RELATIVE ALGEBRAIC STRUCTURES

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Abstract. The concept and some of the algebraic properties of the rejective and non-absorptive sets of a subgroup, subring, and subgroup of a module over a ring are investigated. It is shown that the set theoretic complement of a non-absorptive set in the above mentioned algebraic substructures is a normal subgroup (respectively, (left, right) ideal, submodule) of its underlying algebraic structure. The invariant property of the non-absorptive sets under the operation of inversion in the related underlying algebraic structure is proved. $G \setminus R(H)$, the set theoretic complement of the rejective set of a subgroup H in a group G, is closed under the product in G and whenever $|G|$ the order of the group G is finite, $|R(H)| = (k-s)|H|$ where each of the k and s is the index of H in G and in $G \setminus R(H)$, respectively. For the case of rings and modules, the set theoretic complement of the rejective set of a substructure in the underlying ring is a subring of the underlying ring. For any subring S of a ring R, examples and some of the properties of S-relative (left) ideals and S-relative submodules are given and also it is shown that S is contained in the set theoretic complement of the rejective set of that S-relative (left) ideal (respectively, submodule). Finally, some of the properties of the relative homomorphisms of R-modules, and the rejective (respectively, non-absorptive) sets of the group homomorphisms of R-modules are investigated.

1. Rejective and Non-Absorptive Sets of Some Algebraic Substructures.

Definition 1.1. For a subgroup H of a group G, the set of all elements h in H such that for each h there exists an element g in G with $ghg^{-1} \notin H$ is called the non-absorptive set of H in G and is denoted by $N(H, G)$ or $N(H)$ whenever there is no confusion in the context.

Remark. In the above definition, it is clear that H is a normal subgroup of G if and only if $N(H)$ is the empty set.

Theorem 1.1. Let $N(H)$ be the non-absorptive set of a subgroup H in a group G. Then $H \setminus N(H)$ is a normal subgroup of G.

Proof. Suppose to the contrary that a and b are in $H \setminus N(H)$ and ab is in $N(H)$. Thus, for some $g \in G$, $gabg^{-1} = gag^{-1}gbg^{-1} \notin H$ which is a contradiction since both gag^{-1} and gbg^{-1} are in H. Next, $b \in H \setminus N(H)$ implies $gbg^{-1} \in H$ for all $g \in G$ and this makes $(gbg^{-1})^{-1} = gb^{-1}g^{-1}$ to be in H for all $g \in G$ which implies $b^{-1} \in H \setminus N(H)$. Finally, it remains to show that $H \setminus N(H)$ is normal in G. Suppose for some $a \in H \setminus N(H)$ there exists $g \in G$ such that $gag^{-1} \notin H \setminus N(H)$. Hence, by definition, $gag^{-1} \in N(H)$ and this forces $k(gag^{-1})k^{-1} = (kg)a(kg)^{-1} \notin H$ for some k in G and this is a contradiction to the choice of a in $H \setminus N(H)$.

Theorem 1.2. Let $N(H)$ be the non-absorptive set of a subgroup H in a group G. Then for each $x \in G \setminus N(H)$, x^{-1} is also in $G \setminus N(H)$.

Proof. Suppose to the contrary that $x \in G \setminus N(H)$ and x^{-1} is in $N(H)$. Consequently, x is in $H \setminus N(H)$ which implies $(gxg^{-1})^{-1} = gx^{-1}g^{-1} \in H$ for all $g \in G$ and this is a contradiction to the choice of x^{-1} in $N(H)$.

Remark. From the above theorem, it is clear that $x^{-1} \in N(H)$ whenever $x \in N(H)$. In other words, $N(H)$ is invariant under the group operation of inversion. Furthermore, $N(H)$ can never be closed under the product in the group since e the group identity element is not in $N(H)$.

Corollary 1.1. Let $N(H)$ be the non-absorptive set of a subgroup H in a group G. Then $G \setminus N(H)$ is a subgroup of G if and only if $G \setminus N(H)$ is closed under the product in G.

