33. Mass Distributions on the Ideal Boundaries

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In the previous paper[1] our proof of 3) of Theorem 8 is incomplete. The purpose of the present paper is to introduce a new operation, to give a proof of the above theorem and to consider mass distributions.

Let $U(z)$ be a positive harmonic function in $R-R_0$ such that $U(z)=0$ on ∂R_0 and $D(\min(M, U(z))<\infty$ for $M<\infty$. Let D be a compact or non-compact domain. Let $U_{D_n}(z)$ be a harmonic function in $R-R_0-(D\cap R_n)$ such that $U_{D_n}(z)=0$ on ∂R_0, $U_{D_n}(z)=U(z)$ on $D\cap R_n$ and $U_{D_n}(z)$ has the minimal Dirichlet integral (M.D.I.) over $R-R_0-(D\cap R_n)$. Put $U_D(z)=\lim\limits_n U_{D_n}(z)$. If $U(z)\geqq U_D(z)$ for every domain D, we call $U(z)$ a *superharmonic* function in $\bar{R}$.

In the sequel, we use the same notations and terminologies as in the previous paper. Let $U(z)$ be a positive harmonic function in $R-R_0$ and superharmonic in $\bar{R}$. Then $\max\limits_{z\in R_n} U(z)<M$, whence $D(\min(U(z), M))<\infty$. Let $U_n(z)$ be a harmonic function in R_n-R_0 such that $U_n(z)=U(z)$ in R_n-R_0, and $U_n(z)$ has M.D.I. over $R-R_n$. Then

$$U(z)=\lim_n U_n(z).$$

Operation. ${}_D[U(z)]^*$. Let D be a non-compact domain in R. Let ${}_{D_n}[U(z)]^*$ be a continuous function in R such that $U_n(z)-{}_{D_n}[U(z)]^*$ is harmonic in $\bar{D}\cap\bar{R}_n$, ${}_{D_n}[U(z)]^*$ is harmonic in $R-R_0-(\bar{D}\cap\bar{R}_n)$, ${}_{D_n}[U(z)]^*=0$ on ∂R_0 and ${}_{D_n}[U(z)]^*$ has M.D.I. over $R-R_0-(\bar{D}\cap\bar{R}_n)$. Then since $\bar{D}-\bar{R}_n$ is compact, ${}_{D_n}[U(z)]^*$ is uniquely determined. In fact, since $U_n(z)$ is superharmonic in $\bar{R}$ and $U_n(z)$ has M.D.I. over $R-R_n$, $U_n(z)$ is representable by a unique mass distribution on ∂R_n such that $U_n(z)=\int\limits_{\partial R_n} N(z,p)d\mu_n(p)$. Let ${}_1\mu_n(p)$ be the restriction of $\mu_n(p)$ on $\bar{D}\cap\partial R_n$. Then

$${}_{D_n}[U(z)]^*=\int N(z,p)d_1\mu_n(p).$$

Now ${}_2\mu_n(p)=\mu_n(p)-{}_1\mu_n(p)$ is also a positive mass distribution, which implies that $U_n(z)-{}_{D_n}[U(z)]^*$ is superharmonic $\bar{R}$. Let $\{n'\}$ be a subsequence of $\{n\}$ such that ${}_{D_n}[U(z)]^*$ converges uniformly in R. Put

1) Z. Kuramochi: Mass distributions on the ideal boundary of Riemann surfaces, II, Osaka Math. Jour., **8** (1956).

${}_D[U(z)]^*=\lim_{n'} {}_{D_n}[U(z)]^*$. ${}_D[U(z)]^*$ depends on D and the subsequence $\{n'\}$.

Theorem. *Let D_1 and D_2 be two non-compact domains and $\{n'\}$ be a subsequence such that ${}_{1D_n}[U(z)]^*$ and ${}_{2D_n}[U(z)]^*$ converge uniformly in R. Then*

1) ${}_{1D+2D}[U(z)]^* \leqq {}_{1D}[U(z)]^* + {}_{2D}[U(z)]^*$.

2) ${}_D[CU(z)]^* = C_D[U(z)]^*$ *for a constant* C.

3) ${}_D[U(z)]^* \leqq U_D(z) \leqq U(z)$.

4) *Both* ${}_D[U(z)]^*$ *and* $U(z)-{}_D[U(z)]^*$ *are superharmonic in* $\overline{R}$.

