TOKYO J. MATH.
VOL. 30, NO. 2, 2007

A Note on Finite Simple Groups with Abelian Sylow p-subgroups

Masato SAWABE

Chiba University

(Communicated by K. Shinoda)

Abstract. In this note, we will make a remark on finite simple groups with abelian Sylow p-subgroups using the Classification Theorem of the Finite Simple Groups.

1. Introduction

In our paper Sawabe-Watanabe [6], we verified the Alperin's weight conjecture [1] for the principal block of a finite group X with an abelian Sylow p-subgroup P under the hypothesis (H_1) that $|N_X(P)/C_X(P)| = r$ for a prime r. Our method in [6] is as follows. We first reduce the conjecture under (H_1) to that of finite simple groups, and next try to obtain the result [6, Proposition 6.4]; which is saying that, under (H_1) and X is simple, P must be cyclic, $P \cong C_2 \times C_2$, or $X \cong PSL(2, p^e)$ for $p = 2, 3$. As the conjecture is known to be true in those three cases, we could conclude that the conjecture under (H_1) is verified. Note that to prove [6, Proposition 6.4], we used the Classification Theorem of the finite simple groups. On the other hand, in August 2002, the author was informed by Watanabe[8] that the conjecture for the principal block of a finite group X with an abelian Sylow p-subgroup P, under the another hypothesis (H_2) that $|N_X(P)/C_X(P)| = r^2$ for a prime r, can be also reduced to that of finite simple groups. So it is a frequent occurrence in modular representation theory that a problem on finite groups having abelian Sylow p-subgroups is reduced to that of finite simple groups. So it is quite valuable to investigate, in general, finite simple groups with abelian Sylow p-subgroups. From this reason, the purpose of this note is to prove the following:

THEOREM 1. *Let X be a finite simple group with an abelian Sylow p-subgroup P. Then one of the following holds.*

1. *$N_X(P)/C_X(P)$ contains an involution.*
2. *P is cyclic.*
3. *$P \cong C_2 \times C_2$.*
4. *$X \cong PSL(2, p^e)$.*

Received December 19, 2005; revised June 1, 2006
2000 *Mathematics Subject Classification*: 20D20.
Supported in part by Grant-in-Aid for Young Scientists (B) 16740020, the Ministry of Education, Culture, Sports, Science and Technology, Japan.

5. *$X \cong J_1$ or ${}^2G_2(3^{2m+1})$ with $p = 2$, and $N_X(P)/C_X(P) \cong (7:3)$.*

The following are immediate consequences of Theorem 1.

COROLLARY 1. *Let X be a finite simple group with an abelian Sylow p-subgroup P. Suppose that $|N_X(P)/C_X(P)|$ is a prime. Then one of the following holds.*

1. *P is cyclic.*
2. *$P \cong C_2 \times C_2$.*
3. *$X \cong PSL(2, p^e)$ for $p = 2$ or 3.*

PROOF. Set $\mathcal{E}_X(P) := N_X(P)/C_X(P)$. Suppose that $\mathcal{E}_X(P)$ contains an involution, then $\mathcal{E}_X(P) \cong C_2$. It follows that P is cyclic by Smith-Tyrer[7]. Suppose next that $X \cong PSL(2, p^e)$ with p odd, then $|\mathcal{E}_X(P)| = \frac{1}{2}(p^e - 1)$. If $p \geq 5$ then $\frac{1}{2}(p-1) \neq 1$ and $p^{e-1} + \cdots + p + 1 = 1$. This implies that $e = 1$, and thus P is cyclic. □

COROLLARY 2. *Let X be a finite simple group with an abelian Sylow p-subgroup P. Suppose that $|N_X(P)/C_X(P)| = r^2$ for a prime r. Then one of the following holds.*

1. *$N_X(P)/C_X(P) \cong C_4$ or $C_2 \times C_2$.*
2. *P is cyclic.*
3. *$X \cong PSL(2, 3^e)$.*

PROOF. Set $\mathcal{E}_X(P) := N_X(P)/C_X(P)$. Suppose that $P \cong C_2 \times C_2$, then $\mathcal{E}_X(P)$ is a subgroup of S_3; but this is impossible. Suppose next that $X \cong PSL(2, 2^e)$, then $r^2 = |\mathcal{E}_X(P)| = 2^e - 1$. Note that $e \geq 2$ as $r \neq 1$. Now let $r = 2k + 1$ then we have that $2^e = 4k^2 + 4k + 2$, a contradiction. Finally suppose that $X \cong PSL(2, p^e)$ with p odd, then $r^2 = |\mathcal{E}_X(P)| = \frac{1}{2}(p^e - 1)$. If $p \geq 5$ then $p - 1 = 2n$ for $n \geq 2$, and $p^e - 1 = 2nm$ where $m := p^{e-1} + \cdots + p + 1$. Suppose further that $e \neq 1$. Then $m \neq 1$ and $2r^2 = p^e - 1 = 2nm$. Thus $m = n = r$. But it follows that $p - 1 = 2n = 2m \geq 2(p+1)$, a contradiction. Therefore we have that if $p \geq 5$ then $e = 1$; namely P is cyclic. □

Note that the result [6, Proposition 6.4] mentioned above is exactly Corollary 1, so our result Theorem 1 contains one of the main parts of [6]. Furthermore as indicated earlier, the Alperin's weight conjecture for the principal block of a finite group X with an abelian Sylow p-subgroup P under the hypothesis (H_2) is reduced to that of finite simple groups. So Corollary 2 tells us that, to verify the conjecture under (H_2), it is enough to consider the only three cases described in it.

2. Preliminaries

Throughout this note, denote by $\pi(G)$ the set of primes dividing the order $|G|$ of a finite group G, and by C_n the cyclic group of order n. Furthermore, for a subgroup H of G, we set the factor group $\mathcal{E}_G(H) := N_G(H)/C_G(H)$ called the automizer of H in G. First we prepare the following proposition; which will be used later repeatedly. Although this is shown in [6], we will give a sketch of the proof.