Remark. For a family $\{G_i \mid i \in I\}$ of groups with H_i a subgroup of G_i for each $i \in I$, it is not difficult to show that $\prod N(H_i, G_i) \subseteq N(\prod H_i, \prod G_i)$ and for $|I| = 2$, $(N(H_1) \times H_2) \cup (H_1 \times N(H_2)) = N(H_1 \times H_2)$. In general, $\cup_{i\in I} \prod^j H_i = N(\prod H_i)$ where $\prod^j H_i$ is the Cartesian product of all subgroups H_i $(i \neq j)$ and $N(H_j)$ for $i = j$.

Definition 1.2. For a subgroup H of a group G, the set of all $g \in G$ such that for each g there exists an element h in H with $ghg^{-1} \notin H$ is called the rejective set of H in G and is denoted by $R(H, G)$ or $R(H)$ whenever there is no confusion in the context.

Remark. It is clear that $R(H, G) = \emptyset$ if and only if H is a normal subgroup of G. From the definition, it is obvious that $R(H, G) \subseteq G \setminus H$.

Theorem 1.3. For any subgroup H of a group G, $G \setminus R(H)$ the set theoretic complement of the rejective set of H in G is closed under the group operation of product.

Proof. Suppose to the contrary that $a, b \in G \setminus R(H)$ and $ab \in R(H)$. Thus, for some h in H, $abhb^{-1}a^{-1} = a(bhb^{-1})a^{-1} \notin H$ and this is a contradiction.

Corollary 1.2. Let $R(H)$ be the rejective set of a subgroup H in a group G. Then $G \setminus R(H)$ is a subgroup of G if and only if $R(H)$ is invariant under the operation of inversion in G.

Corollary 1.3. Let $R(H)$ be the rejective set of a subgroup H in a finite group G. Then $G \setminus R(H)$ is a subgroup of G and $|R(H)| = (k-s)|H|$ where k is the index of H in G and s is the index of H in $G \setminus R(H)$.

Proof. Since every non-empty finite subset of a group G is a subgroup of G if and only if it is closed under the product in G, consequently, by applying the above theorem together with Lagrange's Theorem, $G \setminus R(H)$ is a subgroup of G and each of $|G \setminus R(H)|$ and $|G|$ is divisible by $|H|$. Thus, $s|H| = |G \setminus R(H)| = |G| - |R(H)| = k|H| - |R(H)|$ which implies $|R(H)| = (k-s)|H|$ where s is the index of H in $G \setminus R(H)$ and k is the index of H in G.

Example. As an application of the above corollary, it is easy to conclude that any subgroup H of a group G with a finite order $2n$ is normal in G whenever the order of H is n and there exists an element g in $G \setminus H$ such that $ghg^{-1} \in H$ for all $h \in H$.

Definition 1.3. For a subring A of a ring R, the set of all elements $a \in A$ such that for each a there exists an element r in R with $ra \notin A$ (respectively, $ar \notin A$) is called the left (respectively, right) non-absorptive set of A in R and is denoted by $N_l(A, R)$ or $N_l(A)$ (respectively, $N_r(A, R)$ or $N_r(A)$) whenever there is no confusion in the context.

Remark. From the above definition, it is clear that A is a left (respectively, right) ideal in R if and only if $N_l(A)$ (respectively, $N_r(A)$) is the empty set.

Theorem 1.4. Let A be a subring of a ring R. Then $A \setminus N_l(A)$ (respectively, $A \setminus N_r(A)$) is a left (respectively, right) ideal of R.

Proof. Let each of a and b be an element in $A \setminus N_l(A)$ and suppose to the contrary that $(a-b) \notin A \setminus N_l(A)$. Then for some r in R, $r(a-b) = ra - rb \notin A$

which is a contradiction since both ra and rb are in A. Now, for any $a \in A \setminus N_l(A)$ and $r \in R$ if $ra \notin A \setminus N_l(A)$, then ra must be in $N_l(A)$. Hence, for some s in R, $s(ra) = (sr)a \notin A$ which is a contradiction to the choice of a in $A \setminus N_l(A)$. A proof for the case of $N_r(A)$ can be followed analogously.

