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Structure Jacobi Operator of Real Hypersurfaces with Constant Scalar Curvature in a Nonflat Complex Space Form

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Abstract. Let M be a real hypersurface with almost contact metric structure (ϕ, ξ, η, g) in a nonflat complex space form $M_n(c)$. We denote by S be the Ricci tensor of M. In the present paper we investigate real hypersurfaces with constant scalar curvature of $M_n(c)$ whose structure Jacobi operator R_ξ commute with both ϕ and S. We characterize Hopf hypersurfaces of $M_n(c)$.

Introduction

An n-dimensional complex space form $M_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature c. As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}_n$ or a complex hyperbolic space $H_n\mathbf{C}$ according as $c > 0$, $c = 0$ or $c < 0$.

Let M be a real hypersurface of $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure J and the Kaehlerian metric of $M_n(c)$. This structure plays an important role in the study of the geometry of a real hypersurface. The structure vector ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. A real hypersurface is said to be a Hopf hypersurface if the structure vector field ξ of M is principal.

In a complex projective space $P_n\mathbf{C}$, Hopf hypersurfaces with constant principal curvatures are just the homogeneous real hypersurfaces ([7]). Further, Hopf hypersurfaces with constant principal curvatures in a nonflat complex space forms were completely classified as follows:

THEOREM T ([9]). *Let M be a homogeneous real hypersurface of $P_n\mathbf{C}$. Then M is a tube of radius r over one of the following Kaehlerian submanifolds*:

(A_1) *a hyperplane $P_{n-1}\mathbf{C}$, where $0 < r < \frac{\pi}{2}$*,

(A_2) *a totally geodesic $P_k\mathbf{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$*,

(B) *a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$*,

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(C) $P_1\mathbf{C} \times P_{(n-1)/2}\mathbf{C}$, *where* $0 < r < \frac{\pi}{4}$ *and* $n(\geq 5)$ *is odd*,
(D) *a complex Grassmann* $G_{2,5}\mathbf{C}$, *where* $0 < r < \frac{\pi}{4}$ *and* $n = 9$,
(E) *a Hermitian symmetric space* $SO(10)/U(5)$, *where* $0 < r < \frac{\pi}{4}$ *and* $n = 15$.

THEOREM B ([1]). *Let M be a real hypersurface of* $H_n\mathbf{C}$. *Then M has constant principal curvatures and* ξ *is principal if and only if M is locally congruent to one of the following*:
(A_0) *a self-tube*, *that is*, *a horosphere*,
(A_1) *a geodesic hypersphere or* 2 *tube over* 2 *hyperplane* $H_{n-1}(\mathbf{C})$,
(A_2) *a tube over a totally geodesic* $H_k\mathbf{C}(1 \leq k \leq n-2)$,
(B) *a tube over a totally real hyperbolic space* $H_n\mathbf{R}$.

We denote by S and R_ξ be the Ricci tensor and the structure Jacobi operator with respect to the structure vector field ξ of M respectively. Then it is a very important problem to investigate real hypersurfaces satisfying $R_\xi S = SR_\xi$ in $M_n(c)$. From this point of view, Kim, Lee and one of the present authors ([4]) was recently proved the following:

THEOREM KKL ([4]). *Let M be a real hypersurface in a nonflat complex space form* $M_n(c)$. *If it satisfies* $R_\xi\phi = \phi R_\xi$, $R_\xi S = SR_\xi$ *and* $g(S\xi, \xi) = const.$, *then M is a Hopf hypersurface. Further*, *M is locally congruent to one of* (A_1), (A_2) *type if* $c > 0$, *or* (A_0), (A_1), (A_2) *type if* $c < 0$ *provided that* $\eta(A\xi) \neq 0$.

Further, the present authors ([5]) have been also proved the following:

THEOREM KNT ([5]). *Let M be a real hypersurface with* $R_\xi\phi = \phi R_\xi$ *and at the same time* $R_\xi S = SR_\xi$ *in* $M_n(c)$, $c \neq 0$. *If* $(\rho - \lambda)^2 - \frac{c}{4} \neq 0$, *then M is a Hopf hypersurface* (for the definitions of ρ and λ see section 2).