PROPOSITION 1. *Let G be a finite group with an abelian Sylow p-subgroup P.*

1. *If Q is a subgroup of P, then $\mathcal{E}_G(Q)$ is involved in $\mathcal{E}_G(P)$; that is, there exist a subgroup M of $\mathcal{E}_G(P)$ and a normal subgroup N of M such that $\mathcal{E}_G(Q) \cong M/N$. In particular $|\mathcal{E}_G(Q)|$ divides $|\mathcal{E}_G(P)|$.*
2. *If H is an involved group in G with $p \in \pi(H)$, and R is a Sylow p-subgroup of H, then $\mathcal{E}_H(R)$ is involved in $\mathcal{E}_G(P)$. In particular $|\mathcal{E}_H(R)|$ divides $|\mathcal{E}_G(P)|$.*

PROOF. (1) As P is abelian, $P \leq C_G(Q)$. For any $n \in N_G(Q)$, we have that $P^n \leq C_G(Q)^n = C_G(Q) \geq P$, and that there exists $c \in C_G(Q)$ such that $P^{nc^{-1}} = P$. It follows that $N_G(Q) \leq N_G(P)C_G(Q)$, and $N_G(Q) = (N_G(Q) \cap N_G(P))C_G(Q)$ by Modular law. Thus

$$\mathcal{E}_G(Q) \cong N_G(Q) \cap N_G(P)/C_G(Q) \cap N_G(P)\,,$$

and which shows that $\mathcal{E}_G(Q)$ is a homomorphic image of a subgroup $N_G(Q) \cap N_G(P)/C_G(P)$ of $\mathcal{E}_G(P)$. Therefore $\mathcal{E}_G(Q)$ is involved in $\mathcal{E}_G(P)$.

(2) Let $N \trianglelefteq H_1$ be subgroups of G such that $H = H_1/N = \overline{H_1}$, and let $Q \in Syl_p(H_1)$ such that $R = QN/N = \bar{Q}$. Then there are natural surjective homomorphisms from $\mathcal{E}_{H_1}(Q)$ to $\overline{N_{H_1}(Q)}/\overline{C_{H_1}(Q)}$, and from $\overline{N_{H_1}(Q)}/\overline{C_{H_1}(Q)}$ to $\mathcal{E}_{\overline{H_1}}(\bar{Q}) = \mathcal{E}_H(R)$. On the other hand, since $\mathcal{E}_G(Q)$ is involved in $\mathcal{E}_G(P)$ by (1), and since $\mathcal{E}_G(Q)$ possesses a subgroup $N_{H_1}(Q)C_G(Q)/C_G(Q) \cong \mathcal{E}_{H_1}(Q)$, we have that $\mathcal{E}_{H_1}(Q) \cong L/K$ for some $K \trianglelefteq L \leq \mathcal{E}_G(P)$. This implies that there exist surjective homomorphisms $L \to \mathcal{E}_{H_1}(Q) \to \mathcal{E}_H(R)$. Therefore $\mathcal{E}_H(R)$ is involved in $\mathcal{E}_G(P)$. □

LEMMA 1. *Let G be a finite group, and P a p-subgroup of G with $p \notin \pi(Z(G))$. Then $\mathcal{E}_G(P) \cong \mathcal{E}_{\bar{G}}(\bar{P})$ where $\bar{G} = G/Z(G)$.*

PROOF. Straightforward. □

3. Alternating groups and sporadic groups

PROPOSITION 2. *Let X be the alternating group A_n $(n \geq 5)$ with an abelian Sylow p-subgroup P. Then either $\mathcal{E}_X(P)$ contains an involution or P is cyclic; except for $X = A_5 \cong PSL(2,4)$ and $p = 2$, and in which case $\mathcal{E}_{A_5}(P) \cong C_3$ and $P \cong C_2 \times C_2$.*

PROOF. If $p = 2$ then, since P is abelian, we have that $X = A_5$ and $\mathcal{E}_{A_5}(P) \cong C_3$. Now we may assume that $p \geq 3$, and express n as $pk + h$ $(k \in \mathbb{N}, 0 \leq h \leq p-1)$. If $k = 1$ then P is cyclic. Thus we may also assume that $k \geq 2$. Now we can use at least $2p$ letters $i_1^{(1)}, \ldots, i_p^{(1)}, i_1^{(2)}, \ldots, i_p^{(2)}$. For $d = 1, 2$, let $Q_d := \langle (i_1^{(d)}, \ldots, i_p^{(d)}) \rangle \leq X$ and $Q := Q_1 \times Q_2$. Up to conjugacy, we may assume that $Q \leq P$. Furthermore let

$$\alpha_d := (i_1^{(d)}, i_p^{(d)})(i_2^{(d)}, i_{p-1}^{(d)})(i_3^{(d)}, i_{p-2}^{(d)}) \cdots (i_r^{(d)}, i_{r+2}^{(d)}) \quad (d = 1, 2)\,,$$

a permutation on $\{i_1^{(d)}, \ldots, i_p^{(d)}\}$ where $r := \frac{1}{2}(p-1) \geq 1$ as $p \geq 3$. Notice that α_d normalizes Q_d but not centralize Q_d. Then an even permutation $\alpha_1\alpha_2$ is an involution lying in $\mathcal{E}_X(Q)$.

But since $|\mathcal{E}_X(Q)|$ divides $|\mathcal{E}_X(P)|$ by Proposition 1(1), we have that $|\mathcal{E}_X(P)|$ is even. The proof is complete. $\square$

REMARK. Even if P is cyclic, $\mathcal{E}_X(P)$ does not necessarily contain an involution. Indeed, if $p = 2r+1$ is an odd prime then for $C_p \cong P \in Syl_p(A_p)$, $\mathcal{E}_{A_p}(P)$ is of order r. Thus if r is odd then so is $|\mathcal{E}_{A_p}(P)|$.