5) ${}_{1D}[U(z)]^* \leqq {}_{2D}[U(z)]$ *for* ${}_1D \subset {}_2D$.

6) ${}_D[U(z)]^*$ *is representable by a mass distribution on* $\overline{D} \cap B$, *where* $\overline{D}$ *is the closure of* D *with respect to Martin's topology.*

7) ${}_{B_0}[U(z)]^*=0$ *for* $U(z)=\int_{B_0} N(z,p)d\mu(p)$, *where* B_0 *is the set of non-minimal points of* B.

1), 2) and 5) are clear. Proof of 3). $D_D(z)=\lim_n U_{D_n}(z)$, where $U_{D_n}(z)$ is a harmonic function in $R-R_0-(D\cap R_n)$ such that $U(z)=U_{D_n}(z)$ on $D\cap R_n$ and $U_{D_n}(z)$ has M.D.I. over $R-R_0-(D\cap R_n)$. Now ${}_{D_n}[U(z)]^* \leqq U(z)=U_{D_n}(z)$ on $D\cap R_n$. On the other hand, since both $U_{D_n}(z)$ and ${}_{D_n}[U(z)]^*$ have M.D.I. over $R-R_0-(D\cap R_n)$, by the maximum principle ${}_{D_n}[U(z)]^* \leqq U_{D_n}(z)$. Hence

$${}_D[U(z)]^* = \lim_{K=\infty} {}_{D_{n'}}[U(z)]^* \leqq \lim_n U_{D_n}(z) = U_D(z).$$

Proof of 4) and 6). ${}_{D_n}[U(z)]^*$ and $U_n(z)-{}_{D_n}[U(z)]^*$ are representable by positive mass distributions ${}_1\mu_n$ and ${}_2\mu_n$ $(=\mu_n-{}_1\mu_n)$ on $\partial R_n \cap \overline{D}$ and $\partial R_n \cap \overline{CD}$ respectively. But the total masses of ${}_1\mu_n$ and ${}_2\mu_n$ are bounded. We can find a subsequence $\{n'\}$ of $\{n\}$ such that both $\{{}_1\mu_{n'}\}$ and $\{{}_2\mu_{n'}\}$ have weak limits ${}_1\mu$ on $\overline{D}\cap\overline{R}$ and ${}_2\mu$ on $\overline{CD}\cap\overline{R}$ respectively. Clearly by $\{n'\}\subset\{n\}$, $U(z)=\int N(z,p)d\mu(p)$, ${}_D[U(z)]^*=\int N(z,p)d_1\mu(p)$ and $U(z)-{}_D[U(z)]^*=\int N(z,p)d_2\mu(p)$. Hence ${}_D[U(z)]^*$ and $U(z)-{}_D[U(z)]^*$ are superharmonic in $\overline{R}$. Every μ_n is distributed on ∂R_n, whence ${}_1\mu$ and ${}_2\mu$ lie on B, whence ${}_D[U(z)]^*$ and $U(z)-{}_D[U(z)]^*$ are harmonic in R.

Proof of 7). We proved $U_{B_0}(z)^{2)}=0$. Hence by 3) we have 7).

3) *of Theorem* 8. *Every positive harmonic function in* $R-R_0$ *such that* $U(z)=0$ *on* ∂R_0 *and* $U(z)$ *is superharmonic in* $\overline{R}$ *is representable by a positive mass distribution* μ *on* B_1 *such that*

2) 2) of Theorem 8.

$$U(z)=\int_{B_1} N(z,p)d\mu(p),$$

where B_1 is the set of minimal points.

The set Γ_m is defined as the set (possible void) of all points q of such that $\psi(A_m(q))=\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial N_{A_m(q)}(z,q)}{\partial n}ds\leqq\frac{1}{2}$ (this means $\psi(q)=0 =\lim_m \psi(A_m(q)))$, where $A_m(q)=E\left[z\in\bar{R}:\delta(z,q)\leqq\frac{1}{m}\right]$. Then $B_0=\bigcup_{m>0}\Gamma_m$. Suppose $U(z)=\int_{B_0}N(z,p)d\mu(p)$. Then by 2) of Theorem 8, $U_{B_0}(z)=0$. And by 3) $U_{\Gamma_m}(z)=0$ implies $\lim_n {}_{\Gamma_{m,n}}[U(z)]^*=0$, where $\Gamma_{m,n}=E\left[z\in\bar{R}:\delta(z,\Gamma_m)\leqq\frac{1}{n}\right]$. Let z_0 be a point in $R-R_0$. Then for any given positive number ε, there exists a number $n(m)$ such that

$${}_{\Gamma_{m,n}}[U(z_0)]^*\leqq U_{\Gamma_{m,n}}(z_0)\leqq\frac{\varepsilon}{2^m}\quad\text{for } n\geqq n(m).$$