The main purpose of this paper is to establish the following theorem:

THEOREM 3.2. *Let M be a real hypersurface in a nonflat complex space form* $M_n(c)$ *which satisfies* $R_\xi\phi = \phi R_\xi$ *and at the same time* $R_\xi S = SR_\xi$. *If the scalar curvature of M is constant*, *then M is a Hopf hypersurface. Further*, *M is locally congruent to one of* (A_1), (A_2) *type if* $c > 0$, *or* (A_0), (A_1), (A_2) *type if* $c < 0$ *provided that* $\eta(A\xi) \neq 0$.

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

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1. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric $\tilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_Y X = \nabla_Y X + g(AY, X)N\,, \quad \tilde{\nabla}_X N = -AX\,,$$

for any vector fields X and Y on M, where g denoted the Riemannian metric of M induced from $\tilde{g}$ and A is the shape operator of M in $M_n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N\,, \quad JN = -\xi\,.$$

Then we may see that the aggregate (ϕ, ξ, η, g) is an

$$\phi^2 X = -X + \eta(X)\xi\,, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\,,$$

$$\eta(\xi) = 1\,, \quad \phi\xi = 0\,, \quad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M.

Since J is parallel, we find from the Gauss and Weingarten formulas the following:

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi\,, \quad \nabla_X \xi = \phi AX\,. \tag{1.1}$$

The ambient space being of constant holomorphic sectional curvature c, we obtain the following Gauss and Codazzi equations respectively:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY\,, \end{aligned} \tag{1.2}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{1.3}$$

for any vector fields X, Y and Z on M, where R denotes Riemann-Christoffel curvature tensor of M.

NOTATION. In the sequel, we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, $\gamma = \eta(A^3\xi)$, $h_{(2)} = \mathrm{Tr}\, {}^t\!AA$ and $h = \mathrm{Tr}\, A$, and for a function f we denote by ∇f the gradient vector field of f.

Putting $U = \nabla_\xi \xi$, we see that U is orthogonal to ξ. Thus we have

$$\phi U = -A\xi + \alpha\xi\,, \tag{1.4}$$

which leads to $g(U, U) = \beta - \alpha^2$.

From (1.2) the Ricci tensor S of type (1,1) on M is given by

$$S = \frac{c}{4}\{(2n+1)I - 3\eta \otimes \xi\} + hA - A^2\,, \tag{1.5}$$

where I is the identity tensor, which shows that

$$S\xi = \frac{c}{2}(n-1)\xi + hA\xi - A^2\xi\,. \tag{1.6}$$

If we put

$$A\xi = \alpha\xi + \mu W\,, \tag{1.7}$$

where W is a unit vector field orthogonal to ξ. Then we have $U = \mu\phi W$. So we verify that W is also orthogonal to U. Further we have

$$\mu^2 = \beta - \alpha^2\,. \tag{1.8}$$

Therefore, we easily see that ξ is a principal curvature vector, that is, $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$ or $\mu = 0$.

From the definition of U, and (1.1) and (1.7), we verify that

$$g(\nabla_X\xi, U) = \mu g(AW, X)\,. \tag{1.9}$$

Differentiating (1.4) covariantly along M and making use of (1.1), we find

$$\begin{aligned}\eta(X)g(AU + \nabla\alpha, Y) &+ g(\phi X, \nabla_Y U)\\ &= g((\nabla_Y A)X, \xi) - g(A\phi AX, Y) + \alpha g(A\phi X, Y)\,,\end{aligned} \tag{1.10}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha \tag{1.11}$$

because of (1.3) and (1.9). Since W is orthogonal to U, we verify, using (1.1), that

$$\mu g(\nabla_X W, \xi) = g(AU, X)\,. \tag{1.12}$$

Because of (1.1), (1.9) and (1.10), it is seen that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha\,. \tag{1.13}$$

2. Real hypersurfaces satisfying $R_\xi\phi = \phi R_\xi$ and $R_\xi S = SR_\xi$

Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. Then the structure Jacobi operator R_ξ with respect to ξ is given by

$$R_\xi X = R(X, \xi)\xi = \frac{c}{4}(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi \tag{2.1}$$

for any vector X on M, where we have used (1.2).