Theorem 1.5. Let $N_l(A)$ be the left non-absorptive set of a subring A in a ring R. Then for each element $a \in R \setminus N_l(A)$, $-a$ is also in $R \setminus N_l(A)$. In other words, $a \in N_l(A)$ implies $-a \in N_l(A)$.

Proof. Suppose $a \in R \setminus N_l(A)$ and $-a \in N_l(A)$. Then a must be in $A \setminus N_l(A)$ since A is an additive subgroup of R. Thus, $r(-a) = -r(a) \in A$ for all r in R and this is a contradiction to the choice of $-a \in N_l(A)$.

Remark. Let $\{R_i \mid i \in I\}$ be a family of rings with A_i a subring of R_i for each $i \in I$. Then $\prod N_l(A_i, R_i) \subseteq N_l(\prod A_i, \prod R_i)$, and for $|I| = 2$, we have $(N_l(A_1) \times A_2) \cup (A_1 \times N_l(A_2)) = N_l(A_1 \times A_2)$. In general, $\cup_{i \in I} \prod^j A_i = N_l(\prod_{i \in I} A_i)$ where $\prod^j A_i$ is the Cartesian product of all subrings A_i $(i \neq j)$ and $N_l(A_j)$ for $i = j$.

Remark. In any commutative ring, it is obvious that the left and right non-absorptive sets of a subring coincide with each other.

Definition 1.4. For a subring A of a ring R, the set of all $r \in R$ such that for each r there exists an element a in A with $ra \notin A$ (respectively, $ar \notin A$) is called the left (respectively, right) rejective set of A in R and is denoted by $R_l(A, R)$ or $R_l(A)$ (respectively, $R_r(A, R)$ or $R_r(A)$) whenever there is no confusion in the context.

Remark. It is clear that $R_l(A)$ (respectively, $R_r(A)$) is a subset of $R \setminus A$ and A is a left (respectively, right) ideal of R if and only if $R_l(A)$ (respectively, $R_r(A)$) is the empty set. Note that in a commutative ring R, both the left and right rejective sets of any subring A of R coincide with each other.

Theorem 1.6. Let $R_l(A)$ (respectively, $R_r(A)$) be the left (respectively, right) rejective set of a subring A in a ring R. Then $R \setminus R_l(A)$ (respectively, $R \setminus R_r(A)$) is a subring of R.

Proof. Suppose $r, s \in R \setminus R_l(A)$ and $r - s \notin R \setminus R_l(A)$. Then for some a in A, $(r - s)a = ra - sa \notin A$ and this is a contradiction since both ra and sa are in A. Now, suppose for some $r, s \in R \setminus R_l(A)$, rs is not in $R \setminus R_l(A)$. Thus, for some a in A, $(rs)a = r(sa) \notin A$ and this is a contradiction to the choice of r and s.

Defintion 1.5. Let A be a subgroup of an R-module M over a ring R. The set of all elements $a \in A$ such that for each a there exists an element $r \in R$ with $ra \notin A$ is called the non-absorptive set of A in M and is denoted by $N(A, M)$ or $N(A)$ whenever there is no confusion in the context.

Remark. From the above definition, it is clear that A is a submodule of M if and only if $N(A)$ is the empty set.

Theorem 1.7. Let $N(A)$ be the non-absorptive set of a subgroup A in an R-module M over a ring R. Then $A \setminus N(A)$ is a submodule of M.

Proof. Similar to the proof of Theorem 1.4.

Theorem 1.8. Let $N(A)$ be the non-absorptive set of a subgroup A in an R-module M over a ring R. Then $a \in M \setminus N(A)$ implies $-a \in M \setminus N(A)$. In other words, $a \in N(A)$ implies $-a \in N(A)$.

Proof. Similar to the proof of Theorem 1.5.

Remark. Let $\{M_i \mid i \in I\}$ be a family of R-module over a ring R and A_i a subgroup of M_i for each $i \in I$. Then $\prod N(A_i) \subseteq N(\prod A_i)$ and for $|I| = 2$, $(N(A_1) \times A_2) \cup (A_1 \times N(A_2)) = N(A_1 \times A_2)$. In general, we have $\cup_{i \in I} \prod^j A_i = N(\prod_{i \in I} A_i)$ where $\prod^j A_i$ is the Cartesian product of all subgroups A_i $(i \neq j)$ and $N(A_j)$ for $i = j$.