For each m select Γ'_m $(=\Gamma_{m,n})$ in this fashion. Put $C_m=\sum_{i=1}^m\Gamma'_i$. Then C_m is closed and increases as $m\to\infty$. Denote by $\tilde{A}_m$ and A_m the closure of the complement of C_m in B and $\bar{R}$ respectively. Then the distance between $\tilde{A}_m$ and Γ_m is at least $\frac{1}{n(m)}$. Thus $\{\tilde{A}_n\}$, which forms a decreasing sequence, has an intersection $\tilde{A}$ which is closed and, having no point in common with any Γ_m, is a subset B_1. Now

$${}_{C_m}[U(z)]^*\leqq U_{C_m}(z)\leqq\sum_i^m U_{\Gamma'_i}(z)\leqq\sum_i^m 2^{-i}\varepsilon\leqq\varepsilon\quad\text{for } z=z_0.$$

Observing $\tilde{A}_m+\tilde{C}_m=B$, we obtain for a subsequence $\{n'\}$ of $\{n\}$ such that

$${}_{A_m\cap R_{n'}}[U(z)]^*\to{}_{\tilde{A}_m}[U(z)]^*\quad\text{as } n'\to\infty,$$

where $A_m\cap B=\tilde{A}_m$ and A_m is a closed domain in $\bar{R}$.

$${}_{\tilde{A}_m}[U(z)]^*\leqq{}_{\tilde{A}_m}[U(z)]^*+{}_{C_m}[U(z)]^*\leqq{}_{\tilde{A}_m}[U(z)]^*+\varepsilon\geqq U(z)={}_B[U(z)]^*.$$

Now $U(z)-{}_{\tilde{A}_m}[U(z)]^*$ and ${}_{\tilde{A}_m}[U(z)]^*$ are representable by positive mass distributions μ'_m and μ''_m over $C_m\cap B$ and $\tilde{A}_m$ respectively. Let $\{n''\}$ be a subsequence of $\{n'\}$ such that ${}_{A_{m+1}\cap\bar{R}_{n''}}[U(z)]^*\to{}_{\tilde{A}_{m+1}}[U(z)]^*$. Then ${}_{\tilde{A}_{m+1}}[U(z)]^*$ is representable by μ''_{m+1} over $\tilde{A}_{m+1}$ and ${}_{C_m}[U(z_0)]^*<\varepsilon$. Proceeding in this way, then by 5) ${}_{\tilde{A}_m}[U(z)]^*\downarrow{}_{\tilde{A}}[U(z)]^*$ and ${}_{C_m}[(U(z)]^*\uparrow{}_{\bar{C}}[U(z)]^*$, where ${}_{\bar{C}}[U(z_0)]^*<\varepsilon$ by ${}_{C_m}[U(z_0)]^*<\varepsilon$. μ'_m and μ''_m have weak limits μ' and μ'' over $B\cap\bar{C}(=\overline{\sum C_m})$ and $\tilde{A}=\bigcap\tilde{A}_m\subset B_1$ respectively. Hence

$$U(z_0)\leqq{}_{\tilde{A}}[U(z_0)]^*+\varepsilon,$$

where ${}_{\tilde{A}}[U(z)]^*$ and $U(z)-{}_{\tilde{A}}[U(z)]^*$ are superharmonic in $\bar{R}$ and representable by ${}_1\mu'(=\mu')$ and ${}_1\mu''(=\mu'')$. Let ${}_1\mu'''$ be the restriction of ${}_1\mu''$ on B_1 and put

$U_1(z)=\int_{B_0} N(z,p)d({}_1\mu''-{}_1\mu''')$. Then $U_1(z)\leqq U(z)-{}_{\tilde{A}}[U(z)]^*<\varepsilon$ for $z=z_0$

and
$$U(z)-U_1(z)=\int_{B_1} N(z,p)d({}_1\mu'+{}_1\mu''').$$

Repeat the process for $U_1(z)$, writting $U_1(z)=U_2(z)+(U_1(z)-U_2(z))$, where $U_2(z)$ and $U_1(z)-U_2(z)$ are representable by mass distributions on B_0 and B_1 respectively and $U_2(z_0)<\frac{\varepsilon}{2}$.

