36. A Remark on my Paper "A Unique Continuation Theorem of a Parabolic Differential Equation"

By Taira SHIROTA
Osaka University

(Comm. by K. KUNUGI, M.J.A., March 12, 1960)

§1. Introduction. It is well known that real solutions of second order elliptic equations with real coefficients have the property that if the difference of two vanishes sufficiently fast at a point then they are identical in their common range of definition. The question naturally arises what kind of extensions of the unique continuation theorem mentioned above are valid for solutions of parabolic differential equations?

In the present note, we give a simple proof of the theorem[1] in my paper[2] in which I asserted a partial answer of the problem.

§2. Let G be a convex domain of the Euclidean $n+1$ space $R_{t,x}$: $\{-\infty<t<+\infty,\ -\infty<x_i<+\infty\ (i=1,2,\cdots,n)\}$, containing a curve C: $\{(t, x_i(t)) \mid t\in[a,b]\}$, where $x_i(t)\in C^1[a,b]$.

Consider real solutions u of an inequality of the following kind:

$$(2.1)\quad \left|\frac{\partial u(t,x)}{\partial t}-\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i\partial x_j}\right|\leqq M\left\{\sum_{i=1}^{n}\left|\frac{\partial u(t,x)}{\partial x_i}\right|+|u(t,x)|\right\}.$$

Here $(a_{ij}(t,x))$ denote a positive definite, symmetric matrix of real valued functions $a_{ij}(t,x)\in C^2(G)$,[3] and M a constant.

The theorem in my previous paper is the following.

Theorem.[4] *If u is a solution of* (2.1) *in the domain G and if for any $\alpha>0$*

$$(2.2)\quad \lim_{r\to 0}\max_{\substack{|x-x(t)|=r\\ t\in[a,b]\\ i,j=1,2,\cdots,n}}\left\{\left|u(t,x)\right|,\left|\frac{\partial u}{\partial x_i}(t,x)\right|,\left|\frac{\partial^2 u}{\partial x_i\partial x_j}(t,x)\right|\right\}|x-x(t)|^{-\alpha}=0,$$

then u vanishes identically in the horizontal component $G\wedge\{(t,x)\mid t\in[a,b]\}$.

In the following we shall sketch the direct proof of the theorem using notations stated in my paper without repeating definitions of them.

---

1) See below §2.

2) T. Shirota: A unique continuation theorem of a parabolic differential equation, Proc. Japan Acad., **35**, 455-460 (1959).

3) This restriction of the coefficients may be weakened. For instance, we may remove the restriction with respect to $a_{ij}|_{tt}$.

4) More precisely, in my previous paper we assume that $x_i(t)\in C^2[a,b]$ and in (2.2) the term with respect to u_t was inserted, but the refinements of these assumptions in the theorem will be of no essential matter.

§3. To prove the theorem we may assume that $C=[-\varepsilon, 1+\varepsilon]\times\{0\}$ and that

$$L(u)=q(t,x)\frac{\partial u}{\partial t}-\sum_{ij}\bar{a}_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_i b_i\frac{\partial u}{\partial x_i}$$

$$(3.1)\qquad =q(t,x)\frac{\partial u}{\partial t}-\Big(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}+\frac{N}{r^2}\Big)u,$$

where $((\bar{a}_{ij}))$ is positive definite and $\bar{a}_{ij}\in C^0(t,x)$ $(0\leqq r\leqq R)$, $q(t,x)$ $(>\delta>0)$ $\in\bar{C}^1(t,x)$, $b_i(t,x)\in\bar{C}^0(t,x)$, the coefficients of $N\in\bar{C}^1(t, r, \varphi_\sigma)$ and the derivatives of the coefficients of N with respect to $t\in\bar{C}^1(\varphi_\sigma)$ $(0<r\leqq R)$. Furthermore we may assume that for $n>1$.

$$(3.2)\qquad \int Nw_1\cdot w_2 dO_1=\int Nw_2\cdot w_1 dO_1,$$

$$(3.3)\qquad \frac{\partial}{\partial r}\int Nw\cdot wdO_1\leqq m_0\int Nw\cdot wdO_1<0\quad (m_0\geqq 2)$$

for any w_1, w_2 and $w\in C^2(x\mid |x|=1)$, for any $t\in[-\varepsilon, 1+\varepsilon]$ and for any $r\in(0,R]$.