PROPOSITION 3. *Let X be a sporadic simple group with an abelian Sylow p-subgroup P. Then either $\mathcal{E}_X(P)$ contains an involution or P is cyclic; except for the first Janko group $X = J_1$ and $p = 2$, and in which case $\mathcal{E}_{J_1}(P) \cong (7:3)$.*

PROOF. See for example [3] or [5, Section 5]. $\square$

For later use, we prepare the following on the symmetric groups.

PROPOSITION 4. *Let X be the symmetric group S_n $(n \geq 3)$ with an abelian Sylow p-subgroup P with an odd prime p. Then $\mathcal{E}_X(P)$ contains an involution.*

PROOF. As p is odd, we can write p as $2r+1$ for $r \geq 1$. Let $x = (i_1, i_2, \ldots, i_p)$ in P of order p, and let $Q := \langle x \rangle \cong C_p$ be a subgroup of P. Then for an involution

$$\alpha := (i_1, i_p)(i_2, i_{p-1})(i_3, i_{p-2}) \cdots (i_r, i_{r+2})$$

in X, we have that $x^\alpha = x^{-1} \neq x$ as $p \neq 2$. Thus α lies in $\mathcal{E}_X(Q)$. But since $|\mathcal{E}_X(Q)|$ divides $|\mathcal{E}_X(P)|$ by Proposition 1(1), we have that $|\mathcal{E}_X(P)|$ is even. $\square$

4. Some cases of Lie type groups

In this section, we will consider some special cases of Lie type groups. We refer to [2] for their standard property.

PROPOSITION 5 (Defining characteristic). *Let X be a simple group of Lie type over $GF(q)$ where $q = p^e$ for some prime p. Suppose that X possesses an abelian Sylow p-subgroup P. Then $X \cong PSL(2, q)$.*

PROOF. This follows from the Chevalley's commutator formula (see also [6, Proposition 5.1]). $\square$

PROPOSITION 6. *Let X be a simple group of Lie type, X^u a universal version of X, and P an abelian Sylow p-subgroup of X with $p \in \pi(Z(X^u))$ and $p \neq 2$. Then $\mathcal{E}_X(P)$ contains an involution.*

PROOF. As $p \in \pi(Z(X^u))$, it is enough to consider the following (see also in the proof of [6, Proposition 5.2]):

$$A_l(q)(l \geq 1)\,, \quad p|(l+1, q-1)\,; \quad E_6(q)\,, \quad p = 3\,;$$
$${}^2A_l(q^2)(l \geq 2)\,, \quad p|(l+1, q+1)\,; \quad {}^2E_6(q^2)\,, \quad q+1 \equiv 0(3)\,, \quad p = 3\,.$$

CASE. $E_6(q)$, $p = 3$: A Sylow 3-subgroup of $E_6(q)$ is not abelian, since the Weyl group $O^-(6, 2)$ of type E_6 possesses a non-abelian Sylow 3-subgroup.

CASE. $A_l(q) = PSL(l + 1, q)$, $p|(l + 1, q - 1)$: Let $X = PSL(l + 1, q)$. Since $p|(l + 1, q - 1)$, we have that $l + 1 \geq p \geq 3$, and that there exists $t \in GF(q)^\times \cong C_{q-1}$ such that $t^p = 1$ and $t \neq 1$. Let $D := \{\overline{M} = \overline{diag(\alpha_1, \ldots, \alpha_{l+1})} \in X \mid (\overline{M})^p = 1\}$, modulo $Z(X^u)$, be a p-subgroup of X where $diag(\alpha_1, \ldots, \alpha_{l+1})$ is a diagonal matrix in $SL(l+1, q)$. Let $w := \overline{A \oplus B}$ be an involution of X where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $B = diag(1, \ldots, 1, -1) \in GL(l - 1, q)$. Evidently w normalizes D but does not centralize an element $z = \overline{diag(t, t^{-1}, 1, \ldots, 1)}$ in D. Note that $t \neq t^{-1}$ as $p \neq 2$. This implies that an involution w is contained in $\mathcal{E}_X(D)$. But since $|\mathcal{E}_X(D)|$ divides $|\mathcal{E}_X(P)|$ by Proposition 1(1), $|\mathcal{E}_X(P)|$ is even.

CASE. ${}^2A_l(q^2) = PSU(l+1, q^2)$, $p|(l+1, q+1)$: Let $X = PSU(l+1, q^2)$. Recall that $SU(l + 1, q^2) = \{M \in SL(l + 1, q^2) \mid ({}^tM)\theta(M) = I\}$ where θ is an associated field automorphism; which is defined by $\theta(\alpha) = \alpha^q$ for $\alpha \in GF(q^2)^\times$. θ is of order 2. Now since $p|q + 1$, there exists $t \in \{\alpha \in GF(q^2)^\times \mid 1 = \alpha\theta(\alpha) = \alpha^{q+1}\} \cong C_{q+1}$ such that $t^p = 1$ and $t \neq 1$. Then the same argument as above can be applied. Indeed, define D, w, z as in the case of $A_l(q)$. Then D is a p-subgroup of $X = PSU(l + 1, q^2)$ with $z \in D$. Furthermore an involution $w \in X$ lies in $\mathcal{E}_X(D)$. But since $|\mathcal{E}_X(D)|$ divides $|\mathcal{E}_X(P)|$ by Proposition 1(1), $|\mathcal{E}_X(P)|$ is even.

