SOME INVARIANT PROPERTIES OF CURVES IN THE TAXICAB GEOMETRY

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Abstract. Let E_T^2 be the group of all isometries of the 2-dimensional taxicab space R_T^2. For the taxicab group E_T^2, the taxicab type of curves is introduced. All possible taxicab types are found. For every taxicab type, an invariant parametrization of a curve is described. The E_T^2-equivalence of curves is reduced to the problem of the E_T^2-equivalence of paths.

1. Introduction

The 2-dimensional taxicab space can be introduced using the metric $d_T(x,y) = |x_1 - y_1| + |x_2 - y_2|$ instead of the well-known Euclidean metric $d_E(x,y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{\frac{1}{2}}$, where $x = (x_1, x_2), y = (y_1, y_2) \in R^2$. This space will be denoted by R_T^2. R_T^2 is also known as the taxicab plane geometry (shortly, taxicab geometry) [9, 10].

Let $E_T^2 = \left\{F : R_T^2 \to R_T^2 : Fx = gx + b,\ g \in D_4,\ b \in R_T^2\right\}$, where the group D_4 is the (Euclidean) symmetry group of the square.

The 2-dimensional taxicab group is introduced in [14]. For $n \geq 2$, geometric properties in the n-dimensional taxicab space are investigated in [1, 2, 6, 11]. The taxicab arc length of a curve in the 2-dimensional taxicab space is defined in [17].

Invariant parametrizations and global properties of curves and paths in some spaces are considered in papers [3, 8, 12, 13] and some books [5, 7]. Similar problems for taxicab geometry have not yet appeared in the literature. These results are important for the theory of curves, the problems of E_T^2-equivalence of curves and some physical applications. For example, the taxicab geometry plays an important role in ecology, fire-spread simulation with square-cell, grid-based maps [4, 15, 18]. Non-linear differential equations in taxicab geometry are introduced in [16].

This paper is organized as follows: In Section 2, the definitions of taxicab curve, taxicab type and the taxicab arc length function of a curve is given. In section 3, the definition of an invariant parametrization of a curve are given. Invariant parametrization of a curve with a fixed taxicab type are described. In Theorem 3.6, the problems of the E_T^2-equivalence of curves are reduced to that of paths.

Future research could include problems and applications concerning E_T^2-equivalence of curves as well as the complete system of differential invariants of a curve in R_T^2.

2. The Taxicab Type of a Curve

Let R be the field of real numbers and $I = (a, b)$ an open interval of R.

Definition 2.1. A C^∞ mapping $x\colon I \to R_T^2$ will be called an I-path (shortly, a path) in R_T^2.

Definition 2.2. An I_1-path $x\,(t)$ and an I_2-path $y\,(r)$ in R_T^2 will be called D-equivalent if a C^∞ -diffeomorphism $\varphi\colon I_2 \to I_1$ exists such that $\varphi^{'}(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in I_2$. A class of D-equivalent paths in R_T^2 will be called a curve in R_T^2. A path $x \in \alpha$ will be called a parametrization of a curve α.

We denote the group $\left\{F\colon R_T^2 \to R_T^2 : Fx = gx + b,\ g \in D_4,\ b \in R_T^2\right\}$ of all transformations of R_T^2 by E_T^2, where gx is the multiplication of a matrix g and a column vector $x \in R_T^2$.

If $x(t)$ is an I-path then $Fx(t)$ is an I-path in R_T^2 for any $F \in E_T^2$. Let G be a subgroup of E_T^2.

Definition 2.3. Two I-paths $x(t)$ and $y(t)$ in R_T^2 are called G-equivalent if there exists $F \in G$ such that $y(t) = Fx(t)$. This being the case , we write $x(t) \overset{G}{\sim} y(t)$.

Let $\alpha = \{h_\tau, \tau \in Q\}$ be a curve in R_T^2, where h_τ is a parametrization of α. Then $F\alpha = \{Fh_\tau, \tau \in Q\}$ is a curve in R_T^2 for any $F \in E_T^2$.

Definition 2.4. Two curves α and β in R_T^2 are called G-equivalent if $\beta = F\alpha$ for some $F \in G$. This being the case, we write $\alpha \overset{G}{\sim} \beta$.

