32. Correspondence of Sets on the Boundaries of Riemann Surfaces

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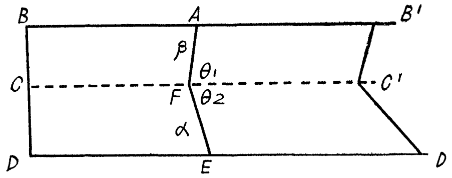
Let D_z be a domain in the z-plane. Let $f(z)=u(z)+iv(z): w=u+iv$ be a topological mapping of D_z into D_w in the w-plane. If $\overline{\lim\limits_{|dz|\to 0}}\frac{|dw|}{|dz|}<M$ in D_z and $f(z)$ is a quasi-conformal mapping almost everywhere in D_z whose dilatation quotient $<K$ in D_z, we say that $f(z)$ is an almost quasi-conformal mapping and abbreviate it to A.Q.C. Let $U(z)$ be a harmonic function in D_z such that the Dirichlet integral $D(U(z))$ is finite and let $f(z)=w$ be an A.Q.C. with dilatation quotient $<K$. Then

$$\frac{1}{K}D(U(z))\leqq D(U(f(w)))\leqq K\,D(U(z)). \tag{1}$$

Example. Let D and D' be simply connected domains whose boundary consists of segments $\overline{AB}$, $\overline{BD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FA}$ and $\overline{AB'}$, $\overline{B'C'}$, $\overline{C'D'}$, $\overline{D'E}$, $\overline{EF}$, $\overline{FA}$, where $A=\alpha e^{i\left(\theta_2+\frac{\pi}{2}\right)}+\beta e^{i\theta_1}$, $B=-r+i\alpha\sin\theta_2+i\beta\sin\theta_1$, $C=-r+i\beta\sin\theta_2$, $D=-r$, $E=0$, $F=\alpha e^{i(\pi-\theta_2)}$, $D'=r$, $C+C'=2F$, $B+B'=2A$.



Put $v(z)=y$, $\dfrac{u(z)+x}{2}=y\cot\theta_2$ in $CDEF$

and $v(z)=y$, $\dfrac{u(z)+x}{2}=(y-h)\cot\theta_1$, $h=\alpha\sin\theta_2$, in $CFAB$.

Then $|dw|=(1+2\cot^2\theta_i(\sin(\psi+2\varphi)))^{\frac{1}{2}}|dz|$, where

$$dx=dz\cos\varphi,\quad dy=dz\sin\varphi \quad\text{and}\quad \psi=\frac{\pi}{2}-\theta_i.$$

Then we see that the above mapping is quasi-conformal in the interior of $CDEF$ and in the interior of $CFAB$ and is an A.Q.C. in the closure of $ABDEF$.

Let R be a Riemann surface with positive boundary and $\{R_n\}$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$ $(n=0,1,2,\cdots)$.

Let B be the ideal boundary of R. Assume that a metric δ is given on $R+B$, for instance, Martin's metric. Let F be a closed set in B. Put $F_m=E\left[z\in R+B:\delta(z,F)\leqq\frac{1}{m}\right]$. Then $F=\bigcap_m F_m$. Let $U_{m,n,n+i}(z)$ be a harmonic function in $R-((R_{n+i}-R_n)\cap F_m)-R_0$ such that $U_{m,n,n+i}(z)=0$ on ∂R_0, $U_{m,n,n+i}(z)=1$ on $\partial((R_{n+i}-R_n)\cap F_m)$ and $\frac{\partial U_{m,n,n+i}(z)}{\partial n}=0$ on $\partial R_{n+i}-F_m$. Then $U_{m,n,n+i}(z)\to U_{m,n}(z)$ in mean as $i\to\infty$, $U_{m,n}(z)\to U_m(z)$ in mean as $n\to\infty$ and $U_m(z)\to U(z)$ in mean as $m\to\infty$. We call $D(U(z))=\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds$ the capacity[1] of F relative $R-R_0$. Then we see

$$\mathrm{Cap}\,(\sum F_i)\leqq\sum\mathrm{Cap}\,(F_i) \tag{2}$$

for closed sets F_i and that $\mathrm{Cap}\,(F)>0$ or $=0$ does not depend on R_0 so long as R_0 is compact.