Now, suppose that $R_\xi\phi = \phi R_\xi$. Then above equation implies that

$$\alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U, X)A\xi\,. \tag{2.2}$$

We set Ω be a set of points such that $\mu(p) \neq 0$ at $p \in M$ and suppose that $\Omega \neq \emptyset$. In what follows we discuss our arguments on the open subset Ω of M unless otherwise stated. Then, it is, using (2.2), clear that $\alpha \neq 0$ on Ω. So a function λ given by $\beta = \alpha\lambda$ is defined. Therefore, replacing X by U in (2.1) and taking account of (1.4), we find

$$\phi AU = \lambda A\xi - A^2\xi\,. \tag{2.3}$$

Further, we assume that $R_\xi S = SR_\xi$. Then we see from (1.6) and (2.1) that

$$\begin{aligned} & g(A^3\xi, Y)g(A\xi, X) - g(A^3\xi, X)g(A\xi, Y) \\ & \quad = g(A^2\xi, Y)g\left(hA\xi - \frac{c}{4}\xi, X\right) - g(A^2\xi, X)g\left(hA\xi - \frac{c}{4}\xi, Y\right) \\ & \quad + \frac{c}{4}h\{g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)\}, \end{aligned}$$

which shows that

$$\alpha A^3\xi = \left(\alpha h - \frac{c}{4}\right)A^2\xi + \left(\gamma - \beta h + \frac{c}{4}h\right)A\xi + \frac{c}{4}(\beta - h\alpha)\xi . \tag{2.4}$$

Combining above two equations and using (1.7), we obtain

$$\mu\{g(A^2\xi, Y)w(X) - g(A^2\xi, X)w(Y)\} = \beta\{\eta(Y)g(A\xi, X) - \eta(X)g(A\xi, Y)\}$$

where an 1-form w is defined by $w(X) = g(W, X)$. Putting $Y = A\xi$ in this, we find

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi , \tag{2.5}$$

where we have put $\mu^2\rho = \gamma - \beta\alpha$ and $\mu^2(\beta - \rho\alpha) = (\beta^2 - \alpha\gamma)$ on Ω, which implies

$$A^3\xi = (\rho^2 - \beta - \rho\alpha)A\xi + \rho(\beta - \rho\alpha)\xi .$$

Comparing this with (2.4), we verify that

$$\mu(h - \rho)\left(\beta - \rho\alpha - \frac{c}{4}\right) = 0 . \tag{2.6}$$

REMARK 2.1. $h - \rho = 0$ on Ω.

In fact, if not, then we see from (2.6) that $\beta = \rho\alpha + \frac{c}{4}$ on a non empty open set $\Omega' = \{x \in \Omega \,|\, (h - \rho)(x) \neq 0\}$. Hence, (2.5) turns out to be $A^2\xi = \rho A\xi + \frac{c}{4}\xi$, which connected to (2.1) implies that $R_\xi A = AR_\xi$. Thus, by Corollary 4.2 of [4], it is seen that $\Omega' = \emptyset$. Hence $h = \rho$ on Ω is proved. In what follows $h = \rho$ is satisfied everywhere.

Since we have $\beta = \alpha\lambda$, (2.5) becomes

$$A^2\xi = hA\xi + \alpha(\lambda - h)\xi . \tag{2.7}$$

Thus, (2.3) implies that

$$AU = (h - \lambda)U . \tag{2.8}$$

We also have by (1.7) and (2.7)

$$AW = \mu\xi + (h - \alpha)W \tag{2.9}$$

because of $\mu \neq 0$.

Differentiating (2.7) covariantly along Ω and making use of (1.1), we find

$$\begin{aligned} g((\nabla_X A)A\xi, Y) + g(A(\nabla_X A)\xi, Y) + g(A^2\phi AX, Y) \\ - hg(A\phi AX, Y) \\ = (Xh)g(A\xi, Y) + hg((\nabla_X A)\xi, Y) \\ + X(\alpha\lambda - \alpha h)\eta(Y) + \alpha(\lambda - h)g(\phi AX, Y) \end{aligned} \tag{2.10}$$

for any vectors X and Y on M, which together with (1.3) and (1.11) yields

$$(\nabla_\xi A)A\xi = hAU - \frac{c}{4}U + \frac{1}{2}\nabla\beta .$$

Putting $X = \xi$ in (2.10) and taking account of (1.11), (2.8) and above equation, we obtain

$$\begin{aligned} \frac{1}{2}\nabla\beta = -A\nabla\alpha + h\nabla\alpha + (\xi h)A\xi + \xi(\alpha\lambda - \alpha h)\xi \\ - \left\{(h-\lambda)(h+\alpha-3\lambda) - \frac{c}{4}\right\}U , \end{aligned} \tag{2.11}$$

which connected to $\beta = \alpha\lambda$ implies that

$$\alpha(\xi\lambda) = (2\alpha - \lambda)\xi\alpha + 2\mu W\alpha . \tag{2.12}$$

