this inner product is given by $\{\tilde{\delta}\} \cup \hat{\mathbf{F}}_q^\times$, where $\tilde{\delta}$ is defined by

$$\tilde{\delta}(z)=\begin{cases}\sqrt{q-1}\,, & \text{if} \quad z=0\,,\\ 0\,, & \text{if} \quad z\neq 0\,.\end{cases}$$

We fix $\chi\in\hat{\mathbf{F}}_q^{\times}$ and $\mu\in\hat{\mathbf{F}}_q^{+}-\{1\}$. G acts on V as follows:

$$(g.f)(z):=\chi(c)\mu(b+dz)f(a+cz)\,,$$

$$\text{for} \quad g=\begin{pmatrix}1 & a & b\\ & c & d\\ & & 1\end{pmatrix}\in G,\ f\in V \quad \text{and} \quad z\in\mathbf{F}_q\,.$$

The representation for G obtained by this action is denoted by $T=T(\chi,\mu)$. Let the character of T be $\eta=\eta(\chi,\mu)$. The degree of η is q. The matrix representation $M=M(\chi,\mu)$ of T with respect to the basis $\{\tilde{\delta}\}\cup\hat{\mathbf{F}}_q^{\times}$ for V can be written in block diagonal form

$$M(g)=\left(\begin{array}{c|c}X(g) & Y_C(g)\\ \hline Z_B(g) & W_{B,C}(g)\end{array}\right)_{B,C\in\hat{\mathbf{F}}_q^{\times}}=\left(\begin{array}{c|c}X^{\chi,\mu}(g) & Y_C^{\chi,\mu}(g)\\ \hline Z_B^{\chi,\mu}(g) & W_{B,C}^{\chi,\mu}(g)\end{array}\right)_{B,C\in\hat{\mathbf{F}}_q^{\times}},$$

for $g\in G$. $X^{\chi,\mu}(g)$ is $1\times 1-$matrix and $W_{B,C}^{\chi,\mu}(g)$ is $(q-1)\times(q-1)-$matrix suffixed by $B, C\in\hat{\mathbf{F}}_q^{\times}$. Namely,

$$T(\chi,\mu)(g)\tilde{\delta}=X^{\chi,\mu}(g)\tilde{\delta}+\sum_{B\in\hat{\mathbf{F}}_q^{\times}}Z_B^{\chi,\mu}(g)B, \text{ and}$$

$$T(\chi,\mu)(g)C=Y_C^{\chi,\mu}(g)\tilde{\delta}+\sum_{B\in\hat{\mathbf{F}}_q^{\times}}W_{B,C}^{\chi,\mu}(g)B \qquad \text{for all} \quad C\in\hat{\mathbf{F}}_q^{\times}\,.$$

THEOREM 3.1. *Let* $\chi\in\hat{\mathbf{F}}_q^{\times}$, $\mu={}^{w}\lambda_0\in\hat{\mathbf{F}}_q^{+}-\{1\}$ *where* $w\in\mathbf{F}_q^{\times}$. *Let* $g=\begin{pmatrix}1 & a & b\\ & c & d\\ & & 1\end{pmatrix}\in G$, *and* B, $C\in\hat{\mathbf{F}}_q^{\times}$. *Then*

(1) $X^{\chi,\mu}(g)=\chi(c)\mu(b)\delta(a)$

(2) $Y_C^{\chi,\mu}(g)=\dfrac{1}{\sqrt{q-1}}\chi(c)\mu(b)C(a)$

(3) $Z_B^{\chi,\mu}(g)=\dfrac{1}{\sqrt{q-1}}\chi(c)\mu\left(b-\dfrac{ad}{c}\right)\bar{B}\left(-\dfrac{a}{c}\right)$

(4) $W_{B,C}^{\chi,\mu}(g)$

$$=\begin{cases}\dfrac{q}{q-1}\chi(c)\mu(b)\bar{B}C(a)B(c)L_B^{\bar{B}C}\left(\dfrac{wad}{c}\right), & \textit{if} \quad a\neq 0\,,\\ \dfrac{1}{q-1}\chi(c)\mu(b)C(c)B\bar{C}(wd)G(\bar{B}C)\,, & \textit{if} \quad a=0\,, \quad d\neq 0\,,\\ \chi(c)\mu(b)C(c)\delta(\bar{B}C)\,, & \textit{if} \quad a=0\,, \quad d=0\,.\end{cases}$$