Theorem 1. *Let D be a domain in the z-plane. Let F_z be a closed set of positive logarithmic capacity on ∂D. Assume that at every point $z\in F_z$, there exists a sector $S(z)$ with its vertex at z, with a positive radius and a positive aperture such that* $\mathrm{int}\,S(z)\subset D$. *Then F_z is a set of positive capacity relative $D-D_0$, where D_0 is a compact disc in D.*

Proof. Let $E_{n,i}$ be the set of points z such that a sector $S(z)$ with its vertex at z and $S(z)$ satisfies the following conditions:

1) $\mathrm{int}\,S(z)\subset D$.

2) radius of $S(z)\geqq\frac{1}{n}$.

3) $\frac{1}{n}\leqq$aperture of $S(z)<\pi-\frac{1}{n}$.

4) $\frac{2\pi}{32n}i\leqq$argument of the half line of $S(z)<\frac{2\pi}{32n}(i+1)$.

Then $F_z=\sum_{n=1}^{\infty}\sum_{i}^{n}E_{n,i}$. Then there exist numbers n_0 and i_0 such that E_{n_0,i_0} is of positive logarithmic capacity. And there exists a closed subset $F'(\subset E_{n_0,i_0})$ of positive logarithmic capacity. Let F'' be the set of points z such that the argument of the half line of $S(z)=\frac{2\pi i_0}{32n_0}$, radius of $S(z)\geqq\frac{1}{n_0}$ and its aperture$\geqq\frac{\pi}{32n}$. Then by 4) $F''\supset E_{n_0,i_0}$. Therefore we can suppose that at every point z of F', there exists a sector $S(z)$ with aperture $2\theta\left(\frac{\pi}{2}>2\theta>0\right)$, radius$=r$ its vertex at z and the argument of the half line is $\frac{\pi}{2}$. We divide the z-plane into an enumerably infinite number of rectangles such that

1) Z. Kuramochi: Mass distributions on the ideal boundary of Riemann surfaces, II, Osaka Math. Jour., **8** (1956).

$$\frac{r}{16}i\cot\theta\leqq x\leqq\frac{r}{16}(i+1)\cot\theta,\quad \frac{r}{16}j\leqq y\leqq\frac{r}{16}(j+1)$$
$$i, j=0,\ \pm1,\ \pm2,\cdots\ .$$

Then there exists at least a rectangle such that the subset F of F' contained in the rectangle is of positive logarithmic capacity. Without loss of generality, we can suppose that the rectangle R is $E\left[z:0\leqq x\leqq\frac{r}{16},\ 0\leqq y\leqq\frac{r}{16}\cot\theta\right]$. Let R be a rectangle $E\left[z:-\frac{r}{16}\leqq x\leqq\frac{2r}{16},\ -\frac{r}{2}\cot\theta\leqq y\leqq\frac{r}{2}\cot\theta\right]$. Let p be a point of F. Then $\operatorname{int}S(p)\subset D$, whence there exists no point of F in R' which has the same projection as that of p. Hence y-coordinate y of p can be considered as a one-valued function $y=f(x)$ of the projection x of p. It is clear

$$\left|\frac{y_1-y_2}{x_1-x_2}\right|\leqq\cot\theta\qquad\text{for } p(z_1) \text{ and } p(z_2)\in F$$

and $S(p):p\in F$ contains the rectangle $E\left[z:-\frac{r}{16}\leqq x\leqq\frac{2r}{16},\ \frac{r}{2}\cot\theta>y>\frac{r}{8}\cot\theta\right]$. (3)

Let Ω be the domain containing every $S(z):z\in F$ and contained in R'. Then Ω is bounded by segments which are boundaries of $S(z):z\in F$ and the boundary of R' and F. By (3) Ω is simply connected $\subset D$ and its boundary is rectifiable. We show that F is of positive capacity relative Ω which implies that $F_z(\supset F)$ is of positive capacity relative D.