Because of (2.9) and (2.11), we also have

$$\alpha W\lambda = (2\alpha - \lambda)W\alpha + 2\mu(\xi h - \xi\alpha) . \tag{2.13}$$

If we take account of (2.7) and (2.8), then (2.11) implies that

$$\begin{aligned} \frac{1}{2}(A\nabla\beta - h\nabla\beta) = -A^2\nabla\alpha + 2hA\nabla\alpha - h^2\nabla\alpha + (\xi\sigma)A\xi \\ + (\sigma\xi h - h\xi\sigma)\xi + \lambda\left\{(h-\lambda)(h+\alpha-3\lambda) - \frac{c}{4}\right\}U . \end{aligned} \tag{2.14}$$

where we have put $\sigma = \alpha(\lambda - h)$.

Now, differentiating (2.9) covariantly along Ω, we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(h-\alpha)W + (h-\alpha)\nabla_X W ,$$

which together with (1.3), (1.12) and (2.8) yields

$$\mu(\nabla_W A)\xi = \left\{(h-\lambda)(h-2\alpha) - \frac{c}{2}\right\}U + \frac{1}{2}\nabla\beta - \alpha\nabla\alpha , \tag{2.15}$$

$$(\nabla_W A)W = -2(h-\lambda)U + \nabla h - \nabla\alpha , \tag{2.16}$$

which shows that

$$W\mu = \xi h - \xi\alpha . \tag{2.17}$$

If we replace X by $A\xi$ in (2.10) and make use of (1.3), (1.7), (1.11), (2.7), (2.8) and the last two equations, we obtain

$$\frac{1}{2}(A\nabla\beta - h\nabla\beta) + \alpha^2\nabla\lambda + \mu^2\nabla h$$
$$= g(A\xi, \nabla h)A\xi + g(A\xi, \nabla\sigma)\xi + \left\{(h-\lambda)(2h\lambda - 3\alpha h + 2\alpha\lambda) + \frac{c}{4}(3\alpha - 2\lambda)\right\}U\,.$$

Substituting (2.14) into this, we find

$$\begin{aligned}&\alpha^2\nabla\lambda + \mu^2\nabla h - A^2\nabla\alpha + 2hA\nabla\alpha - h^2\nabla\alpha\\ &\quad= \{g(A\xi, \nabla h) - \xi\sigma\}A\xi + \{g(A\xi, \nabla\sigma) + h(\xi\sigma) - (\beta - h\alpha)\xi h\}\xi\\ &\quad\quad+ \left\{(h-\lambda)(h\lambda - 3\alpha h + \alpha\lambda + 3\lambda^2) + \frac{c}{4}(3\alpha - \lambda)\right\}U\,.\end{aligned} \tag{2.18}$$

Now, it is, using (2.1), verified that

$$\alpha\phi A\phi AX + \alpha A^2X = hg(A\xi, X)A\xi + \sigma\eta(X)A\xi - g(AU, X)U$$

because of properties of almost contact metric structure.

On the other hand, we have from (1.10)

$$\nabla_X U + g(A^2\xi, X)\xi = \phi(\nabla_X A)\xi + \phi A\phi AX + \alpha AX\,,$$

which together with (2.7) and the last equation yields

$$\begin{aligned}\nabla_X U + \{hg(A\xi, X) + \alpha(\lambda - h)\eta(X)\}\xi &= \phi(\nabla_X A)\xi + \alpha AX - A^2X\\ &\quad+ \frac{1}{\alpha}\{hg(A\xi, X) + \alpha(\lambda - h)\eta(X)\}A\xi - \frac{h-\lambda}{\alpha}g(U, X)U\,.\end{aligned}$$

If we put $X = U$ in this and take account of (2.8), then we obtain

$$\nabla_U U = \phi(\nabla_U A)\xi + (h-\lambda)(2\alpha - h)U\,. \tag{2.19}$$

If we differentiate (2.8) covariantly, we find

$$(\nabla_X A)U + A\nabla_X U = X(h-\lambda)U + (h-\lambda)\nabla_X U\,, \tag{2.20}$$

which together with (1.3), (1.13), (2.2) and (2.8) implies that

$$\begin{aligned}\phi(\nabla_U A)\xi = &-\left\{3(\lambda - h)(\lambda - \alpha) - \frac{c}{4} - \frac{1}{\alpha}U\alpha\right\}U - \mu(\xi h - \xi\lambda)W\\ &-(h-\lambda)(\nabla\alpha - (\xi\alpha)\xi) + A\nabla\alpha - \frac{1}{\alpha}g(A\xi, \nabla\alpha)A\xi\,.\end{aligned}$$