CASE. ${}^2E_6(q^2)$, $q+1 \equiv 0(3)$, $p = 3$: Let $X = {}^2E_6(q^2)$, and then $H = PSU(6, q^2)$ is involved in X. Since $p = 3|(6, q + 1)$, we have that p divides $|Z(SU(6, q^2))|$. Then applying the unitary case above, we have that $|\mathcal{E}_H(Q)|$ is even for $Q \in Syl_p(H)$, and thus so is $|\mathcal{E}_X(P)|$. The proof is complete. □

PROPOSITION 7 (Weyl groups). *Let $X = {}^dX_l(q^d)$ be a universal group of Lie type, and P a Sylow p-subgroup of X with $p \neq 2$, $p \nmid q$ and $p \notin \pi(W(X_l))$ where $W(X_l)$ is the Weyl group of type X_l. Then $\mathcal{E}_X(P)$ contains an involution.*

REMARK. For formality of notation, ${}^1X_l(q^1)$ implies the untwisted group $X_l(q)$. In the case of Suzuki and Ree groups, namely ${}^dX_l = {}^2B_2, {}^2F_4, {}^2G_2$, we set ${}^dX_l(q^d) = {}^2B_2(q)$ $(q = 2^{2m+1})$, ${}^2F_4(q)$ $(q = 2^{2m+1})$, ${}^2G_2(q)$ $(q = 3^{2m+1})$. The twisted group ${}^dX_l(q^d)$ $(d \geq 2)$ is a set of elements of $X_l(q^d)$; which is fixed by a graph-field automorphism of order d in Aut$(X_l(q^d))$. (In the Atlas [3], ${}^dX_l(q^d)$ is denoted by ${}^dX_l(q, q^d)$, and a abbreviated notation ${}^dX_l(q) := {}^dX_l(q^d)$ is also used there.)

PROOF. Concerning the Sylow structure of P, we follow the argument in the proof of [5, (10-1)]. See [5] for the details. Let m be the multiplicative order of q modulo p, and set

$Y := X_l(q^{dm})$ a universal group. Then there exists a group $F = \langle \rho, \beta \rangle \leq \mathrm{Aut}(Y)$ generated by a graph-field automorphism $\rho = \sigma\theta$ of order d and a field automorphism β of order m such that $X \cong C_Y(F)$. (As mentioned above, in the case of Suzuki and Ree groups, we have that $Y = B_2(q^{2m})$, $F_4(q^{2m})$, $G_2(q^{2m})$, and thus β is of order $2m$ in these cases.) We identify X with $C_Y(F)$. Furthermore we may assume that up to conjugacy, P is contained in a Sylow p-subgroup R of a Cartan subgroup H of Y; in particular P is abelian. Then we have that $P = C_R(F)$ as $P = C_P(F) \leq C_R(F) \leq X$ and $P \in Syl_p(X)$. Recall that up to conjugacy, $H = \langle h_r(t) \,|r \in \Pi,\ t \in GF(q^{md})^\times \rangle$ where Π is a set of fundamental roots of Y and $h_r(t)$ is a standard generator of H. Thus letting E the unique Sylow p-subgroup of the multiplicative group $GF(q^{md})^\times$, we have that

$$R = \langle h_r(t) \mid r \in \Pi,\ t \in E \rangle\,.$$

Now let $\{\omega_r \mid r \in \Pi\}$ be a set of standard generators of the Weyl group of Y. Then setting $N = \langle \omega_r, H \mid r \in \Pi \rangle \leq Y$, we have that $N/H \cong W(X_l)$.

CASE. X is untwisted: Since $F = \langle \beta \rangle$ in this case, $P = C_R(\beta) = \langle h_r(t) \mid r \in \Pi,\ t \in E,\ t^\beta = t \rangle$. Take any $r \in \Pi$. Since $[\omega_r, \beta] = 1$ and $h_s(t)^{\omega_r} = h_{\omega_r(s)}(t)$ for any root s, we have that ω_r is in X and also normalizes $C_R(\beta) = P$; namely $\omega_r \in N_X(P)$. On the other hand, for $t \in E$ with $t \neq 1$ and $t^\beta = t$, an element $h_r(t)$ lies in $P \backslash \{1\}$, and we have that $h_r(t)^{\omega_r} = h_{-r}(t) = h_r(t)^{-1} \neq h_r(t)$ as $p \neq 2$. This implies that $\omega_r \notin C_X(P)$. Furthermore since $\omega_r^2 \in X \cap H \leq X \cap C_Y(P) = C_X(P)$, $\mathcal{E}_X(P)$ contains an involution $\omega_r C_X(P)$.

CASE. X is twisted: First we recall some ρ-invariant subgroups of $C_Y(\rho) \cong {}^dX_l((q^m)^d)$. For a σ-orbit J on Π, set $W(J) = \langle \overline{\omega_r} = \omega_r H \mid r \in J \rangle \leq N/H$. Then there exists a unique element $\overline{w_0(J)}$ of order 2 in $W(J)$ such that $\overline{w_0(J)}^\rho = \overline{w_0(J)}$. Then $\langle \overline{w_0(J)} \mid J = \sigma\text{-orbit on } \Pi \rangle$ is the Weyl group of $C_Y(\rho)$; which is isomorphic to $N^1/H^1 \cong N^1H/H$ where $N^1 = C_Y(\rho) \cap N$ and $H^1 = C_Y(\rho) \cap H$. We may assume that $w_0(J) \in N^1$. Recall that $\overline{w_0(J)}$ is a reflection along the vector $a(r)$ where $r \in J$ and $a(r)$ is the average of the vectors in the σ-orbit J. Next define an element of H^1 as follows:

$$\begin{array}{lll} h_J(t) := h_r(t) & \text{for } t \in GF(q^{dm})^\times \text{ with } \bar{t} = t\,, & \text{if } J = \{r\}\,, \\ h_J(t) := h_r(t)h_{\bar{r}}(\bar{t}) & \text{for } r \in J \text{ and } t \in GF(q^{dm})^\times\,, & \text{if } |J| = 2\,, \\ h_J(t) := h_r(t)h_{\bar{r}}(\bar{t})h_{\bar{\bar{r}}}(\bar{\bar{t}}) & \text{for } r \in J \text{ and } t \in GF(q^{dm})^\times, & \text{if } |J| = 3\,, \end{array}$$