Let $x(t) = (x_1(t), x_2(t))$ be an I-path in R_T^2, $x^{'}(t) = (x_1^{'}(t), x_2^{'}(t))$ be the derivative of the path $x(t)$. For $p, q \in I = (a, b), p < q$, we let

$$l_x(p, q) = \int\limits_p^q (|x_1'(t)| + |x_2'(t)|)dt.$$

Obviously, the finite and infinite limits $l_x(a, q) = \lim_{p\to a} l_x(p, q) \leq +\infty$ and $l_x(p, b) = \lim_{q\to b} l_x(p, q) \leq +\infty$ exist. We have the following four possibilities:

$$l_x(a, q) < +\infty, \quad l_x(p, b) < +\infty \tag{2.1}$$

$$l_x(a,q) < +\infty, \quad l_x(p,b) = +\infty \tag{2.2}$$

$$l_x(a,q) = +\infty, \quad l_x(p,b) < +\infty \tag{2.3}$$

$$l_x(a,q) = +\infty, \quad l_x(p,b) = +\infty. \tag{2.4}$$

Suppose that the case (2.1) holds for some $p, q \in I$. Then $l = l_x(a,q) + l_x(p,b) - l_x(p,q)$, where $0 \leq l \leq +\infty$, does not depend on $p, q \in I$. In this case we say that x belongs to the taxicab type of $(0, l)$. In cases (2.2), (2.3), and (2.4), we say that x has taxicab types $(0, +\infty)$, $(-\infty, 0)$, and $(-\infty, +\infty)$, respectively. The taxicab type of a path x will be denoted by $L(x)$.

Remark 2.5. The following examples 2.6–2.9 below show that there exist paths of all types $(0, l)$, where $l < +\infty$, $(0, +\infty)$, $(-\infty, 0)$, $(-\infty, +\infty)$.

Example 2.6. Consider the I-path $x(t) = (rcost, rsint)$ in E_T^2, where $I = (0, \pi/2)$ and $r > 0$. Then

$$l_x(p,q) = r\int_p^q (sint + cost)dt = r(-cosq + sinq + cosp - sinp)$$

for all $0 < p < q < \frac{\pi}{2}$.

Since $l_x(0,q) = \lim_{p\to 0} l_x(p,q) < +\infty$ and $l_x(p, \frac{\pi}{2}) = \lim_{q\to\frac{\pi}{2}} l_x(p,q) < +\infty$, the type of the path is $(0, l)$.

Example 2.7. Consider the I-path $x(t) = (t, e^t)$ in E_T^2, where $I = (0, +\infty)$. Then

$$l_x(p,q) = \int_p^q (1 + e^t)dt = q - p + e^q - e^p$$

for all $0 < p < q$.

Since $l_x(0,q) = \lim_{p\to 0} l_x(p,q) < +\infty$ and $l_x(p, +\infty) = \lim_{q\to+\infty} l_x(p,q) = +\infty$, the type of the path is $(0, +\infty)$.

Example 2.8. Consider the I-path $x(t) = (t, t^2)$ in E_T^2, where $I = (-\infty, 0)$. Then

$$l_x(p,q) = \int_p^q (1 - 2t)dt = q - p - q^2 + p^2$$

for all $-\infty < p < q < 0$.

Since $l_x(-\infty, q) = \lim_{p\to-\infty} l_x(p,q) = +\infty$ and $l_x(p, 0) = \lim_{q\to 0} l_x(p,q) < +\infty$, the type of the path is $(-\infty, 0)$.

Example 2.9. Consider the I-path $x(t) = (t, t^2)$ in E_T^2, where $I = (-\infty, +\infty)$. Then

$$l_x(p,q) = \int_p^q (1 + 2|t|)dt = q - p + q^2 + p^2$$

for all $p < 0 < q$.

Since

$$l_x(-\infty, q) = \lim_{p \to -\infty} l_x(p,q) = +\infty$$
$$\text{and} \quad l_x(p, +\infty) = \lim_{q \to \infty} l_x(p,q) = +\infty,$$

the type of the path is $(-\infty, +\infty)$.