Substituting this into (2.19), we find

$$
\begin{aligned}
\nabla_U U = &\left\{(h-\lambda)(3\lambda-\alpha-h)+\frac{c}{4}+\frac{1}{\alpha}U\alpha\right\}U + A\nabla\alpha-(h-\lambda)\nabla\alpha \\
&+\{(h-\lambda)\xi\alpha-g(A\xi,\nabla\alpha)\xi\}-\mu\left\{\xi h-\xi\lambda+\frac{1}{\alpha}g(A\xi,\nabla\alpha)\right\}W\,,
\end{aligned}
\tag{2.21}
$$

which tells us that

$$
\begin{aligned}
A(\nabla_U U)-(h-\lambda)\nabla_U U = &\, A^2\nabla\alpha-2(h-\lambda)A\nabla\alpha+(h-\lambda)^2\nabla\alpha \\
&+\{(h-\lambda)\xi\alpha-g(A\xi,\nabla\alpha)\}\{A\xi-(h-\lambda)\xi\} \\
&-\mu\left(\xi h-\xi\lambda+\frac{1}{\alpha}g(A\xi,\nabla\alpha)\right)\{AW-(h-\lambda)W\}\,.
\end{aligned}
$$

Because of (1.3) and (1.4), the relationship (2.20) implies that

$$
\begin{aligned}
&\frac{c}{4}\mu\{\eta(Y)w(X)-\eta(X)w(Y)\}+g(AX,\nabla_Y U)-g(AY,\nabla_X U) \\
&\quad = Y(h-\lambda)u(X)-X(h-\lambda)u(Y) \\
&\qquad +(h-\lambda)\{(\nabla_Y u)(X)-(\nabla_X u)(Y)\}\,,
\end{aligned}
$$

where an 1-form u is defined by $u(X)=g(U,X)$.

If we replace X by U in this and make use of (2.8), then we obtain

$$
A(\nabla_U U)-(h-\lambda)\nabla_U U=\mu^2(\nabla\lambda-\nabla h)+U(h-\lambda)U\,,
$$

which together with (2.21) gives

$$
\begin{aligned}
&A^2\nabla\alpha-2(h-\lambda)A\nabla\alpha+(h-\lambda)^2\nabla\alpha \\
&\quad=\{g(A\xi,\nabla\alpha)-(h-\lambda)\xi\alpha\}\{A\xi-(h-\lambda)\xi\} \\
&\qquad+\mu\left\{\xi h-\xi\lambda+\frac{1}{\alpha}g(A\xi,\nabla\alpha)\right\}\{AW-(h-\lambda)W\} \\
&\qquad+\mu^2(\nabla\lambda-\nabla h)+U(h-\lambda)U\,.
\end{aligned}
\tag{2.22}
$$

Substituting (2.18) into (2.22) and using (2.11), we find

$$
\begin{aligned}
&2\mu^2(\nabla h-\nabla\lambda)+U(\lambda-h)U-3(\lambda-\alpha)\left\{(h-\lambda)^2-\frac{c}{4}\right\}U \\
&\quad=\{g(A\xi,\nabla h)-\xi\sigma-2\lambda(\xi h)\}A\xi+\{g(A\xi,\nabla\sigma)+(h-2\lambda)\xi\sigma-\sigma(\xi h)\}\xi \\
&\qquad+\{g(A\xi,\nabla\alpha)-(h-\lambda)\xi\alpha\}\{A\xi-(h-\lambda)\xi\} \\
&\qquad+\mu\left\{\xi h-\xi\lambda+\frac{1}{\alpha}g(A\xi,\nabla\alpha)\right\}\{AW-(h-\lambda)W\}\,.
\end{aligned}
\tag{2.23}
$$

Since $A\xi$ and AW are orthogonal to U, it follows from the last equation that

$$U(h-\lambda) = 3(\lambda-\alpha)\left\{(h-\lambda)^2 - \frac{c}{4}\right\}.$$

Using this, (1.7) and (2.9), the equation (2.23) can be written as

$$\mu^2(\nabla h - \nabla\lambda) = \mu^2(a\xi + bW) + 3(\lambda-\alpha)\left\{(h-\lambda)^2 - \frac{c}{4}\right\}U$$

for some functions a and b, which shows that $a = \xi h - \xi\lambda$ and $b = W(h-\lambda)$. Since $\lambda - \alpha$ does not vanish on Ω, we verify that

$$\alpha(\nabla h - \nabla\alpha) = \alpha(a\xi + bW) + 3\left\{(h-\lambda)^2 - \frac{c}{4}\right\}U\,. \tag{2.24}$$