PROOF. (1), (2), and (3) can be easily checked by the fact that $\{\tilde{\delta}\}\cup\hat{\mathbf{F}}_q^\times$ is an orthogonal normal basis for V, and by the definitions of the inner product and the action of G on V. For the same reason, and since $\mu(dz)=\zeta^{wdz}$, we also have

$$W_{B,C}^{\chi,\mu}(g)=\frac{1}{q-1}\chi(c)\mu(b)\sum_z\zeta^{wdz}C(a+cz)\bar{B}(z)\,.$$

When $a\neq 0$, replace $(c/a)z$ by z. Then

$$\begin{aligned}W_{B,C}^{\chi,\mu}(g)&=\frac{1}{q-1}\chi(c)\mu(b)\sum_z\zeta^{\frac{wad}{c}z}C(a+az)\bar{B}\left(\frac{a}{c}z\right)\\&=\frac{1}{q-1}\chi(c)\mu(b)\bar{B}C(a)B(c)\sum_z\bar{B}(z)C(1+z)\zeta^{\frac{wad}{c}z}\\&=\frac{q}{q-1}\chi(c)\mu(b)\bar{B}C(a)B(c)L_B^{\bar{B}C}\left(\frac{wad}{c}\right).\end{aligned}$$

When $a=0$,

$$W_{B,C}^{\chi,\mu}(g)=\frac{1}{q-1}\chi(c)\mu(b)C(c)\sum_z\bar{B}C(z)\zeta^{wdz}\,.$$

if $d\neq 0$, replace wdz by z. Then

$$\begin{aligned}W_{B,C}^{\chi,\mu}(g)&=\frac{1}{q-1}\chi(c)\mu(b)C(c)\sum_z\bar{B}C\left(\frac{z}{wd}\right)\zeta^z\\&=\frac{1}{q-1}\chi(c)\mu(b)C(c)B\bar{C}(wd)G(\bar{B}C)\,.\end{aligned}$$

If $d=0$,

$$\begin{aligned}W_{B,C}^{\chi,\mu}(g)&=\frac{1}{q-1}\chi(c)\mu(b)C(c)\sum_z\bar{B}C(z)\\&=\chi(c)\mu(b)C(c)\delta(\bar{B}C)\,.\end{aligned}$$

□

PROPOSITION 3.2. *$T=T(\chi,\mu)$ is irreducible.*

PROOF. Let

$$U=\left\{\begin{pmatrix}1&a&b\\&1&\\&&1\end{pmatrix}\middle|\,a,b\in\mathbf{F}_q\right\}\quad\text{and}$$

$$L=\left\{\begin{pmatrix}1&&\\&c&d\\&&1\end{pmatrix}\middle|\,c\in\mathbf{F}_q^\times,d\in\mathbf{F}_q\right\}.$$

Hence U is an abelian normal subgroup and L is a subgroup of G. G is the semi-direct product of U and L. We can therefore determine all irreducible characters of G using the "Method of little groups" (See [8]). In particular, for $\chi\in\hat{\mathbf{F}}_q^{\times}$, $\mu\in\hat{\mathbf{F}}_q^{+}-\{1\}$, and

$$H=\left\{\begin{pmatrix}1 & a & b\\ & c & \\ & & 1\end{pmatrix}\;\middle|\; a, b\in\mathbf{F}_q, c\in\mathbf{F}_q^{\times}\right\},$$

we have the linear character $\rho(\chi, \mu)$ of H defined by

$$\rho(\chi, \mu)\begin{pmatrix}1 & a & b\\ & c & \\ & & 1\end{pmatrix}=\chi(c)\mu(b).$$

Thus, the induced character of $\rho(\chi, \mu)$ from H to G is irreducible, and we can show that it is equal to $\eta(\chi, \mu)$. Hence $\eta(\chi, \mu)$ is irreducible.

We can also prove that $\langle\eta, \eta\rangle_G=1$ directly by making use of Theorem 3.1. But this calculation is tedious, so we omit it. □

In general, the following relation with respect to the matrix representation is well-known. For example, see [8].

LEMMA 3.3. *Let (ρ, V) be an irreducible representation of a group G and $\varepsilon=\{v_1, \cdots, v_n\}$ be a basis of V. We denote the matrix representation of ρ with respect to ε by $M(g)=(r_{ij}(g))_{1\le i,j\le n}$ for all $g\in G$. Then*

$$\sum_{g\in G} r_{ij}(g^{-1})r_{kl}(g)=\frac{|G|}{n}\delta_{il}\delta_{jk}.$$ □

By utilizing Lemma 3.3, we can obtain the following relation with respect to Laguerre character sums.