where $\bar{r} = r^\sigma$ for a root r, and $\bar{t} = t^\theta$ for $t \in GF(q^{dm})$. (Note that if the characteristic of Suzuki-Ree groups $C_Y(\rho)$ is 2 or 3, then $h_J(t)$ is defined respectively as $h_r(t)h_{\bar{r}}(\bar{t^2})$ or $h_r(t)h_{\bar{r}}(\bar{t^3})$ for a short root r in J.) Any element of H^1 can be uniquely expressed as a product $\Pi h_J(t)$ where J runs through all σ-orbits on Π. Thus a Sylow p-subgroup R^1 of H^1 is as follows:

$$R^1 = \left\langle h_J(t) \;\middle|\; \begin{array}{l} J = \sigma\text{-orbit on } \Pi,\ t \in E \text{ such that } \bar{t} = t \\ \text{if } t \text{ is a coefficient of } h_J(t) \text{ with } |J| = 1 \end{array} \right\rangle$$

where $E \in Syl_p(GF(q^{dm})^{\times})$. Then $P = C_R(F) = C_{R^1}(\beta)$.

Return back to the proof of Proposition 7. Take a σ-orbit J on Π. Then we may assume that $[w_0(J), \rho] = [w_0(J), \beta] = 1$, and thus $w_0(J) \in X$. Furthermore since $w_0(J)$ normalizes R^1, a unique Sylow p-subgroup of H^1, we can see that $w_0(J)$ acts on $C_{R^1}(\beta) = P$; namely $w_0(J) \in N_X(P)$. Now we may assume that $|J| \geq 2$. Then, for $t \in E$ with $t \neq 1$ and $t^{\beta} = t$, an element $h_J(t)$ lies in $P \backslash \{1\}$. But $h_J(t)^{w_0(J)} = h_{-J}(t) \neq h_J(t)$ as $p \neq 2$; which implies that $w_0(J) \notin C_X(P)$. Furthermore since $w_0(J)^2 \in X \cap H \leq X \cap C_Y(P) = C_X(P)$, $\mathcal{E}_X(P)$ contains an involution $w_0(J)C_X(P)$. The proof is complete. □

PROPOSITION 8 (Primes p with $p|q-1$). *Let $X = {}^dX_l(q^d)$ be a universal group of Lie type, and P an abelian Sylow p-subgroup of X with $p \neq 2$ and $p|q-1$. Then $\mathcal{E}_X(P)$ contains an involution.*

PROOF. As $p|q-1$, p divides the order of a Cartan subgroup H of X. But we have shown in Proposition 7 implicitly that $|\mathcal{E}_X(Q)|$ is even for $Q \in Syl_p(H)$, and so is $|\mathcal{E}_X(P)|$. (see also [6, Propositions 5.3, 5.4]). □

Finally, we mention simple groups with abelian Sylow 2-subgroups (See [4, Chapter 16.6]):

PROPOSITION 9 (Abelian Sylow 2-subgroups). *Let X be a nonabelian simple group with an abelian Sylow 2-subgroup P. Then one of the followings holds.*

1. $X \cong PSL(2,q)$ *with* $q > 3$ *and* $q \equiv 3, 5 (mod\ 8)$, *or* $q = 2^e$.
2. $X \cong J_1$*; the first Janko group.*
3. $X \cong {}^2G_2(3^{2m+1})$*; the Ree group.*

Note that if $X \cong PSL(2,q)$ with $q > 3$ and $q \equiv 3, 5 (\mathrm{mod}\ 8)$ then $P \cong C_2 \times C_2$, and that if $X \cong J_1$ or ${}^2G_2(3^{2m+1})$ then $\mathcal{E}_X(P) \cong (7:3)$.

5. Classical groups

The aim of this section is to show the following:

PROPOSITION 10. *Let X be a classical simple group, and P an abelian Sylow p-subgroup of X with $p \neq 2$ and $p \nmid q$. Then either $\mathcal{E}_X(P)$ contains an involution or P is cyclic.*

PROPOSITION 11 (Untwisted classical). *Let $X = X_l(q)$ be one of universal groups $A_l(q)(l \geq 1)$, $B_l(q)(l \geq 2, q \equiv 1(2))$, $C_l(q)(l \geq 2)$, $D_l(q)(l \geq 4)$, and P an abelian Sylow p-subgroup of X with $p \neq 2$ and $p \nmid q$. Then $\mathcal{E}_X(P)$ contains an involution.*

PROOF. Let $W(X_l)$ be the Weyl group of type X_l. By Proposition 7, we may assume that $p \in \pi(W(X_l))$. Recall $W(A_l) \cong S_{l+1}$, $W(B_l) \cong W(C_l) \cong 2^l S_l$, and $W(D_l) \cong 2^{l-1} S_l$. As $p \neq 2$, p divides the order of the symmetric group S_n ($n = l$ or $l+1$). Then $|\mathcal{E}_{S_n}(Q)|$ is even for $Q \in Syl_p(S_n)$ by Proposition 4. But since $|\mathcal{E}_{S_n}(Q)|$ divides $|\mathcal{E}_X(P)|$ by Proposition 1(2), we have that $|\mathcal{E}_X(P)|$ is even. The proof is complete. $\square$

Let $X = {}^2X_l(q^2)$ be a universal version of a classical group. Then the order of X is expressed as

$$|X| = q^N \Pi_{m \in \mathcal{O}({}^2X_l)} \Phi_m(q)^{r_m}$$

where $\Phi_m(q)$ the cyclotomic polynomial for the mth roots of unity, $\mathcal{O}({}^2X_l)$ a set of positive integers depending on ${}^2X_l$, N the number of positive roots in the root system corresponding to X, and r_m a positive integer (see [5, Section 10] for the details). Note that r_m is known as in Table 1:

TABLE 1. r_m ([5, Table 10:1])