On the other hand, if we take the inner product (2.23) with W, and straightforward calculation, then we obtain

$$\alpha^2 Wh = 3\alpha\mu\xi h + \alpha(4\alpha - 3\lambda)W\alpha - \mu(4\alpha-\lambda)\xi\alpha\,,$$

where we have used (2.12), (2.13) and the fact that $\sigma = \alpha(\lambda - h)$. Comparing this with (2.12) and (2.13), we see that $\alpha W(h-\lambda) = \mu\xi(h-\lambda)$, that is, $b\alpha = \mu a$. From this and (1.7), the equation (2.24) turns out to be

$$\alpha(\nabla h - \nabla\lambda) = aA\xi + 3\left\{(h-\lambda)^2 - \frac{c}{4}\right\}U\,.$$

Further, we can verify that $a = 0$ and hence

$$\alpha(\nabla h - \nabla\lambda) = 3\left\{(h-\lambda)^2 - \frac{c}{4}\right\}U\,.$$

(for detail, see [4]).

If we assume that $(h-\lambda)^2 - \frac{c}{4} \neq 0$ on an open subset Ω'' of Ω, then we have from the last equation

$$(Y\alpha)u(X) - (X\alpha)u(Y) = \alpha du(Y, X)$$

and

$$\lambda\nabla\alpha - \alpha\nabla\lambda = 2\left\{(h-\lambda)^2 + (h-\lambda)(\alpha - 2\lambda) - \frac{c}{4}\right\}U\,,$$

(for detail, see [5]). Using above two equations, we can verify that $du(Y, X) = 0$, where the exterior derivative du of 1-form u is given by

$$du(X, Y) = Yu(X) - Xu(Y) - u([X, Y])\,.$$

Therefore we have

$$\left\{(h-\lambda)^2 - \frac{c}{4}\right\}du(Y, X) = 0\,. \tag{2.25}$$

on Ω. Therefore, we see, using (1.9), (1.13) and (2.8), that

$$du(\xi, X) = (3\lambda - 2h)\mu w(X) + g(\phi\nabla\alpha, X) \tag{2.26}$$

for any vector X.

We prepare the following without proof in order to prove our Theorem 3.3 (See Lemma 3.5 of [4]).

REMARK 2.2. Let M be a real hypersurface in $M_n(c)$, $c \neq 0$ such that $R_\xi\phi = \phi R_\xi$ and $R_\xi S = SR_\xi$. If $du = 0$, then Ω is void.

3. Proof ot Theorem

We will continue our arguments under the same hypotheses $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi S = SR_\xi$ as in section 2. Because of Theorem KNT and Remark 2.2, we may only consider the case where $\theta = 3(h-\lambda)^2 - \frac{3}{4}c = 0$ and hence

$$(h-\lambda)^2 = \frac{c}{4} \tag{3.1}$$

by virtue of (2.25). From (1.6), (2.7) and Remark 2.1, it follows that

$$g(S\xi, \xi) = \frac{c}{2}(n-1) + (h-\lambda)\alpha\,,$$

which together with (3.1) implies that $g(S\xi, \xi) =$ const. if α is constant.

According to Theorem KKL, we have

LEMMA 3.1. *Let M be a real hypersurface with* (3.1) *satisfying $R_\xi\phi = \phi R_\xi$, and $R_\xi S = SR_\xi$ in $M_n(c)$, $c \neq 0$. If α is constant, then $\Omega = \emptyset$.*