Case 1. F is of positive linear measure. In this case map Ω conformally into $|\xi|<1$. Then F is mapped onto a set of positive linear measure. Hence $\lim\limits_m \omega_m(z)>0$, where Ω_0 is a compact set in Ω and $\omega_m(z)$ is a harmonic function in $\Omega-\Omega_0-F_m$ such that $\omega_m(z)=0$ on $\partial\Omega_0+\partial\Omega-F_m$ and $\omega_m(z)=1$ on ∂F_m and $F_m=E\left[z:\operatorname{dist}(z, F)\leqq\frac{1}{m}\right]$, whence $U(z)=\lim\limits_m U_m(z)\geqq\lim\limits_m \omega_m(z)>0$, where $U_m(z)$ is a harmonic function in $\Omega-\Omega_0-F_m$ such that $U_n(z)=0$ on $\partial\Omega_0$, $\frac{\partial U_m(z)}{\partial n}=0$ on $\partial\Omega-F_m$ and $U_m(z)=1$ on ∂F_m. Thus F is of positive capacity.

Case 2. F is of linear measure zero. Let F_x be the projection of F. Then the function $y(x)$ $(x\in F_x)$ satisfies the Lipsitz's condition, whence F_x is also closed. Now the complementary set of F with respect to $y=0$, $0\leqq x\leqq\frac{r}{16}$ is composed of an enumerably infinite number of open intervals I_i $(a_i<x<b_i)$. Let Ω_i be the subdomain of Ω lying between $x=a_i$ and $x=b_i$. Let Γ be the boundary of $\partial\Omega$ consisting of F and segments which are boundaries of $S(z):z\in F$. Then Γ can be

considered as a graph of $y=g(x): -\frac{r}{16}<x<\frac{2r}{16}$ and it is clear that $g(x)$ also satisfies the Lipsitz's condition

$$\left|\frac{g(x_1)-g(x_2)}{x_1-x_2}\right|<\cot\theta.$$

Let Γ_1, Γ_2 and Γ_3 be boundaries of Ω lying on $y=\frac{r}{2}\cot\theta$, $x=-\frac{r}{16}$ and $x=\frac{2r}{26}$ respectively. Let $w=f(z)=u(z)+iv(z): u(z)=x$, $v(z)=2g(x)-y$ be the mapping. Then $f(z)$ is continuous and univalent and $\sup_{|dz|\to 0}\left|\frac{dw}{dz}\right|<M_1(\theta)=(1+2\cdots\cot\theta/1+\cot^2\theta)^{\frac{1}{2}}$ and A.Q.C with dilatation quotient$\leqq M_2(\theta)=((1+2\cot\theta/\mathrm{cosec}^2\theta)/(1-2\cot\theta/\mathrm{cosec}^2\theta))^{\frac{1}{2}}$ in $\sum_i \mathrm{int}\,\Omega_i$. On the other hand, $\overline{\sum\partial\Omega_i}$ is a set of areal measure zero. Thus $f(z)$ is an A.Q.C. $f(\Omega)$ contains a rectangle: $-\frac{r}{16}\leqq x\leqq\frac{2r}{16}$, $\frac{-r\cot}{4}\leqq y<0$ by $r\cot\theta\left(1-2\left(\frac{1}{2}-\frac{1}{8}\right)\right)=-\frac{r}{4}\cot$.

Assume that F is of capacity zero relative Ω. Then $D(U_n(z))\to 0$, where $U_n(z)$ is a harmonic function in $\Omega-F_n: F=E\left[z: \mathrm{dist}\,(z, F)\leqq\frac{1}{n}\right]$ such that $U_n(z)=0$ on $\Gamma_1+\Gamma_2+\Gamma_3$, $U_n(z)=1$ on ∂F_n and $\frac{\partial U_n(z)}{\partial n}=0$ on $\Gamma-F_n$. Now $\partial F_n+f(\partial F_n)$ encloses F. Put $\widetilde{U}_n(z)=U_n(z)$ in $\Omega-F_n$, $\widetilde{U}_n(z)=U_n(f^{-1}(z))$ in $f(\Omega-F_n)$. Then $\widetilde{U}_n(z)$ is continuous in $(\Omega-F_n+f(\Omega-F_n))$, $\widetilde{U}_n(z)=0$ on $\Gamma_1+\Gamma_2+\Gamma_3+f(\Gamma_1+\Gamma_2+\Gamma_3)$ and $\widetilde{U}_n(z)=1$ on $\partial F_n+f(\partial F_n)$. Then by (1) $D(\widetilde{U}_n(z))\leqq D(U_n(z))(1+M_2(\theta))$. Let $U_n^*(z)$ be a harmonic function in $(\Omega-F_n)+f(\Omega-F_n)$ such that $U_n^*(z)=0$ on $\Gamma_1+\Gamma_2+\Gamma_3+f(\Gamma_1+\Gamma_2+\Gamma_3)$ and $U_n^*(z)=1$ on $\partial F_n+f(\partial F_n)$. Then by the Dirichlet principle