THEOREM 3.4. *For B, $C\in\hat{\mathbf{F}}_q^{\times}$,*

$$\sum_x \zeta^x L_B^{\bar{B}C}(x)L_C^{\bar{C}B}(x)=\frac{q-2}{q}BC(-1).$$

PROOF. Since T is irreducible, we apply Lemma 3.3 to

$$M(g)=\left(\begin{array}{c|c} X(g) & Y_C(g)\\ \hline Z_B(g) & W_{B,C}(g)\end{array}\right)_{B,C\in\hat{\mathbf{F}}_q^{\times}}. \quad \text{Then}$$

$$\sum_{g\in G} W_{B,C}(g^{-1})W_{D,E}(g)=q^2(q-1)\delta_{B,E}\delta_{C,D}.$$

On the other hand,

$$\sum_{g\in G} W_{B,C}(g^{-1})W_{D,E}(g) = \sum_{\substack{b,d\\ a,c\neq 0}} W_{B,C}(g^{-1})W_{D,E}(g) \tag{1}$$

$$+ \sum_{\substack{b\\ c,d\neq 0,a=0}} W_{B,C}(g^{-1})W_{D,E}(g) \tag{2}$$

$$+ \sum_{\substack{b\\ c\neq 0,a=d=0}} W_{B,C}(g^{-1})W_{D,E}(g)\,. \tag{3}$$

For $g=\begin{pmatrix} 1 & a & b \\ & c & d \\ & & 1 \end{pmatrix}$, $g^{-1}=\begin{pmatrix} 1 & -ac^{-1} & adc^{-1}-b \\ & c^{-1} & -dc^{-1} \\ & & 1 \end{pmatrix}$.

By Theorem 3.1(4),

$$\begin{aligned}(1) &= \sum_{\substack{b,d\\ a,c\neq 0}} \frac{q}{q-1}\chi(c^{-1})\mu(adc^{-1}-b)\bar{B}C(-ac^{-1})B(c^{-1})L_B^{\bar{B}C}\left(\frac{wad}{c}\right)\\ &\quad\times\frac{q}{q-1}\chi(c)\mu(b)\bar{D}E(a)D(c)L_D^{\bar{D}E}\left(\frac{wad}{c}\right)\\ &= \frac{q^2}{(q-1)^2}BC(-1)\sum_{\substack{b,d\\ a,c\neq 0}}\bar{B}C\bar{D}E(a)\bar{C}D(c)\mu\left(\frac{ad}{c}\right)L_B^{\bar{B}C}\left(\frac{wad}{c}\right)L_D^{\bar{D}E}\left(\frac{wad}{c}\right)\end{aligned}$$

Replace $\frac{wa}{c}d$ by x, so $\mu\left(\frac{ad}{c}\right)={}^{w}\lambda_0(w^{-1}x)=\zeta^x$. Therefore

$$\begin{aligned}(1) &= \frac{q^3}{(q-1)^2}BC(-1)\sum_{a\neq 0}\bar{B}C\bar{D}E(a)\sum_{c\neq 0}\bar{C}D(c)\sum_x\zeta^x L_B^{\bar{B}C}(x)L_D^{\bar{D}E}(x)\\ &= q^3BC(-1)\delta(\bar{B}C\bar{D}E)\delta(\bar{C}D)\sum_x\zeta^x L_B^{\bar{B}C}(x)L_D^{\bar{D}E}(x)\,.\end{aligned}$$

By Theorem 3.1(4),

$$\begin{aligned}(2) &= \sum_{\substack{b\\ c,d\neq 0,a=0}} \frac{1}{q-1}\chi(c^{-1})\mu(-b)C(c^{-1})B\bar{C}(-wdc^{-1})G(\bar{B}C)\\ &\quad\times\frac{1}{q-1}\chi(c)\mu(b)E(c)D\bar{E}(wd)G(\bar{D}E)\\ &= \frac{1}{(q-1)^2}BC(-1)B\bar{C}D\bar{E}(w)\sum_{\substack{b\\ c,d\neq 0}}\bar{B}E(c)B\bar{C}D\bar{E}(d)G(\bar{B}C)G(\bar{D}E)\\ &= qBC(-1)B\bar{C}D\bar{E}(w)\delta(\bar{B}E)\delta(B\bar{C}D\bar{E})G(\bar{B}C)G(\bar{D}E)\,.\end{aligned}$$