Definition 1.6. For a group A of an R-module M over a ring R, the set of all elements $r \in R$ such that for each r there exists an element a in A with $ra \notin A$ is called the rejective set of A in M and is denoted by $R(A, M)$ or $R(A)$ whenever there is no confusion in the context.

Remark. For the above definition, it is clear that A is a submodule of M if and only if $R(A)$ is the empty set.

Theorem 1.9. If $R(A)$ is the rejective set of a subgroup A in an R-module M over a ring R, then $R \setminus R(A)$ is a subring of R.

Proof. Similar to the proof of Theorem 1.6.

Theorem 1.10. For any two subgroups (respectively, subrings, subgroups) A and B of a group (respectively, a ring, an R-module), $A - N(A) \subseteq B - N(B)$ (respectively, $A - N_l(A) \subseteq B - N_l(B)$, $A - N(A) \subseteq B - N(B)$) whenever $A \subseteq B$.

Proof. We just give a proof for the subgroups A and B of a group G and leave the other cases to the reader. Suppose to the contrary that there exists an element $a \in A - N(A)$ with $a \notin B - N(B)$. Thus, $a \in N(B)$ and this implies $gag^{-1} \notin B$ for some $g \in G$. Consequently, gag^{-1} is not in A which implies $a \in N(A)$ and this is a contradiction to the choice of a in $A - N(A)$.

2. Relative Ideals.

Definition 2.1. Let S be a subring of a ring R. A subring A of R is an S-relative left (respectively, right) ideal of R provided $s \in S$ and $a \in A$ imply $sa \in A$ (respectively, $as \in A$). A is an S-relative ideal of R if it is both an S-relative left and an S-relative right ideal of R. A subring A of R is said to be a strictly S-relative left (respectively, right) ideal of R whenever A is an S-relative left (respectively, right) ideal of R and it is not an R-relative left (respectively, right) ideal of R

Remark. Whenever a statement is made about the S-relative left ideals, it is to be understood that the analogous statement holds for the S-relative right ideals. It is clear that any left ideal A of a ring R is an S-relative left ideal of R for any subring S of R. Also, A contains S whenever 1_R the identity element of R is in A.

Example. Let M be the ring of all 2×2 matrices over a ring R. Then A the subring of all matrices of the form

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

is neither a left nor a right ideal of M. Let S be the set of all 2×2 matrices with zero $(2,1)$ entries and T the set of all 2×2 matrices with zero $(1,2)$ entries. Now, it is not difficult to show that A is an S-relative left and a T-relative right ideal of M. Furthermore, A is neither an S-relative right ideal nor a T-relative left ideal of M.

Theorem 2.1. If S and A are two subrings of a ring R with A an S-relative left ideal of R, then $R \setminus R_l(A)$ contains S and A is also an $R \setminus R_l(A)$-relative left ideal of R where $R_l(A)$ is the left rejective set of A in R.

Proof. See Theorem 1.6.

Corollary 2.1. For a subring S of a ring R if $\{A_i \mid i \in I\}$ is a family of S-relative left ideals of R, then A_l is a $\cap_{i\in I} R \setminus R_l(A_i)$-relative left ideal of R for each i in I.

Theorem 2.2. The following results can be proved directly from the definition.