Proposition 2.10. *Let $x(t)$ be an I-path in R_T^2. Then $l_x(p,q) = l_{gx}(p,q)$ for all $g \in D_4$*

Proof. Since $x(t)$ is an I-path in R_T^2, $gx(t)$ is an I-path in R_T^2 for all $g \in D_4$. Since the derivative of the I-path $x(t)$ is $x^{'}(t) = (x_1^{'}(t), x_2^{'}(t))$, we have $[gx(t)]^{'} = gx^{'}(t)$ for all $g \in D_4$ and for all $t \in I$. Then an I-path $gx^{'}(t)$ can be written in forms $(x_1^{'}(t), x_2^{'}(t))$, $(-x_1^{'}(t), x_2^{'}(t))$, $(x_1^{'}(t), -x_2^{'}(t))$, $(-x_1^{'}(t), -x_2^{'}(t))$, $(x_2^{'}(t), x_1^{'}(t))$, $(-x_2^{'}(t), x_1^{'}(t))$, $(x_2^{'}(t), -x_1^{'}(t))$, $(-x_2^{'}(t), -x_1^{'}(t))$. Clearly, $l_x(p,q) = l_{gx}(p,q)$ for all $g \in D_4$. □

Corollary 2.11. *Let $x(t)$ be an I-path in R_T^2. Then $l_x(p,q) = l_{Fx}(p,q)$ for all $F \in E_T^2$.*

Proof. It follows from Proposition 2.10. □

Proposition 2.12. *Let $x(t)$ and $y(t)$ be two I-paths in R_T^2. Then*

(i) *if $x \overset{E_T^2}{\sim} y$ then $L(x) = L(y)$.*
(ii) *if x, y are parametrizations of a curve α then $L(x) = L(y)$.*

Proof. It is obvious. □

The taxicab type of a path $x \in \alpha$, will be called the taxicab type of the curve α and denoted by $L(\alpha)$. $L(\alpha)$ is an E_T^2-invariant of a curve α.

Definition 2.13. An I-path $x(t)$ is called regular if $x^{'}(t) \neq 0$ for all $t \in I$.

If $x(t)$ is a regular path and a path $y(t)$ for all $t \in I$ is D-equivalent to $x(t)$, the $y(t)$ is also a regular path for all $t \in I$. A curve α is called regular if it contains a regular path.

3. Invariant Parametrization of the Taxicab Curve

Now we define an invariant parametrization of a regular curve in R_T^2. Let $I = (a, b)$ and $x(t)$ be a regular I-path in R_T^2. We define the taxicab arc length function $s_x(t)$ for each taxicab type as follows. We put $s_x(t) = l_x(a, t)$ for the case $L(x) = (0, l)$, where $l \leq +\infty$, and $s_x(t) = -l_x(t, b)$ for the case $L(x) = (-\infty, 0)$. Let $L(x) = (-\infty, +\infty)$. We choose a fixed point in every interval $I = (a, b)$ of R and denote it by a_I. Let $a_I = 0$ for $I = (-\infty, +\infty)$. We set $s_x(t) = l_x(a_I, t)$.

Since $s_x^{'}(t) > 0$ for all $t \in I$, the inverse function of $s_x(t)$ exists. Let us denote it by $t_x(s)$. The domain of $t_x(s)$ is $L(x)$ and $t_x^{'}(s) > 0$ for all $s \in L(x)$.

Proposition 3.1. *Let $I = (a, b)$ and x be a regular I-path in R_T^2. Then*

(*i*) *$s_{Fx}(t) = s_x(t)$ and $t_{Fx}(s) = t_x(s)$ for all $F \in E_T^2$;*

(*ii*) *the equalities $s_{x(\varphi)}(r) = s_x(\varphi(r)) + s_0$ and $\varphi(t_{x(\varphi)}(s + s_0)) = t_x(s)$ hold for any C^∞-diffeomorphism $\varphi : J = (c, d) \to I$ such that $\varphi^{'}(r) > 0$ for all $r \in J$, where $s_0 = 0$ for $L(x) \neq (-\infty, +\infty)$ and $s_0 = l_x(\varphi(a_J), a_I)$ for $L(x) = (-\infty, +\infty)$.*

Proof. The statement (*i*) is obvious. Let us prove statement (*ii*). Let $L(x) = (-\infty, +\infty)$. Then we have

$$\begin{aligned} s_{x(\varphi)}(r) &= \int_{a_J}^{r} \left(\left| \frac{d}{dr} x_1(\varphi(r)) \right| + \left| \frac{d}{dr} x_2(\varphi(r)) \right| \right) dr \\ &= \int_{a_J}^{r} \frac{d\varphi}{dr} \left(\left| \frac{d}{d\varphi} x_1(\varphi(r)) \right| + \left| \frac{d}{d\varphi} x_2(\varphi(r)) \right| \right) dr \\ &= l_x(\varphi(a_J), \varphi(r)) = l_x(a_I, \varphi(r)) + l_x(\varphi(a_J), a_I). \end{aligned}$$

Thus, $s_{x(\varphi)}(r) = s_x(\varphi(r)) + s_0$, where $s_0 = l_x(\varphi(a_J), a_I)$. This implies that $\varphi(t_{x(\varphi)}(s + s_0)) = t_x(s)$. For $L(x) \neq (-\infty, +\infty)$, it is easy to see that $s_0 = 0$. □

Let α be a regular curve, $x \in \alpha$. Then $x(t_x(s))$ is a parametrization of α.