$$D(U_n^*(z))\leqq D(\widetilde{U}_n(z))\to 0 \quad \text{as } n\to\infty.$$

On the other hand, $\partial F_n+f(\partial F_n)$ encloses F in a domain $\Omega+f(\Omega)$ and the distance between $(\Gamma_1+\Gamma_2+\Gamma_3+f(\Gamma_1+\Gamma_2+\Gamma_3))$ and F is positive. This contradicts that F is of positive logarithmic capacity. Hence F is of positive capacity relative Ω. This implies that $F_z(\supset F)$ is of positive capacity relative D.

Let G be a non-compact domain in a Riemann surface R whose relative boundary G consists of at most an enumerably infinite number of compact or non-compact analytic curves clustering nowhere in R. We can construct another Riemann surface $\widehat{G}$ by the process of symmetrization. We proved the following

Theorem.[2] *Let R be a Riemann surface with null-boundary.*

2) Z. Kuramochi: On covering surfaces, Osaka Math. Jour., **6** (1954).

Let G be a non-compact domain. Then $G+\hat{G}$ is a Riemann surface with null-boundary.

As an inverse of the above theorem we have by Theorem 1 the following

Theorem 2. *Let R be a Riemann surface of finite genus with positive boundary. Let B' be a closed subset of B such that B' is of positive capacity relative R. If there exists a sector $S(z)\subset G$ with its vertex at $z\in B'$ at every point of B', then $G+\hat{G}$ is a Riemann surface with positive boundary.*

Let $w=f(z): z\in R$ be an analytic function in a Riemann surface R and suppose that the covering surface of $f(z)$ is spread over the w-sphere K. Let a be a point of K and K_ρ be a spherical disc of radius ρ with a as its centre. Let $n(a)$ be the number of zero of $f(z)-a$ in R.

If
$$\lim_{\rho\to 0}\sup_{w\in K_\rho} n(w)<\infty,$$
then a is called a boundedly covered point.

Let F be a closed set on the ideal boundary on R. We call $H(f(z))=\bigcap_n \overline{\underset{z\in F_n}{f(z)}}$ the cluster set of $f(z)$ at F, where $F_n=E\left[z\in R: \text{dist}\,(z,F)\leqq\frac{1}{n}\right]$. Then

Theorem 3. *Let F be a closed set of positive capacity relative R and $w=f(z)$ be a non-constant analytic function. If every point of $H(f(z))$ is boundedly covered, then $H(f(z))$ is a set of positive logarithmic capacity.*

Since at every point a of $H(f(z))$, there exists a circle $C(a)$ with radius $\frac{1}{n}$ such that $\sup_{w\in C(a)} n(w)\leqq m(n,a)$ and since $H(f(z))$ is closed, $H(f(z))$ is contained in the interior of sum of a finite number of circles $C(a_i)$. Hence $n(w)\leqq m$ in $G=\sum_i \text{int}\, C(a_i)$ and
$$\text{dist}\,(H(f(z)),\partial G)\geqq\delta>0. \tag{4}$$
G may consist of a finite number of components. Without loss of generality, we can suppose that G does not cover a disc in the w-sphere. Then $f^{-1}(G)$ does not fall in a compact set D_0. On the other hand, $z=f^{-1}(w): w\in\partial G$ does not tend to F. If it were not so, $H(f(z))\cap\partial G\neq 0$. This contradicts (4). Let $G_n=E\left[w:\text{dist}\,(w,H(f(z)))\leqq\frac{1}{n}\right]$. Then $f^{-1}(G_n)$ covers a neighbourhood of F and $\text{dist}\,(F,f^{-1}(\partial G_n))\geqq\delta_n>0$, as above and $f^{-1}(\partial G_n)$ separates F from D_0. Assume that $H(f(z))$ is of logarithmic capacity zero. Then $D(U_n(w))\to 0$ as $n\to\infty$, where $U_n(w)$ is a harmonic function in $G-G_n$ such that $U_n(w)=0$ on ∂G and $U_n(w)=1$ on ∂G_n. Consider a continuous function $\tilde{U}_n(z)$ in $R-D_0$ such that