a) If A is an S-relative left ideal of a ring R, then $S \cap R_l(A) = \emptyset$.
b) If $f\colon R \to T$ is a homomorphism of rings and A is an S-relative left ideal of R, then $f(A)$ is an $f(S)$-relative left ideal of T.
c) If A is both an S-relative left and a T-relative right ideal of a ring R, then A is an $S \cap T$-relative ideal of R.
d) Let $\{S_i \mid i \in I\}$ be a family of subrings of a ring R. Then A is a $\cap_{i\in I} S_i$-relative left ideal of R if A is an S_i-relative left ideal of R for each $i \in I$.
e) If $S_1 \subseteq S_2$ are two subrings of a ring R and A is an S_2-relative left ideal of R, then A is an S_1-relative left ideal of R.
f) For any ascending chain $\{S_i \mid i \in I\}$ of subrings S_i of a ring R, A is a $\cup_{i\in I} S_i$-relative left ideal of R if and only if A is an S_i-relative left ideal of R for each $i \in I$.
g) For a family $\{S_i \mid i \in I\}$ of subrings S_i of a ring R, $\cap_{i\in I} A_i$ is a $\cap_{i\in I} S_i$-relative left ideal of R whenever A_i is an S_i-relative left ideal of R for each $i \in I$.
h) Let $\{R_i \mid i \in I\}$ be a family of rings and S_i a subring of R_i for each $i \in I$. If A_i is an S_i-relative left ideal of R_i for each $i \in I$, then $\prod_{i\in I} A_i$ is a $\prod_{i\in I} S_i$-relative left ideal of $\prod_{i\in I} R_i$ the direct product of the rings.

Example. In the ring R of $n \times n$ matrices over a division ring D, let I_k be the set of all matrices that have nonzero entries only in column k and J_k' the set of all matrices with zero kth rows. Then I_k is a left ideal and a J_k'-relative right ideal but not a right ideal of R. If J_k consists of those matrices with nonzero entries only in row k and I_k' the set of all matrices with zero kth columns, then J_k is a right ideal and an I_k'-relative left ideal but not a left ideal in R.

Theorem 2.3. For any subring A of a ring R, the left rejective set of A in R is $R \setminus A$ whenever A contains 1_R the identity element of R.

Proof. $A = R \setminus R_l(A)$ since A is always contained in $R \setminus R_l(A)$ and $1_R \in A$ implies $R \setminus R_l(A) \subseteq A$.

Example. As an application of the above theorem let $R[X]$ be the ring of all polynomials over an integral domain R and A the ring of all polynomials with zero

X-coefficients, then $R(A, R[X])$ the rejective set of A in $R[X]$ is $R[X] \setminus A$ which is exactly the set of all polynomials with nonzero X-coefficients.

Definition 2.2. Let X be a subset of a ring R and S a subring of R. If $\{A_i \mid i \in I\}$ is the family of all S-relative left ideals of R containing X, then $\cap_{i \in I} A_i$ is called the S-relative left ideal generated by X in R and is denoted by $(X)_S$. The elements of X are called S-relative generators of $(X)_S$. If $X = \{x_1, x_2, \dots, x_n\}$, then the S-relative left ideal $(X)_S$ is denoted by $(x_1, x_2, \dots, x_n)_S$ and is said to be an S-relative finitely generated left ideal. An S-relative left ideal $(x)_S$ generated by a single element x is called an S-relative principal left ideal of R.

Theorem 2.4. Let S be a subring of a ring R, a an element in R, and K the set of all elements of the form $ra + as + na + \sum_{i=1}^{m} r_i a s_i$ where $r, s, r_i, s_i \in S$, n an integer, and m runs over the set of non-negative integers. Then we have the following results:

1) $K \subseteq (a)_S$ the S-relative principal ideal generated by a in R. Moreover, $a \in S$ implies $(a)_S = K = (a)^S$ the principal ideal generated by a in S.
2) $x \in (a)_S \setminus K$ implies $-x \in (a)_S \setminus K$.
3) If R is a commutative ring and $a \in S$, then $(a)_S$ consists of all elements of the form $sa + na$ where $s \in S$ and $n \in Z$ the ring of rational integers.

Proof. The proof is an immediate consequence of the definition and we leave it to the reader as an exercise.

Theorem 2.5. For a subring S of a ring R if A is an S-relative ideal of R, then

1) $S + A = \{s + a \mid s \in S,\ a \in A\}$ is also an S-relative ideal in R.
2) $S \cup A$ is a multiplicative system in R.
3) $S \cap A$ is an S-relative ideal of R, and also it is an S-relative left ideal of R whenever A is an S-relative left ideal of R.

Proof. The proof is a direct consequence of the definition and we leave it to the reader.

