31. Commutativity of Some Continuous Magnitude

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A *magnitude* M is a set of elements a, b, c, $\cdots$ and a binary operation called *sum* satisfying the following Conditions I–IV.

I. For every pair a, b of M, the sum $a+b$ exists.

II. For every a, b, c, $(a+b)+c=a+(b+c)$.

Definition. If $a, b \in M$, then $a<b$ means that there is an element x such that $a+x=b$. $a>b$ means $b<a$.

III. a, $b<a+b$ for every a, b.

IV. If a, b are distinct, then either $a<b$ or $b<a$.

Therefore, any magnitude M is linear ordered set.

A magnitude M is *Archimedean*, if $a>b$, then there is a positive integer n such that $nb>a$.

Since O. Hölder, some mathematicians, R. Baer, H. Cartan, F. Loonstra and F. A. Behrend [1], have proved that *any Archimedean magnitude is commutative*: $a+b=b+a$ for every element a, b (see H. G. Forder [2]).

A magnitude M is *continuous*, if every bounded subset has the least upper bound.

In this Note, we shall prove the following

Theorem. *Any continuous magnitude is commutative.*

To prove it, we shall show that a continuous magnitude is Archimedean.

Remark. For the proof, Condition III is essential.

Proof. Suppose that M is not Archimedean, then there are two elements a, b such that $a<b$ and $na\leq b$ for $n=1, 2, \cdots$.

Consider the set of elements na $(n=1, 2, \cdots)$, then the set is bounded, and has a least upper bound c such that $na\leq c$ for all $n=1, 2, \cdots$ and for $c'<c$, there is an integer m such that $ma>c'$. From $a<c$, there is an element x such that $a+x=c$. By Condition II, we have $x<a+x=c$. Therefore there is an integer m such that $ma>x$. Hence we have $a+ma>a+x$. (This is proved without commutative law.) This shows $(m+1)a>c$, which is a contradiction. Therefore we complete the proof.

References

[1] F. A. Behrend: A system of independent axioms for magnitudes, Journal and Proceedings of Royal Society of New South Wales, **87**, 27–30 (1953).

[2] H. G. Forder: The Foundations of Euclidean Geometry, New York (1927).