By Theorem 3.1(4),

$$(3)=\sum_{\substack{b\\ c\neq 0, a=d=0}} \chi(c^{-1})\mu(-b)C(c^{-1})\delta(\bar{B}C)\cdot\chi(c)\mu(b)E(c)\delta(\bar{D}E)$$
$$=q\left(\sum_{c\neq 0}\bar{C}E(c)\right)\delta(\bar{B}C)\delta(\bar{D}E)$$
$$=q(q-1)\delta(\bar{C}E)\delta(\bar{B}C)\delta(\bar{D}E).$$

Therefore

$$q^3BC(-1)\delta(\bar{B}C\bar{D}E)\delta(\bar{C}D)\sum_x \zeta^x L_B^{\bar{B}C}(x)L_D^{\bar{D}E}(x)$$
$$=q^2(q-1)\delta_{B,E}\delta_{C,D}$$
$$-qBC(-1)B\bar{C}D\bar{E}(w)\delta(\bar{B}E)\delta(B\bar{C}D\bar{E})G(\bar{B}C)G(\bar{D}E)$$
$$-q(q-1)\delta(\bar{C}E)\delta(\bar{B}C)\delta(\bar{D}E).$$

In particular, let $B=E$, $C=D$, then

$$q^3BC(-1)\sum_x \zeta^x L_B^{\bar{B}C}(x)L_C^{\bar{C}B}(x)=q^2(q-1)-qBC(-1)G(\bar{B}C)G(\bar{C}B)-q(q-1)\delta(\bar{B}C).$$

If $B\neq C$, then $G(\bar{B}C)G(\bar{C}B)=qBC(-1)$. The result follows. □

The following proposition is easily checked.

PROPOSITION 3.5. *$T=T(\chi,\mu)$ is unitary. i.e.*

$$\langle T(g)f, T(g)f'\rangle=\langle f, f'\rangle \quad \text{for all} \quad g\in G, \quad \text{and} \quad f, f'\in V. \qquad \square$$

THEOREM 3.6. *For $A, B\in\hat{\mathbf{F}}_q^\times$ and $x\in\mathbf{F}_q$,*

$$\frac{q^2}{(q-1)^2}\sum_C L_A^{\bar{A}C}(x)\overline{L_B^{\bar{B}C}(x)}=\delta_{A,B}-\frac{1}{q-1}AB(-1).$$

PROOF. From Proposition 3.5, $M(g)^t\overline{M(g)}=I$ for all $g\in G$. So in particular, for $A, B\in\hat{\mathbf{F}}_q^\times$,

$$Z_A(g)\overline{Z_B(g)}+\sum_C W_{A,C}(g)\overline{W_{B,C}(g)}=\delta_{A,B}.$$

When $g=\begin{pmatrix}1 & a & b\\ & c & d\\ & & 1\end{pmatrix}$, $a\neq 0$, from Theorem 3.1(4),

$$
\begin{aligned}
&\sum_C W_{A,C}(g)\overline{W_{B,C}(g)} \\
&=\sum_C \frac{q}{q-1}\chi(c)\mu(b)\bar{A}C(a)A(c)L_A^{\bar{A}C}\left(\frac{wad}{c}\right)\overline{\frac{q}{q-1}\chi(c)\mu(b)\bar{B}C(a)B(c)L_B^{\bar{B}C}\left(\frac{wad}{c}\right)} \\
&=\frac{q^2}{(q-1)^2}\bar{A}B\left(\frac{a}{c}\right)\sum_C L_A^{\bar{A}C}\left(\frac{wad}{c}\right)\overline{L_B^{\bar{B}C}\left(\frac{wad}{c}\right)}.
\end{aligned}
$$

On the other hand, from Theorem 3.1(3),

$$
\begin{aligned}
Z_A(g)\overline{Z_B(g)}&=\frac{1}{\sqrt{q-1}}\chi(c)\mu\left(b-\frac{ad}{c}\right)\bar{A}\left(-\frac{a}{c}\right)\cdot\overline{\frac{1}{\sqrt{q-1}}\chi(c)\mu\left(b-\frac{ad}{c}\right)\bar{B}\left(-\frac{a}{c}\right)} \\
&=\frac{1}{q-1}\bar{A}B\left(-\frac{a}{c}\right).
\end{aligned}
$$

Let $w=a=c=1$ and replace d by x, then the result follows. □

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