Proceeding in this way $U_n(z)=U_{n+1}(z)-(U_n(z)-U_{n+1}(z))$, where $U_{n+1}(z)$ and $U_n(z)-U_{n+1}(z)$ are representable by mass distributions over B_0 and B_1 respectively and $U_{n+1}(z_0)<\frac{\varepsilon}{2^n}$. Then

$$U(z)=U(z)-U_1(z)+\sum_{n=1}^{\infty}(U_n(z)-U_{n+1}(z))$$

and $U(z)$ is represented by a mass distribution μ over B_1. Let $U(z)$ be a positive harmonic in R and superharmonic in $\overline{R}$. Then $U(z)=\int_B N(z,p)d\mu(p)$. Let μ' be the restriction of μ over B_0. Then μ' can be replaced by a distribution over B_1 and μ can be replaced by another distribution μ^* on B_1 without any change of $U(z)$. Hence we have the theorem.

$\overset{*}{M}$*inimal point.* Let p be a point in $\overline{R}-R_0$ and let $F_m=E\left[z\in\overline{R}:\delta(z,p)\leqq\frac{1}{m}\right]$. If there exists a sequence ${}_{F_m}[N(z,p)]^*\downarrow 0$ as $m\to\infty$, then we call p a non-$\overset{*}{\text{m}}$inimal point, otherwise a $\overset{*}{\text{m}}$inimal point.

Theorem. *Denote by* B_1^* *and* B_0^* *the set of minimal points and non-minimal points. Then*

$$B_0=B_0^* \quad and \quad B_1=B_1^*.$$

Let $p\in B_0$. Then $N_{F_m}(z,p)\downarrow 0$ as $m\to\infty$ by Theorem 9. Then by 3) ${}_{F_m}[N(z,p)]^*\downarrow 0$, whence $p\in B_0^*$ and $B_0\subset B_0^*$.

Let $p\in B_1$. Then

$${}_{B\cap F_m}[N(z,p)]^*\leqq N(z,p)\leqq{}_{B\cap F_m}[N(z,p)]^*+{}_{B\cap\overline{CF}_m}[N(z,p)]^*.$$

Now ${}_{B\cap F_m}[N(z,p)]^*$, ${}_{B\cap\overline{CF}_m}[N(z,p)]^*$, $N(z,p)-{}_{B\cap F_m}[N(z,p)]^*$ and $N(z,p)-{}_{B\cap\overline{CF}_m}[N(z,p)]^*$ are superharmonic in $\overline{R}$. By the minimality of $N(z,p)_{B\cap F_m}[N(z,p)]=KN(z,p)$ and ${}_{B\cap CF_m}[N(z,p)]=K'N(z,p)$. Assume $K<1$, then $K'>0$, which implies ${}_{B\cap\overline{CF}_m}[N(z,p)]^*=K'N(z,p)$ is represented by a positive mass distribution μ_m over $\overline{CF}_m$. Assume that μ_m is not a point mass, we can find a mass distribution μ' such that $0<\mu_n'<\mu_m$ and $\int N(z,p)d\mu_n'(p)$ is not a multiple of $N(z,p)$, because if for every μ'_n, $\int N(z,p)d\mu'_n(p)=K'N(z,p)$, we can select a descending sequence of closed subsets T_n in $\overline{CF}_m$ such that diameter of $T_n\to 0$ and $\bigcap T_n=q$. Put $\mu_n=\dfrac{\mu \text{ on } T_n}{\text{total mass of } \mu}$ on T_n. Then $\{\mu_n\}$ has a weak limit mass at q and $N(z,p)=N(z,q): q\in\overline{CF}_m$. This implies $p=q$. This is a contradiction. If μ_m is a point mass, then we have the same contradiction. Hence ${}_{B\cap F_m}[N(z,p)]^*=N(z,p)$ for every m. Thus $p\in B_1^*$ and $B_0=B_0^*$ and $B_1=B_1^*$.