Moreover we suppose that u satisfies the conditions (2.1) and (2.2) with $x_i(t)=0$ for $t\in[-\varepsilon, 1+\varepsilon]$. Let D_{r_0, K_0} $(r_0\leqq R)$ be the domain $\{(t,x)\mid 0<t<1 \text{ and } |x|<r_0\wedge K_0^{-1}t\wedge K_0^{-1}(1-t)\}$. Moreover let $\rho(r)$ and $\varphi(t)$ be the smooth functions such that

$$\rho(r)=1\quad\text{for}\quad r:0\leqq r\leqq\frac{3}{4}$$

$$\rho(r)=0\quad\text{for}\quad r:r\geqq\frac{4}{5}$$

$$0\leq\rho(r)\leq 1\quad\text{for any } r,$$

and such that

$$\begin{aligned}\varphi(t)&=K_0^{-1}t\quad\text{for}\quad t:0\leqq t\leqq K_0r_0-\varepsilon\\ &=r_0\quad\text{for}\quad t:K_0r_0+\varepsilon\leqq t\leqq 1-K_0r_0-\varepsilon\\ &=1-K_0^{-1}t\quad\text{for}\quad t:1-K_0r_0+\varepsilon\leqq t\leqq 1,\ \text{and}\end{aligned}$$

$$\Big(\frac{2}{3}r_0\Big)\wedge\Big(\frac{3}{2}K_0\Big)^{-1}t\wedge\Big(\frac{3}{2}K_0\Big)^{-1}(1-t)\leqq\varphi(t)\leqq r_0\wedge K_0^{-1}t\wedge K_0^{-1}(1-t).$$

Then $v=u\Psi(t,x)=u\cdot\rho(r\cdot\varphi(t)^{-1})\in\mathfrak{K}$ where $\mathfrak{K}$ is the class of functions v such that $v\in C^2(x)\cap C^1(t)$ in D_{r_0, K_0} with the carrier contained in $\bar{D}_{\frac{4}{5}r_0, \frac{5}{4}K_0}$ and such that v satisfies the following condition:

$$(3.4)\qquad \lim_{r\to 0}\max_{\substack{|x|=r\\ t\in[0,1]\\ i,j=1,2,\cdots,n}}\{|v(t,x)|, |v_{|x_i}(t,x)|, |v_{|x_ix_j}(t,x)|, |v_{|t}(t,x)|\}r^{-\alpha}=0.$$

Our proof of theorem follows immediately from the following lemma.[5)]

Lemma. Let $\Phi(t)$ be the smooth function such that

$$\Phi(t)=t\qquad\text{for}\quad t:\ 0\leqq t\leqq\frac{1}{5}$$

5) See 2).

$$=1 \quad \text{for} \quad t: \frac{2}{5} \leqq t \leqq \frac{3}{5}$$

$$=1-t \quad \text{for} \quad t: \frac{4}{5} \leqq t \leqq 1.$$

Then for sufficiently small r_0 and sufficiently large K_0 with $r_0 K_0 \leqq \frac{1}{5}$, there are constants α_0 and K_1 such that for any $\alpha > \alpha_0$ and for any $v \in \mathfrak{K}$,

$$\iiint (Lv)^2 r^{3-2\alpha} \Phi(t)^{3\alpha}\, dr dO_1\, dt \geqq K_1 \alpha^3 \iiint (v)^2 r^{-2\alpha} \Phi(t)^{3\alpha}\, dr dO_1\, dt \tag{3.5}$$

$$\iint (Lv)^2 r^{4-2\alpha-n} \Phi(t)^{3\alpha} dx dt \geqq K_1 \alpha \iint \Big(|v|^2 + \sum_{i=1}^{n} |v_{|x_i}|^2 \Big) r^{4-2\alpha-n} \Phi(t)^{3\alpha}\, dx dt. \tag{3.6}$$

The inequality (3.3) follows from (3.2) by the Cordes' method and the inequality (3.2) is derived from (3.2), (3.3) and the following inequalities:

$$\iiint (Lv)^2 r^{3-\alpha} \Phi(t)^{3\alpha}\, dr dO_1\, dt$$

$$\geqq \iiint \{2\alpha r z \cdot L^* z - \alpha^2 r^2 z^2 + (L^{**} z)^2 + 2L^* z L^{**} z\} r^{-1} \Phi(t)^{3\alpha}\, dr dO_1\, dt,\text{[6]}$$

where $z = v r^{-\alpha}$, $L^* z = \alpha(\alpha + n - 2) z + N z + r^2 z_{|rr}$,

$$L^{**} z = (2\alpha + n - 1) r z_{|r} - q r^2 z_{|t},$$

$$| r \Phi_{|t} / \Phi | \leqq \Big(\frac{1}{K_0} \vee r_0 \Big) k_1 \quad \text{for} \quad (t,x) \in D_{r_0, k_0}$$

and

$$\iiint q r z_{|t} \cdot N z\, \Phi(t)^{3\alpha} dr dO_1 dt$$

$$\leqq k_2 r_0 \iiint (z_{|t})^2 r^3\, \Phi(t)^{3\alpha}\, dr dO_1 dt - (k_3 + k_1 \cdot k_2 (K_0^{-1} \vee r_0) \alpha) \iiint z \cdot N z\, \Phi(t)^{3\alpha}\, dr dO_1 dt,$$

where k_1 is a positive constant depending only on $\Phi_{|t}$, k_2 a positive constant depending only on the derivatives of $a_{ij}(t,x)$ of order $\leqq 2$ with respect to x and of order 1 with respect to t and k_3 a positive constant depending on the derivatives of a_{ij} of order $\leqq 2$ with respect to x and t.

6) Li Der-Yuan: Uniqueness of Cauchy's problem for a parabolic type of equation, Doklady Akad. Nauk, **129**, 979-982 (1959).