Theorem 2.6. In a commutative ring R, let each of $S_1, S_2, \dots, S_n$ be a subring of R and A_i an S_i-relative ideal of R for each $i = 1, 2, \dots, n$, respectively. Then $A_1 A_2 A_3 \cdots A_n$ is an $S_{i_1} S_{i_2} \cdots S_{i_k}$-relative ideal of R where $\{i_1, i_2, \dots, i_k\}$ is a subset of the set $\{1, 2, \dots, n\}$.

Proof. The proof follows directly from the definition and we leave it to the reader.

Definition 2.3. Let S be a subring of a ring R. An S-relative left ideal P of R is said to be an S-relative prime left ideal of R if $P \neq R$ and for any S-relative left ideals A and B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Theorem 2.7. For a subring S of a ring R if P is an S-relative left ideal of R such that $P \neq R$ and for all elements $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$, then P is an S-relative prime left ideal of R. Conversely if R is commutative and P is an S-relative prime ideal of R, then for any $a, b \in S$, $ab \in P$ implies either $a \in P$ or $b \in P$.

Proof. Let A and B be two S-relative left ideals of R such that $AB \subseteq P$. Suppose $A \not\subseteq P$, then there exists an element $a \in A$ with $a \ni P$. Now, for each $b \in B$, $ab \in AB \subseteq P$ implies $a \in P$ or $b \in P$ which implies $b \in P$ and consequently $B \subseteq P$. Conversely, $ab \in P$ implies $(ab)_S \subseteq P$. Now since R is commutative and $a, b \in S$, then $(a)_S(b)_S \subseteq (ab)_S \subseteq P$ which implies the desired conclusion.

Remark. From the above result it is clear that if P is an S-relative prime ideal of a commutative ring R, then $S \setminus P$ is a multiplicative system in R.

Remark. Any S-relative (left) ideal A of a ring R is a (left) S-module.

3. Relative Submodules.

Definition 3.1. For a ring R, let M be an R-module and S a subring of R. A non-empty subset A of M is an S-relative submodule of M provided that A is an additive subgroup of M and $sa \in A$ for all $s \in S$ and $a \in A$.

Example. A subring S of a ring R is an S-relative submodule of R. In general, any S-relative (left) ideal of a ring R is an S-relative submodule of R whenever R is assumed to be an R-module over itself.

Remark. An S-relative submodule B of an R-module A over a ring R need not be a subring of R whenever $A = R$.

Example. Let A be an S-relative left ideal of a ring R and M an R-module. If X is a non-empty subset of M, then $AX = \{\sum_{i=1}^{n} a_i x_i \mid a_i \in A,\ x_i \in X, \text{ and } n \text{ a positive integer}\}$ forms an S-relative submodule of M. Similarly, for any $x \in M$, $Ax = \{ax \mid a \in A\}$ is an S-relative submodule of M.

Theorem 3.1. For a subring S of a ring R if A is an S-relative submodule of an R-module M, then A is an $R \setminus R(A)$-relative submodule of M and S is contained in $R \setminus R(A)$ where $R(A)$ is the rejective set of A in M.

Proof. See Theorem 1.9.

Corollary 3.1. For a subring S of a ring R if $\{A_i \mid i \in I\}$ is a family of S-relative submodules of an R-module M, then A_i is a $\cap_{i\in I}(R \setminus R(A_i))$-relative submodule of M for each i in I.

Theorem 3.2. The following results can be proved directly from the definition.