$\tilde{U}_n(z)=0$ in $R-D_0-f(G)$ and $\tilde{U}_n(z)=U_n(f^{-1}(w))$ in $f^{-1}(G-G_n)$. Then $D(\tilde{U}_n(z))\leqq m(D(U_n(z)))$. Let $V_n(z)$ be a harmonic function in $R-D_0-F_{\delta_n}$ such that $V_n(z)=0$ on ∂D_0, $V_n(z)=1$ on ∂F_{δ_n} and $V_n(z)$ has the minimal Dirichlet integral, where $F_{\delta_n}=E[z\in R: \mathrm{dist}\,(z,F)\leqq\delta_n]$ and $F_{\delta_n}\subset f^{-1}(G_n)$. Then by the Dirichlet principle $D(\tilde{U}_n(z))\geqq D(V_n(z))$. Let $n\to\infty$. Then $D(V(z))\to 0$. This means that F is a set of capacity zero relative R. This is a contradiction. Hence we have the theorem.

Theorem 4. *Let D_z be a simply connected domain in the z-plane. Let E_z be a closed set of positive logarithmic capacity. Suppose at every point z of E_z, there exists a sector $S(z)$ such that* $\mathrm{int}\,S(z)\subset D$. *Map D conformally onto $|w|<1$. Then the image E_w of E_z is also of positive logarithmic capacity.*

Let $w=f(z)$ be the mapping function. Let $\Omega(\subset D)$ be a simply connected domain in the proof of Theorem 1. Put $E=\partial\Omega\cap E_z$. Then E is closed and is of positive capacity relative Ω. Then Ω is mapped onto a domain $f(\Omega)$ in the circle $|w|<1$. Let $l(p)(\subset\Omega): p\in E$ be a path tending to p. Then $f(l(p))$ tends to a point q on $|w|=1$ by Riesz's theorem. Let E'_w be the set of points q such that there exists a curve $(\Omega\supset)l(p): p\in E$ and $\lim_{\substack{z\to p\\ z\in l(p)}} f(z)=q$. Then E'_w is closed. In fact, $F_w=\overline{f(\partial\Omega)}\cap\Gamma:\Gamma=[|w|=1]$ is closed. Clearly $E'_w\subset F_w$. Let q be a point of F_w. Then there exists a sequence $\{w_i\}: w_i=f(z_i)$, $w_i\in f(\Omega)$ and $\lim_i w_i=q$. Consider $f^{-1}(w_i)\subset\Omega$. Then $f^{-1}(w_i)$ has limit points only on E. Choose a subsequence $f^{-1}(w'_i)$ of $\{f^{-1}(w_i)\}$ such that $f^{-1}(w_i)\to z_0\in E$. Since every point of $\partial\Omega$ is accessible, connect $f^{-1}(w'_i), f^{-1}(w''_i)\cdots$ by a curve $l\subset\Omega$. Then $f(z)$ has limit q as z tends to z_0 along l. Hence $F_w\subset E'_w$. Next we show $\bigcap_{n>0}\overline{\underset{z\in(E_n\cap\Omega)}{f(z)}}=E'_w$, where $E_n=E\left[z: \mathrm{dist}\,(z,E)\leqq\frac{1}{n}\right]$. It is clear $E'_w\subset H(f(z))$. Let $w_0\in H(f(z))$. Then there exists a sequence $\{z_i\}$ such that $\{z_i\}$ tends to E and $f(z_i)\to w_0$. Since E is closed, we can find a point $z_0\in E$ and a subsequence $\{z'_i\}$ of $\{z_i\}$ such that $\lim_{i=\infty} z'_i=z_0$. Connect z'_i by a curve l in Ω such that l tends to z_0, for every point of $\partial\Omega$ is accessible in Ω. Then $f(z)\to w_0$ as $z\to z$ along l. Hence $w_0\in E'_w$ and $E'_w=H(f(z))$. Now $E(\subset E_z)$ is a set of positive capacity relative Ω by Theorem 1, since E is a set of positive logarithmic capacity. Hence E'_w is of positive logarithmic capacity by Theorem 3. Hence the image $E_w(\supset E'_w)$ of E_z is of positive logarithmic capacity.