Because of (3.1), the equations (2.11), (2.21) and (2.22) are reduced respectively to

$$A\nabla\alpha - h\nabla\alpha = -\frac{1}{2}\nabla\beta + (\xi h)A\xi + (\lambda - h)(\xi\alpha)\xi + (h-\lambda)(2\lambda - \alpha)U\,, \tag{3.2}$$

$$\begin{aligned}\nabla_U U = &\left\{(h-\lambda)(2\lambda-\alpha) + \frac{1}{\alpha}U\alpha\right\}U + A\nabla\alpha - (h-\lambda)\nabla\alpha \\ &+ \{(h-\lambda-\alpha)\xi\alpha - \mu W\alpha\}\xi - \{\mu\xi\alpha + (\lambda-\alpha)W\alpha\}W\,,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&A^2\nabla\alpha + 2(\lambda-h)A\nabla\alpha + (h-\lambda)^2\nabla\alpha \\ &\quad = \{g(A\xi, \nabla\alpha) - (h-\lambda)\xi\alpha\}\{A\xi - (h-\lambda)\xi\} \\ &\qquad + \frac{\mu}{\alpha}g(A\xi, \nabla\alpha)\{AW - (h-\lambda)W\}\,.\end{aligned} \tag{3.4}$$

Now, differentiating (1.7) covariantly, we find

$$(\nabla_X A)\xi + A\phi AX = (X\alpha)\xi + \alpha\phi AX + (X\mu)W + \mu\nabla_X W\,, \tag{3.5}$$

from which, taking the trace and using (2.17) we get

$$div W = 0 . \tag{3.6}$$

Putting $X = \mu W$ in (3.5) and making use of (1.8), (2.9), (2.15) and (3.1), we obtain

$$\begin{aligned} &\mu^2 \nabla_W W + \mu(W\mu)W \\ &= \frac{1}{2}\nabla\beta - \alpha\nabla\alpha - \mu(W\alpha)\xi + \{(h-\lambda)(2\lambda - 3\alpha) - \alpha(h-\alpha)\}U . \end{aligned} \tag{3.7}$$

By the way, from $\mu W = -\phi U$ we have

$$(X\mu)W + \mu\nabla_X W = g(AX, U)\xi - \phi\nabla_X U ,$$

where we have used (1.1), which shows that

$$-\mu\phi\nabla_W U = \mu^2\nabla_W W + \mu(W\mu)W .$$

From this and (3.7) it follows that

$$\mu\phi\nabla_W U = \alpha\nabla\alpha - \frac{1}{2}\nabla\beta + \mu(W\alpha)\xi + \{(h-\lambda)(3\alpha - 2\lambda) + \alpha(h-\alpha)\}U . \tag{3.8}$$

Differentiating $\mu\phi W = U$ covariantly and using (1.1), we also find

$$\nabla_X U = (X\mu)\phi W - \mu g(AX, W)\xi + \mu\phi\nabla_X W .$$

Putting $X = U$ in this, we obtain

$$\nabla_U U = \frac{1}{\mu}(U\mu)U + \mu\phi\nabla_U W ,$$

which together with (3.8) implies that

$$\begin{aligned} \mu\phi(\nabla_W U + \nabla_U W) = {} & \alpha\nabla\alpha - \frac{1}{2}\nabla\beta + \mu(W\alpha)\xi + \nabla_U U - \frac{1}{\mu}(U\mu)U \\ & + \{(h-\lambda)(3\alpha - 2\lambda) + \alpha(h-\alpha)\}U . \end{aligned}$$

Substituting (3.3) into this, we get

$$\begin{aligned} \mu\phi(\nabla_W U + \nabla_U W) = {} & A\nabla\alpha + (\lambda - h + \alpha)\nabla\alpha - \frac{1}{2}\nabla\beta \\ & + \left\{\frac{1}{\alpha}U\alpha - \frac{1}{\mu}U\mu + \alpha(3h - 2\lambda - \alpha)\right\}U \\ & + (h - \lambda - \alpha)(\xi\alpha)\xi - \{\mu\xi\alpha + (\lambda - \alpha)W\alpha\}W , \end{aligned}$$

or, using (3.2),

$$\begin{aligned} \mu\phi(\nabla_W U + \nabla_U W) = {} & \alpha(\nabla\alpha - \nabla h) + (\xi h - \xi\alpha)A\xi - (\lambda - \alpha)(W\alpha)W \\ & + \left\{\frac{1}{\alpha}U\alpha - \frac{1}{\mu}U\mu + 2h\alpha - \lambda\alpha - \alpha^2 + 2h\lambda - 2\lambda^2\right\}U . \end{aligned} \tag{3.9}$$

On the other hand, from (1.7) and (2.2) we have

$$(A\phi - \phi A)X + \eta(X)U + u(X)\xi + \tau(w(X)U + u(X)W) = 0\,, \tag{3.10}$$

where we have put

$$\alpha\tau = \mu\,. \tag{3.11}$$

From the last relationship, we see that

$$\mu\alpha\nabla\tau = \mu\nabla\mu - (\lambda - \alpha)\nabla\alpha\,. \tag{3.12}$$