a) If A is an S-relative submodule of a module M over a ring R, then $S \cap R(A) = \emptyset$ where $R(A)$ is the rejective set of A in M.
b) If R is a ring and $f\colon M \to N$ an R-module homomorphism, then the homomorphic image (respectively, inverse image) of any S-relative submodule of M (respectively, N) is again an S-relative submodule of N (respectively, M).
c) For a ring R if $\{S_i \mid i \in I\}$ is a family of subrings of R, A an R-module, and B_i an S_i-relative submodule of A for each $i \in I$, then $\cap_{i\in I} B_i$ is a $\cap_{i\in I} S_i$-relative submodule of A.
d) If $S_1 \subseteq S_2$ are two subrings of a ring R and A is an S_2-relative submodule of an R-module M, then A is an S_1-relative submodule of M.
e) For any ascending chain $\{S_i \mid i \in I\}$ of subrings S_i of a ring R, A is a $\cup_{i\in I} S_i$-relative submodule of an R-module M if and only if A is an S_i-relative submodule of M for each $i \in I$.
f) For a family $\{S_i \mid i \in I\}$ of subrings S_i of a ring R, $\cap_{i\in I} A_i$ is a $\cap_{i\in I} S_i$-relative submodule of an R-module whenever A_i is an S_i-relative submodule of M for each $i \in I$.
g) For a family of rings $\{R_i \mid i \in I\}$, assume S_i is a subring of R_i, M_i an R_i-module, and A_i an S_i-relative submodule of M_i for each $i \in I$, then $\prod_{i\in I} A_i$ is a $\prod_{i\in I} S_i$-relative submodule of $\prod_{i\in I} M_i$.
h) Let S be a subring of a ring R and $\{A_i \mid i \in I\}$ an ascending chain of subgroups of an R-module M. Then $\cup_{i\in I} A_i$ is an S-relative submodule of M whenever A_i is an S-relative submodule of M for each $i \in I$.

Definition 3.2. If X is a subset of a module M over a ring R and S is a subring of R, then the intersection of all S-relative submodules of M containing X is called the S-relative submodule generated by X or spanned by X and is denoted by $\langle X\rangle_S$. If X is finite and X generates the S-relative submodule A in M, then A is said to be S-relative finitely generated. If $X = \{a\}$, then $\langle a\rangle_S$ is called the S-relative cyclic submodule generated by a. Finally if $\{B_i \mid i \in I\}$ is a family of S-relative submodules of M, then the S-relative submodule generated by $X = \cup_{i\in I} B_i$ is called the sum of the S-relative submodules B_i. If the index set I is finite, then the sum of $B_1, B_2, \ldots, B_n$ is denoted by $B_1 + B_2 + \cdots + B_n$.

Theorem 3.3. Let S be a subring of a ring R, A an R-module, X a subset of A, $\{B_i \mid i \in I\}$ a family of S-relative submodules of A, a an element in A, and $Sa = \{sa \mid s \in S\}$.

1) Sa is an S-relative submodule of A and the map $S \to Sa$ given by $s \vdash sa$ is an S-module epimorphism.
2) The S-relative cyclic submodule C generated by a is $\{sa + na \mid s \in S$ and $n \in \mathbb{Z}$ the ring of integers$\}$. If S has an identity 1_S and $1_S a = a$, then $C = Sa$.
3) The S-relative submodule D generated by X is the set of all elements of the form $\sum_{i=1}^{n} s_i a_i + \sum_{j=1}^{m} n_j b_j$ where n, m are non-negative integers, $n_j \in \mathbb{Z}$, $s_i \in S$ and $a_i, b_j \in X$. If S has an identity 1_S and for each $x \in X$, $1_S x = x$, then $D = SX = \{\sum_{i=1}^{n} s_i a_i \mid s_i \in S,\ a_i \in X,$ and n a non-negative integer$\}$.
4) The sum of the family $\{B_i \mid i \in I\}$ consists of all finite sums $b_{i_1} + b_{i_2} + \cdots + b_{i_n}$ where b_{i_k} is an element of B_{i_k}.

Proof. The proof follows directly from the definition.

Definition 3.3. Let S and T with $S \subseteq T$ be two subrings of a ring R, A an R-module and B a T-module. A group homomorphism $f\colon A \to B$ is said to be an S-relative homomorphism of modules if for all $s \in S$ and $a \in A$, $f(sa) = sf(a)$.

Theorem 3.4. Let S be a subring of a ring R and B an S-relative submodule of a module A over R. Then the quotient group A/B is an S-module with the action of S on A/B given by $s(a + B) = sa + B$ for all $s \in S$ and $a \in A$. The map $\pi_S\colon A \to A/B$ given by $a \vdash a + B$ is an S-relative epimorphism of modules with the kernel B. The map π_S is called the S-relative canonical epimorphism or projection.