Definition 3.2. The parametrization of the form $x(t_x(s))$ of a regular curve α is called an invariant parametrization of α.

Denote the set of all invariant parametrizations of α by $I_p(\alpha)$. Every $y \in I_p(\alpha)$ is a J-path, where $J = L(\alpha)$.

Proposition 3.3. *Let α be a regular curve, $x \in \alpha$ and x be a J-path, where $J = L(\alpha)$. Then the following conditions are equivalent:*

(i) *x is an invariant parametrization of α;*
(ii) $\left|x_1^{'}(s)\right| + \left|x_2^{'}(s)\right| = 1$ *for all $s \in L(\alpha)$;*
(iii) $s_x(s) = s$ *for all $s \in L(\alpha)$.*

Proof. $(i) \to (ii)$. Let $x \in I_p(\alpha)$. Then there exists $y \in \alpha$ such that $x(s) = y(t_y(s))$. By Proposition 3.1, $s_x(s) = s_{y(t_y)}(s) = s_y(t_y(s)) + s_0 = s + s_0$, where s_0 is as in Proposition 3.1. Since s_0 does not depend on s, we have $\frac{ds_x(s)}{ds} = \left|x_1^{'}(s)\right| + \left|x_2^{'}(s)\right| = 1$ for all $s \in L(\alpha)$.

$(ii) \to (iii)$. Let $\left|x_1^{'}(s)\right| + \left|x_2^{'}(s)\right| = 1$ for all $s \in L(\alpha)$. Using the definition of $s_x(t)$, we get $\frac{ds_x(s)}{ds} = \left|x_1^{'}(s)\right| + \left|x_2^{'}(s)\right| = 1$. Therefore $s_x(s) = s + c$ for some $c \in R$. In the case $L(x) \neq (-\infty, +\infty)$, conditions $s_x(s) = s + c$ and $s_x(s) \in L(\alpha)$ for all $s \in L(\alpha)$ implies $c = 0$, that is, $s_x(s) = s$. In the case $L(\alpha) = (-\infty, +\infty)$, equalities $s_x(s) = l_x(a_J, s) = l_x(0, s) = s + c$ implies $0 = l_x(0,0) = c$, that is , $s_x(s) = s$.

$(iii) \to (i)$. Since $s_x(s) = s$ implies $t_x(s) = s$, we get $x(s) = x(t_x(s)) \in I_p(\alpha)$. □

Proposition 3.4. *Let α be a regular curve and $L(\alpha) \neq (-\infty, +\infty)$. Then there exists a unique invariant parametrization of α.*

Proof. Let $x, y \in \alpha$, x be an I_1-path. Then there exists a C^∞- diffeomorphism $\varphi : I_2 \to I_1$ such that $\varphi'(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in I_2$. By Proposition 3.3 and $L(\alpha) \neq (-\infty, +\infty)$, we obtain $y(t_y(s)) = x(\varphi(t_y(s)) = x(\varphi(t_{x(\varphi)}(s))) = x(t_x(s))$. □

Proposition 3.5. *Let α be a regular curve, $L(\alpha) = (-\infty, +\infty)$ and $x \in I_p(\alpha)$. Then $I_p(\alpha) = \{y : y(s) = x(s + c), c \in (-\infty, +\infty)\}$.*

Proof. Let $x, y \in I_p(\alpha)$. Then there exist $h, k \in \alpha$ such that $x(s) = h(t_h(s))$, $y(s) = k(t_k(s))$, where h is an I_1-path and k is an I_2-path. Since $h, k \in \alpha$ there exists $\varphi : I_2 \to I_1$ such that $\varphi'(r) > 0$ and $k(r) = h(\varphi(r))$ for all $r \in I_2$. By Proposition 3.1, $y(s) = k(t_k(s)) = h(\varphi(t_k(s)) = h(\varphi(t_{h(\varphi)}(s)) = h(t_h(s - s_0)) = x(s - s_0)$.