${}^2A_l$	$r_m = \left[\dfrac{l+1}{lcm(2,m)}\right]$	if $m \not\equiv 2(4)$
	$r_m = \left[\dfrac{2(l+1)}{m}\right]$	if $m \equiv 2(4),\ m > 2$
	$r_2 = l$	
${}^2D_l$	$r_m = \left[\dfrac{2l}{lcm(2,m)}\right]$	if $m \nmid l$
	$r_m = \left[\dfrac{2l}{lcm(2,m)}\right] - 1$	if $m \mid l$

Let e be the smallest positive integer such that $p|\Phi_e(q)$, and $m_p(X)$ the maximal p-rank of a Sylow p-subgroup of X. Set

$$\pi := \{p \in \pi(X) \mid p \neq 2,\ \ p \nmid q,\ \ p \notin \pi(Z(X))\}\,.$$

LEMMA 2 ((10-2) in [5]). *For $p \in \pi$, we have that $m_p(X) = m_p(X/Z(X)) = r_e$.*

We will keep the above notation throughout this section.

PROPOSITION 12 (Unitary groups). *Let $X = {}^2A_l(q^2) \cong SU(l+1, q^2)(l \geq 2)$ a universal group with an abelian Sylow p-subgroup P for $p \in \pi$. Then either $\mathcal{E}_X(P)$ contains an involution or P is cyclic.*

PROOF. Set $l = 2k$ or $2k-1$ for $k \geq 1$.

STEP 1. We may assume that $p \notin \pi(S_k)$:

Suppose that $p \in \pi(S_k)$, and let $Q \in Syl_p(S_k)$. Then $\mathcal{E}_{S_k}(Q)$ contains an involution by Proposition 4. But since S_k is involved in X as the (twisted) Weyl group, we have that $|\mathcal{E}_X(P)|$ is even by Proposition 1(2). Thus we may assume that $p \notin \pi(S_k)$.

STEP 2. We may assume that $e > 1$ and $r_e > 1$:

If $e = 1$ then $p|\Phi_1(q) = q - 1$ and thus $|\mathcal{E}_X(P)|$ is even by Proposition 8. On the other hand if $r_e = 1$ then $m_p(X) = r_e = 1$ by Lemma 2 and thus an abelian Sylow p-subgroup P is cyclic.

STEP 3. If $e = 2i$ and $i \geq 2$ is even then $\mathcal{E}_X(P)$ contains an involution:

Since $e \not\equiv 2(4)$ and $2 \leq r_e = [\frac{l+1}{e}]$, we have that $2e \leq l+1$ and $e \leq \frac{l+1}{2} = k$ or $k + \frac{1}{2}$; which follows that $e \leq k$ and $\pi(S_e) \subseteq \pi(S_k)$. Let $H = {}^2A_{e-1}(q^2) \cong SU(e, q^2)$ $(e \geq 4)$ be a subgroup of X. As $r_e = [\frac{e}{e}] = 1$ for H, $p \in \pi(H)$. But since $\pi(W(A_{e-1})) = \pi(S_e) \subseteq \pi(S_k)$, $p \notin \pi(W(A_{e-1}))$ by Step 1. Thus $|\mathcal{E}_H(Q)|$ is even for $Q \in Syl_p(H)$ by Proposition 7. Now $|\mathcal{E}_H(Q)|$ divides $|\mathcal{E}_X(P)|$ by Proposition 1(2), and hence $|\mathcal{E}_X(P)|$ is even.

STEP 4. If $e = 2i$ and $i \geq 1$ is odd then $\mathcal{E}_X(P)$ contains an involution:

Suppose $e = 2$; that is, $p|\Phi_2(q) = q+1$. Let $H = {}^2A_1(q^2) \cong SU(2, q^2) \cong SL(2, q)$ be a subgroup of X. As $|H| = q(q-1)(q+1)$, $p \in \pi(H)$. Then $|\mathcal{E}_H(Q)|$ is even for $Q \in Syl_p(H)$ by Proposition 11. Thus we may assume that $i \geq 3$.

Since $e \equiv 2(4)$ and $e > 2$, we have that $2 \leq r_e = [\frac{2(l+1)}{e}]$ and $i = \frac{e}{2} \leq \frac{l+1}{2} = k$ or $k + \frac{1}{2}$; which follows that $i \leq k$ and $\pi(S_i) \subseteq \pi(S_k)$. Let $H = {}^2A_{i-1}(q^2) \cong SU(i, q^2)$ $(i \geq 3)$ be a subgroup of X. As $r_e = [\frac{2i}{e}] = 1$ for H, $p \in \pi(H)$. But since $\pi(W(A_{i-1})) = \pi(S_i) \subseteq \pi(S_k)$, $p \notin \pi(W(A_{i-1}))$ by Step 1. Thus $|\mathcal{E}_H(Q)|$ is even for $Q \in Syl_p(H)$ by Proposition 7.

STEP 5. If $e \geq 3$ is odd then $\mathcal{E}_X(P)$ contains an involution:

Since $e \not\equiv 2(4)$, we have that $2 \leq r_e = [\frac{l+1}{2e}]$ and $2e \leq \frac{l+1}{2} = k$ or $k + \frac{1}{2}$; which follows that $2e \leq k$ and $\pi(S_{2e}) \subseteq \pi(S_k)$. Let $H = {}^2A_{2e-1}(q^2) \cong SU(2e, q^2)$ $(2e \geq 6)$ be a subgroup of X. As $r_e = [\frac{2e}{2e}] = 1$ for H, $p \in \pi(H)$. But since $\pi(W(A_{2e-1})) = \pi(S_{2e}) \subseteq \pi(S_k)$, $p \notin \pi(W(A_{2e-1}))$ by Step 1. Thus $|\mathcal{E}_H(Q)|$ is even for $Q \in Syl_p(H)$ by Proposition 7. The proof is complete. $\square$

PROPOSITION 13 (Orthogonal groups of type $-$). *Let $X = {}^2D_l(q^2) \cong \Omega^-(2l, q)$ $(l \geq 4)$ a universal group with an abelian Sylow p-subgroup P for $p \in \pi$. Then either $\mathcal{E}_X(P)$ contains an involution or P is cyclic.*

PROOF. STEP 1. We may assume that $p \notin \pi(S_{l-1})$:

Suppose that $p \in \pi(S_{l-1})$, and let $Q \in Syl_p(S_{l-1})$. Then $\mathcal{E}_{S_{l-1}}(Q)$ contains an involution by Proposition 4. But since $2^{l-1}S_{l-1}$ is involved in X as the (twisted) Weyl group, we have that $|\mathcal{E}_X(P)|$ is even by Proposition 1(2). Thus we may assume that $p \notin \pi(S_{l-1})$.