Proof. If $a + B = a' + B$, then $a - a' \in B$. Since B is an S-relative submodule of A, then $sa - sa' = s(a - a')$ is an element in B for all s in S. Thus, $sa + B = sa' + B$ which implies that the action of S on A/B is well defined. The remainder of the proof is left to the reader.

Definition 3.4. Let A and B be two R-modules over a ring R and $f\colon A \to B$ a group homomorphism. The set of all r in R such that for each r there exists an element a in A with $f(ra) \neq rf(a)$ is called the rejective set of f in R and is denoted by $R(f, R)$ or $R(f)$ whenever there is no confusion in the context.

Remark. From the above definition, it is clear that $f\colon A \to B$ is an R-module homomorphism if and only if $R(f,R)$ is the empty set. Note that zero is always in $R \setminus R(f)$ since $f(0a) = f(0) = 0 = 0f(a)$. In addition if $f\colon A \to B$ is a group homomorphism of two unitary R-modules A and B, then $f(1_R a) = f(a) = 1_R f(a)$ which implies $1_R \in R \setminus R(f)$.

Theorem 3.5. Assume each of A and B is an R-module over a ring R. If $f\colon A \to B$ is a group homomorphism, then $R \setminus R(f)$ the set theoretic complement of the rejective set of f in R is a subring of R and f is an $R \setminus R(f)$-relative homomorphism of A and B. In addition, $R \setminus R(f)$ is a subfield of R whenever R is a field and A and B are unitary R-modules.

Proof. For any $r, s \in R \setminus R(f)$ and $a \in A$, $f((r-s)a) = f(ra - sa) = f(ra) + f(-sa) = rf(a) - sf(a) = (r-s)f(a)$ which implies $(r-s)$ is in $R \setminus R(f)$. Similarly, $f((rs)a) = f(r(sa)) = rf(sa) = (rs)f(a)$ implies rs is in $R \setminus R(f)$. Now suppose R is a field and r is an arbitrary nonzero element of $R \setminus R(f)$. Thus, for any a in A, $f(a) = f(rr^{-1}a) = rf(r^{-1}a)$ which implies $r^{-1}f(a) = f(r^{-1}a)$.

Corollary 3.2. Let S be a subring of a ring R and $f\colon A \to B$ a group homomorphism of the R-modules A and B. Then f is an $R \setminus R(f)$-relative homomorphism of the R-modules A and B and S is contained in $R \setminus R(f)$ whenever f is an S-relative homomorphism of A and B.

Definition 3.5. Let A and B be two R-modules over a ring R and $f\colon A \to B$ a homomorphism of the groups. The set of all a in A such that for each a there exists an element r in R with $f(ra) \neq rf(a)$ is called the non-absorptive set of f in R and is denoted by $N(f,R)$ or $N(f)$ whenever there is no confusion in the context.

Remark. In the above definition, it is clear that f is an R-module homomorphism of A and B if and only if $N(f)$ is the empty set.

Theorem 3.6. Let A and B be two R-modules over a ring R and $f\colon A \to B$ a group homomorphism. Then $A \setminus N(f)$ the set theoretic complement of the non-absorptive set of f in A is a submodule of A.

Proof. Note that $A \setminus N(f)$ is a non-empty set since it contains the zero element $f(r0) = f(0) = 0 = rf(0)$ for any r in R. For any $r \in R$ and $a, b \in A \setminus N(f)$, $f(r(a-b)) = f(ra-rb) = f(ra) + f(-rb) = rf(a) + rf(-b) = rf(a-b)$ which implies $a-b$ is in $A \setminus N(f)$. Now suppose for some a in $A \setminus N(f)$ there exists an r in R such that $ra \ni A \backslash N(f)$. Then there exists $s \in R$ such that $f(s(ra)) \neq sf(ra) = (sr)f(a)$ which is a contradicition to the choice of a in $A \setminus N(f)$.

In conclusion, it should be noted that the above ideas are new to the author and a search of the literature found no mention of such a concept as presented here. It is entirely possible, however, that a reader might know of a source of similar ideas.

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