Let $x \in I_p(\alpha)$ and $s' \in (-\infty, +\infty)$. We proof $x(\theta) \in I_p(\alpha)$, where $\theta(s) = s + s'$. By Proposition 3.3, $\left|x_1^{'}(s)\right| + \left|x_2^{'}(s)\right| = 1$ and $s_x(s) = s$. Put $z(s) = x(\theta(s))$. Since θ is a C^∞-diffeomorphism of $(-\infty, +\infty)$ onto $(-\infty, +\infty)$, then $z = x(\theta) \in \alpha$. Using Proposition 3.1 and $s_x(s) = s$, we get $s_z(s) = s_{x(\theta)}(s) = s_x(\theta(s)) + s_1 = (s + s') + s_1$, where

$$s_1 = \int_{\theta(0)}^{0} \left(\left|x_1^{'}(s)\right| + \left|x_2^{'}(s)\right|\right) ds$$

for $s \in L(\alpha)$.

This, in view of $\left|x_1'(s)\right| + \left|x_2'(s)\right| = 1$, implies $s_1 = -\theta(0) = -s'$. Then $s_z(s) = (s+s') - s' = s$. By Proposition 3.3, $z \in I_p(\alpha)$. □

Theorem 3.6. *Let α, β be regular curves and $x \in I_p(\alpha), y \in I_p(\beta)$. Then*

(*i*) *for $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{E_T^2}{\sim} \beta$ if and only if $x \overset{E_T^2}{\sim} y$;*

(*ii*) *for $L(\alpha) = L(\beta) = (-\infty, +\infty)$, $\alpha \overset{E_T^2}{\sim} \beta$ if and only if $x \overset{E_T^2}{\sim} y(\psi_c)$ for some $c \in (-\infty, +\infty)$, where $\psi_c(s) = s + c$.*

Proof. (*i*). Let $\alpha \overset{E_T^2}{\sim} \beta$ and $h \in \alpha$. Then there exists $F \in E_T^2$ such that $\beta = F\alpha$. This implies $Fh \in \beta$. Using Propositions 3.1–3.4, we get $x(s) = h(t_h(s)), y(s) = (Fh)(t_{Fh}(s))$ and $Fx(s) = F(h(t_h(s))) = (Fh)(t_h(s)) = (Fh)(t_{Fh}(s)) = y(s)$. Thus $x \overset{E_T^2}{\sim} y$. Conversely, let $x \overset{E_T^2}{\sim} y$, that is, there exists $F \in E_T^2$ such that $Fx = y$. Then $\alpha \overset{E_T^2}{\sim} \beta$.

(*ii*). Let $\alpha \overset{E_T^2}{\sim} \beta$. Then there exist J-paths $h \in \alpha, k \in \beta$ and $F \in E_T^2$ such that $k(t) = Fh(t)$. We have $k(t_k(s)) = k(t_{Fh}(s)) = k(t_h(s)) = (Fh)(t_h(s))$. By Proposition 3.5, $x(s) = k(t_k(s+s_1)), y(s) = h(t_h(s+s_2))$ for some $s_1, s_2 \in (-\infty, +\infty)$. Therefore, $x(s - s_1) = Fy(s - s_2)$. This implies that $x \overset{E_T^2}{\sim} y(\psi_c)$, where $\psi_c(s) = s + c$ and $c = s_1 - s_2$. Conversely, let $x \overset{E_T^2}{\sim} y(\psi_c)$ for some $c \in (-\infty, +\infty)$, where $\psi_c = s + c$. Then there exists $F \in E_T^2$ such that $y(s+c) = Fx(s)$. Since $y(s+c) \in \beta$, then $\alpha \overset{E_T^2}{\sim} \beta$. □

Theorem 3.6 reduces problems of the E_T^2-equivalence regular curves to that of paths only for the case $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$. Let G be a subgroup of E_T^2.

Definition 3.7. J-paths $x(t)$ and $y(t)$ will be called $[G, (-\infty, +\infty)]$-equivalent, if there exist $g \in G$ and $d \in (-\infty, +\infty)$ such that $y(t) = gx(t+d)$ for all $t \in J$.

Theorem 3.6 reduces problems of the G-equivalence of regular curves to $[G, (-\infty, +\infty)]$-equivalence of paths for the case $L(\alpha) = L(\beta) = (-\infty, +\infty)$.

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