STEP 2. We may assume that $e > 1$ and $r_e > 1$:

By the same reason as in the proof of Step 2 in Proposition 12.

STEP 3. If $e = 2i$ is even then $\mathcal{E}_X(P)$ contains an involution:

Suppose $e = 2$ or 4; that is $p|\Phi_2(q) = q+1$ or $p|\Phi_4(q) = q^2+1$. Let $H = {}^2D_2(q^2) \cong A_1(q^2)$ be a subgroup of X. As $|H| = q^2(q^2-1)(q^2+1)$, $p \in \pi(H)$. Then $|\mathcal{E}_H(Q)|$ is even for $Q \in Syl_p(H)$ by Proposition 11. But since $|\mathcal{E}_H(Q)|$ divides $|\mathcal{E}_X(P)|$ by Proposition 1(2), we have that $|\mathcal{E}_X(P)|$ is even. Thus we may assume that $i \geq 3$.

Since $1 < r_e \leq [\frac{2l}{e}]$, we have that $e < 2l$ and $i = \frac{e}{2} < l$; which follows that $i \leq l-1$ and $\pi(S_i) \subseteq \pi(S_{l-1})$. Let $H = {}^2D_i(q^2) \cong \Omega^-(2i, q)$ $(i \geq 3)$ be a subgroup of X. (Note that ${}^2D_3(q^2) \cong {}^2A_3(q^2)$.) As, for H, $r_e = [\frac{2i}{e}] = 1$ if $i \geq 4$ and $r_e = [\frac{2(3+1)}{e}] = 1$ if $i = 3$, we have that $p \in \pi(H)$. Furthermore if $i \geq 4$ then since $\pi(W(D_i)) = \pi(2^{i-1}S_i) \subseteq \pi(S_{l-1})$ we have that $p \notin \pi(W(D_i))$ by Step 1, and if $i = 3$ then since $\pi(W(A_3)) = \pi(S_4)$ and $p > l-1 \geq i = 3$ we have that $p \notin \pi(W(A_3))$. In either case, p does not divide the order of the Weyl group $W(D_i)$ or $W(A_3)$ of H. Thus $\mathcal{E}_H(Q)$ is even for $Q \in Syl_p(H)$ by Proposition 7.

STEP 4. If e is odd then $\mathcal{E}_X(P)$ contains an involution:

Since $2 \leq r_e \leq [\frac{2l}{2e}]$, we have that $e \leq \frac{l}{2} < l-1$ and $e+1 \leq l-1$; which follows that $\pi(S_{e+1}) \subseteq \pi(S_{l-1})$. Let $H = {}^2D_{e+1}(q^2)$ $(e+1 \geq 4)$ be a subgroup of X. As $r_e = [\frac{2(e+1)}{2e}] = 1$ for H, $p \in \pi(H)$. But since $\pi(W(D_{e+1})) = \pi(2^e S_{e+1}) \subseteq \pi(S_{l-1})$, $p \notin \pi(W(D_{e+1}))$ by Step 1. Thus $|\mathcal{E}_H(Q)|$ is even for $Q \in Syl_p(H)$ by Proposition 7. The proof is complete. □

PROOF OF PROPOSITION 10. Let X^u be a universal version of X. By Proposition 6, we may assume that $p \notin \pi(Z(X^u))$. Then we have, by Propositions 11, 12, 13, that either $|\mathcal{E}_{X^u}(R)|$ is even or R is cyclic for $R \in Syl_p(X^u)$. But this implies that, for $P := \bar{R} \in Syl_p(X)$ modulo $Z(X^u)$, either $|\mathcal{E}_X(P)| = |\mathcal{E}_{X^u}(R)|$ is even by Lemma 1, or $P \cong R$ is cyclic, as desired. □

6. Exceptional groups

The aim of this section is to show the following:

PROPOSITION 14. *Let X be an exceptional simple group, and P an abelian Sylow p-subgroup of X with $p \neq 2$ and $p \nmid q$. Then either $\mathcal{E}_X(P)$ contains an involution or P is cyclic.*

PROPOSITION 15 (Untwisted exceptional). *Let $X = X_l(q)$ be one of universal groups $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, $G_2(q)$, and P an abelian Sylow p-subgroup of X with $p \neq 2$ and $p \nmid q$. Then $\mathcal{E}_X(P)$ contains an involution.*

PROOF. Let $W(X_l)$ be the Weyl group of type X_l. By Proposition 7, we may assume that $p \in \pi(W(X_l))$. Recall $W(E_6) \cong PSp(4,3)2$, $W(E_7) \cong 2 \times Sp(6,2)$, $W(E_8) \cong 2\Omega^+(8,2)2$, $W(F_4) \cong (2^3 S_4)S_3$, and $W(G_2) \cong D_{12}$. As $p \neq 2$, p divides the order of a group H; which is a classical group, the symmetric group, or the dihedral group D_{12}. Thus $|\mathcal{E}_H(Q)|$ is even for $Q \in Syl_p(H)$ by Propositions 4 or 11. But since $|\mathcal{E}_H(Q)|$ divides $|\mathcal{E}_X(P)|$ by Proposition 1(2), we have that $|\mathcal{E}_X(P)|$ is even. The proof is complete. □

PROPOSITION 16 (Twisted exceptional). *Let $X = {}^dX_l(q^d)$ be one of universal groups ${}^3D_4(q^3)$, ${}^2E_6(q^2)$, ${}^2F_4(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, ${}^2B_2(2^{2m+1})$, and P an abelian Sylow p-subgroup of X with $p \neq 2$, $p \nmid q$, $p \notin \pi(Z(X))$. Then either $\mathcal{E}_X(P)$ contains an involution or P is cyclic.*

PROOF. If $X = {}^2G_2(3^{2m+1})$ or ${}^2B_2(2^{2m+1})$ then an abelian Sylow p-subgroup P of X is always cyclic (see [5, (10-2)] or Lemma 2). Thus we may assume that X is otherwise.