Using (1.7) and (2.8), the equation (1.13) turns out to be

$$\nabla_\xi U = \mu(3\lambda - 3h + \alpha)W + \alpha(\alpha - \lambda)\xi + \phi\nabla\alpha\,. \tag{3.13}$$

Differentiating (3.10) covariantly and using (1.1), we find

$$\begin{aligned}(\nabla_k A^r_j)\phi^r_i &+ (\nabla_k A_{ir})\phi^r_j + A^2_{jk}\xi_i - A_{ki}(A_{jr}\xi^r) + A^r_{ik}\xi_j - A_{kj}(A_{ir}\xi^r)\\ &+ \nabla_k U_j(\xi_i + \tau w_i) + \nabla_k U_i(\xi_j + \tau w_j) + U_j\nabla_k\xi_i + U_i\nabla_k\xi_j\\ &+ \tau_k(U_j W_i + U_i W_j) + \tau(U_j\nabla_k W_i + U_i\nabla_k W_j) = 0\,.\end{aligned}$$

Now we define the function $h_{(2)}$ by $h_{(2)} = A^i_j A^j_i$. Then, taking $\sum g^{ki}$ on the last equation and summing for k and i, we obtain

$$\begin{aligned}-\frac{c}{2}(n-1)\xi &- \phi\nabla h - hA\xi + h_{(2)}\xi + \tau(\nabla_W U + \nabla_U W) + \mu(3\lambda - 3h + \alpha)W\\ &+ \alpha(\alpha - \lambda)\xi + \phi\nabla\alpha + divU(\xi + \tau W) - (h - \lambda)\mu W\\ &+ (W\tau)U + (U\tau)W = 0\,,\end{aligned}$$

where we have used (1.3), (2.8), (3.6) and (3.13), which tells us that

$$\begin{aligned}\alpha\phi(\nabla\alpha - \nabla h) &+ \mu(\nabla_W U + \nabla_U W) + \alpha(W\tau)U\\ &= \alpha\left\{\frac{c}{2}(n-1) + h\alpha - h_{(2)} + \alpha(\lambda - \alpha) - divU\right\}\xi\\ &\quad + \{\mu\alpha(5h - 4\lambda - \alpha) - \mu divU - \alpha(U\tau)\}W\end{aligned} \tag{3.14}$$

by virtue of (3.11). If we apply this by ϕ and make use of (2.17), (3.9) and (3.12), then we obtain

$$divU = (h - \lambda)(3\alpha - 2\lambda)\,. \tag{3.15}$$

Since we have

$$g(\nabla_W U + \nabla_U W, \xi) = \mu(\alpha - \lambda)$$

because of (1.1), (2.8) and (2.9), by taking the inner product (3.14) with ξ, we also find

$$divU = \frac{c}{2}(n-1) + h\alpha - h_{(2)} + \lambda^2 - \alpha\lambda\,.$$

From this and (3.15), it follows that

$$(h-\lambda)(3\alpha-2\lambda)=\frac{c}{2}(n-1)+h\alpha-h_{(2)}+\lambda^2-\alpha\lambda\,,$$

From this and (3.15), it follows that

$$(h-\lambda)(3\alpha-2\lambda)=\frac{c}{2}(n-1)+h\alpha-h_{(2)}+\lambda^2-\alpha\lambda\,,$$

which together with (3.1) implies that

$$\nabla h_{(2)}-2h\nabla h=2(\lambda-h)\nabla\alpha\,. \tag{3.16}$$

However, the scalar curvature r of M is given by

$$r=c(n^2-1)+h^2-h_{(2)}$$

since we have (1.5). Thus, (3.16) is reduced to

$$\nabla r=2(h-\lambda)\nabla\alpha\,.$$

Now, we assume that the scalar curvature of M is constant. Then we have

$$\nabla\alpha=0\,, \tag{3.17}$$

since $h-\lambda\neq 0$.

So, using Lemma 3.1, we finally have

THEOREM 3.2. *Let M be a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $R_\xi\phi=\phi R_\xi$ and at the same time $R_\xi S=SR_\xi$. If the scalar curvature of M is constant, then M is a Hopf hypersurface. Further, M is locally congruent to one of* (A_1), (A_2) *type if* $c>0$, *or* (A_0), (A_1), (A_2) *type if* $c<0$ *provided that* $\eta(A\xi)\neq 0$.

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