27. On the mod p Hopf Invariant

By Tsuneyo YAMANOSHITA
Department of Mathematics, Musashi Institute of Technology, Tokyo
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J. F. Adams [1] has proved that there is no element of Hopf invariant one in $\pi_{2n-1}(S^n)$ $(n\geqq 16)$.

In other words, his result may be expressed as follows:

If $p=2$, mod p Hopf invariant homomorphism

$$H_p:\pi_{m+n-1}(S^m)\to Z_p,\quad n=2t(p-1)$$

is trivial for $t\geqq p^3$.

In case of mod p (p: odd prime), we have the following

Theorem 1. *If p is an odd prime, the* mod p *Hopf invariant homomorphism is trivial for $t\geqq p$.*

The special case of this theorem, corresponding to $t=p$ was proved by Toda [2].

We shall adopt the definition of the stable secondary cohomology operation of Adams [1]. Then we have a similar result to the theorem of Adams [1] on Sq^{2^k} $(k\geqq 4)$.

Theorem 2. *$\mathcal{P}^{p^k}$ $(k\geqq 1)$ can be represented in the form $\sum a_i\Phi_i$ where Φ_i are stable secondary cohomology operations and a_i are elements of Steenrod algebra with positive degrees.*

Theorem 1 is easily deduced from Theorem 2. The special case of Theorem 2 for $k=1$ was also proved by Toda [2, 3].

We shall denote the Steenrod algebra over Z_p by A and denote the A free module with the symbolic base $[c(\Delta)]$, $[c(\mathcal{P}^1)],\cdots,[c(\mathcal{P}^{p^k})]$ by C_1^k $(k\geqq 0)$. Moreover, define the element $z_{-1,k}$ $(k\geqq 1)$ of C_1^k as follows:

$$z_{-1,k}=c(\Delta)[c(\mathcal{P}^{p^k})]-c(\Delta,\mathcal{P}^{p^k-1})[c(\mathcal{P}^1)]-c(\mathcal{P}^{p^k})[c(\Delta)],$$

where Δ is the Bockstein operator associated with the exact sequence $0\to Z_p\to Z_{p^2}\to Z_p\to 0$ and c is the conjugacy operation [2]. Let d be the A-homomorphism of C_1^k into $A=C_0$ such that $d[c(\Delta)]=c(\Delta)$, $d[c(\mathcal{P}^{p^i})]=c(\mathcal{P}^{p^i})$, $i=0,1,\cdots,k$. Then $z_{-1,k}$ is a d-cycle, i.e. $d(z_{-1,k})=0$. The stable secondary cohomology operation associated with $(d,z_{-1,k})$ will be denoted with $\Phi_{z_{-1,k}}$. This is uniquely determined [1, Theorem 3]. Let ε be the augmentation (A-homomorphism) of A into $H^+(X,Z_p)=\sum_{i>0}H^i(X,Z_p)$ which maps A free base 1 into an element u of $H^q(X,Z_p)$. Then we have $\varepsilon d=0$, if $u\in\bigcap_{i=0}^k \mathrm{Ker}\,c(\mathcal{P}^{p^i})\frown\mathrm{Ker}\,c(\Delta)=\bigcap_{i=0}^k\mathrm{Ker}\,\mathcal{P}^{p^i}\frown\mathrm{Ker}\,\Delta$, in which case $\Phi_{z_{-1,k}}(u)$ is defined.

Consider the effect of $\Phi_{z_{-1,k}}$ for element $y^{p^{k+1}n}$ in $H^{2p^{k+1}n}(P,Z_p)$, where P is infinite dimensional complex projective space and y is a generator of $H^2(P, Z_p)$. Then we have the following propositions.

Proposition 1.

$$\Phi_{z_{-1},1}(y^{p^2 n}) = -ny^{p(np+p-1)} \qquad (\text{mod } zero).$$

Proof of this proposition is performed by utilizing a formula expressing $\Phi_{z_{-1},1}c(\mathcal{P}^{p(p-1)})$ in stable secondary cohomology operations, and other formulas for stable secondary cohomology operations defined for cohomology classes contained in Ker $\mathcal{P}^{p+1} \frown$ Ker $\mathcal{P}^1 \frown$ Ker Δ.

Proposition 2. *For $k \geqq 2$, we have*

$$\Phi_{z_{-1},k}(y^{p^{k+1}n}) = -ny^{p^k(np+p-1)} \qquad (\text{mod } zero).$$

In the proof of this proposition we use a formula for the composite operation $\Phi_{z_{-1},k}c(\mathcal{P}^{p^k(p-1)})$.

Now, we can obtain Theorem 2 from the above propositions and the following relations:

$$c(\mathcal{P}^{p^k(p-1)}\Delta) = c(\mathcal{P}^1 \Delta \mathcal{P}^{p^k(p-1)-1}) + c(\Delta \mathcal{P}^{p^k(p-1)}) \quad (k \geqq 2),$$

$$c(\mathcal{P}^{p(p-1)}\Delta) = -c(\mathcal{P}^2 \Delta \mathcal{P}^{p(p-1)-2}) + c(\Delta \mathcal{P}^{p(p-1)}) \quad (k=1).$$

Detailed proof will be published elsewhere. After completion of this note, the author received a communication from Prof. N. Shimada that he has also obtained the same result in utilizing the method of "functional operations".

References

[1] J. F. Adams: On the nonexistence of elements of Hopf invariant one, Bull. Amer. Math. Soc., **64**, 279–282 (1958).

[2] H. Toda: p-primary components of homotopy groups, I, II, Mem. Coll. Sci., Univ. Kyoto, **31**, 129–142, 143–160 (1958).

[3] H. Toda: p-primary components of homotopy groups, III, Mem. Coll. Sci., Univ. Kyoto, **31**, 191–210 (1958).