Now let $W(X_l)$ be the Weyl group of type X_l. By Proposition 7, we may assume that $p \in \pi(W(X_l))$.

CASE. $X = {}^3D_4(q^3)$: Since $p \in \pi(W(D_4)) = \pi(2^3S_4) = \{2, 3\}$, we have that $p = 3$. Note that X possesses $W(G_2) \cong D_{12}$ as the (twisted) Weyl group, and $|\mathcal{E}_{D_{12}}(Q)|$ is even for $Q \in Syl_3(D_{12})$. But since $|\mathcal{E}_{D_{12}}(Q)|$ divides $|\mathcal{E}_X(P)|$ by Proposition 1(2), we have that $|\mathcal{E}_X(P)|$ is even.

CASE. $X = {}^2F_4(q)$ $(q = 2^{2m+1}, m \geq 1)$: Since $p \in \pi(W(F_4)) = \pi(W(D_4)S_3) = \{2, 3\}$, we have that $p = 3$. Let $H = SL(2, q)$ be a subgroup of X. As $|H| = q(q-1)(q+1)$, $p \in \pi(H)$. (Note that if $p = 3$ does not divide $q - 1$ then $q + 1$ is divisible by p.) Thus $|\mathcal{E}_H(Q)|$ is even for $Q \in Syl_p(H)$ by Proposition 11.

CASE. $X = {}^2E_6(q^2)$: Since $p \in \pi(W(E_6)) = \pi(PSp(4, 3)2) = \{2, 3, 5\}$, we have that $p = 3$ or 5. Note that X possesses $W(F_4) \cong (2^3S_4)S_3$ as the (twisted) Weyl group. So if $p = 3$ then, for an involved group S_3, we have that $\mathcal{E}_{S_3}(R) \cong C_2$ for $R \in Syl_3(S_3)$. Thus $|\mathcal{E}_X(P)|$ is even, and we may assume that $p = 5$.

Let $H = F_4(q)$ be a subgroup of X of order

$$|H| = q^{24}\Phi_1(q)^4\Phi_2(q)^4\Phi_3(q)^2\Phi_4(q)^2\Phi_6(q)^2\Phi_8(q)\Phi_{12}(q)\,,$$

where $\Phi_m(q)$ is the cyclotomic polynomial for the mth roots of unity (see [5, Table 4-1] for the existence of $F_4(q)$ in X). Now it is easy to see that if $p = 5$ does not divide both $\Phi_1(q) = q - 1$ and $\Phi_2(q) = q + 1$ then $\Phi_4(q) = q^2 + 1$ is divisible by p. Thus p always divides $|H|$. But since $\pi(W(F_4)) = \pi((2^3S_4)S_3)$, $p = 5 \notin \pi(W(F_4))$. Thus $|\mathcal{E}_H(Q)|$ is even for $Q \in Syl_p(H)$ by Proposition 7. The proof is complete. □

PROOF OF PROPOSITION 14. The same as in that of Proposition 10. □

7. Proof of Theorem 1

Suppose that X is the alternating group or a sporadic group. Then by Propositions 2 and 3, $|\mathcal{E}_X(P)|$ is even; (1), P is cyclic; (2), $P \cong C_2 \times C_2$; (3), or $X = J_1$; (5).

Suppose next that X is a Lie type group ${}^dX_l(q^d)$. If $p = 2$ then by Proposition 9, $P \cong C_2 \times C_2$; (3), $X \cong PSL(2, p^e)$; (4), or $X \cong {}^2G_2(3^{2m+1})$; (5). If $p|q$ then by Proposition 5, $X \cong PSL(2, p^e)$; (4). Thus we may assume that $p \neq 2$ and $p \nmid q$. Then by Propositions 10 and 14, $|\mathcal{E}_X(P)|$ is even; (1), or P is cyclic; (2).

Finally we consider the Tits simple group $X = {}^2F_4(2)'$ of order $2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$. Then it easy to see that $|\mathcal{E}_G(P)|$ is even; (1), or P is cyclic; (2), (see [3]). The proof is complete. □

References

[1] J. ALPERIN, Weights for finite groups, Proc. Symposia in Pure Math. **47** (1987), 369–379.
[2] R. CARTER, *Simple Groups of Lie Type*, Wiley-Interscience, New York, 1972.

[3] J. CONWAY, R. CURTIS, S. NORTON, R. PARKER and R. WILSON, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[4] D. GORENSTEIN, *Finite groups*, Harper and Row, New York, 1968.

[5] D. GORENSTEIN and R. LYONS, The local structure of finite groups of characteristic 2 type, Memoirs of Amer. Math. Soc. **276** (1983).

[6] M. SAWABE and A. WATANABE, On the principal blocks of finite groups with abelian Sylow p-subgroups, J. Algebra **237** (2001), 719–734.

[7] S. SMITH and A. TYRER, On finite groups with a certain Sylow normalizer II, J. Algebra **26** (1973), 366–367.

[8] A. WATANABE, private communication in August 2002.

Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION,
CHIBA UNIVERSITY,
CHIBA, 263–8522 JAPAN.
e-mail: sawabe@faculty.chiba-u.jp