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A localization principle for biholomorphic mappings between the Fock-Bargmann-Hartogs domains

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ABSTRACT. In this paper, we prove that a localization principle for biholomorphic mappings between equidimensional Fock-Bargmann-Hartogs domains holds. As an application of this, we show that any proper holomorphic mapping between two equidimensional Fock-Bargmann-Hartogs domains satisfying some condition is necessarily a biholomorphic mapping.

1. Introduction and results

Let D_1 and D_2 be two domains in $\mathbb{C}^N$. Then we say that *the localization principle for biholomorphic mappings between D_1 and D_2 holds* if the following (†) is fulfilled:

(†) For some open subsets U_1, U_2 in $\mathbb{C}^N$ with $U_1 \cap \partial D_1 \neq \emptyset$, $U_2 \cap \partial D_2 \neq \emptyset$, any biholomorphic mapping $f : U_1 \to U_2$ satisfying

$$f(U_1 \cap D_1) \subset D_2, \qquad f(U_1 \cap \partial D_1) \subset \partial D_2$$

extends to a biholomorphic mapping $F : D_1 \to D_2$.

Of course, the localization principle for biholomorphic mappings does not hold, in general, without any other assumptions on the domains D_1 or D_2. Indeed, as a typical example, consider the following domains D_1, D_2 in $\mathbb{C}^2$ and a mapping $h : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$D_1 = \{(z, w) \in \mathbb{C}^2; |z|^2 + |w|^4 < 1\}, \qquad D_2 = \{(u, v) \in \mathbb{C}^2; |u|^2 + |v|^2 < 1\}$$

$$\text{and} \qquad (u, v) = h(z, w) = (z, w^2) \qquad \text{for } (z, w) \in \mathbb{C}^2.$$

Take a point $(z_o, w_o) \in \partial D_1$ with $w_o \neq 0$ and let U_1 be a sufficiently small open neighborhood of (z_o, w_o) in $\mathbb{C}^2$. Then h gives rise to a biholomorphic mapping, say $f : U_1 \to U_2 := h(U_1)$ satisfying the condition in (†); while f does not extend to a biholomorphic mapping from D_1 onto D_2. However,

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there already exist several articles showing the existence of domains D_1, D_2 in $\mathbb{C}^N$ for which the localization principle (†) holds. See, for instance, Alexander [1, 2], Pinchuk [18, 19, 20], Dini-Primicerio [8] and Kodama [13].

The main purpose of this paper is to prove that the localization principle for biholomorphic mappings between equidimensional Fock-Bargmann-Hartogs domains in $\mathbb{C}^N$ holds. In order to state our precise results, let us define the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ according to Yamamori [27] as follows:

$$D_{n,m}(\mu) = \left\{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N; \|w\|^2 < e^{-\mu\|z\|^2}\right\},$$

where $0 < \mu \in \mathbb{R}$ and $n, m \in \mathbb{N}$ with $N = n + m$. This is an unbounded strictly pseudoconvex domain in $\mathbb{C}^N$ with real analytic boundary. Since the complex Euclidean space $\mathbb{C}^n$ is now imbedded in $D_{n,m}(\mu)$ in the canonical manner, it is not hyperbolic in the sense of Kobayashi [12].

Now we can state our results as follows:

THEOREM 1. *Let $D_1 = D_{n_1,m_1}(\mu_1)$, $D_2 = D_{n_2,m_2}(\mu_2)$ be two equidimensional Fock-Bargmann-Hartogs domains in $\mathbb{C}^N$ with $p_1 \in \partial D_1$, $p_2 \in \partial D_2$. Assume that*

(1) *$m_1 \geq 2$, $m_2 \geq 2$;*

(2) *there are open neighborhoods U_1 of p_1, U_2 of p_2 in $\mathbb{C}^N$ and a biholomorphic mapping $f : U_1 \to U_2$ such that $f(p_1) = p_2$, $f(U_1 \cap D_1) \subset D_2$ and $f(U_1 \cap \partial D_1) \subset \partial D_2$.*

Then f extends to a biholomorphic mapping from D_1 onto D_2. In particular, we have $(n_1, m_1) = (n_2, m_2)$.

Recall that any proper holomorphic mapping $f : D_1 \to D_2$ between two equidimensional Fock-Bargmann-Hartogs domains D_1, D_2 in $\mathbb{C}^N$ extends holomorphically to an open neighborhood of $\overline{D_1}$, the closure of D_1 in $\mathbb{C}^N$, by Tu-Wang [23; Theorem 2.5]. Then, as an application of Theorem 1, we can prove the following:

THEOREM 2. *Let $D_1 = D_{n_1,m_1}(\mu_1)$, $D_2 = D_{n_2,m_2}(\mu_2)$ be two equidimensional Fock-Bargmann-Hartogs domains in $\mathbb{C}^N$. Assume that $m_1 \geq 2$. Then every proper holomorphic mapping $f : D_1 \to D_2$ is necessarily a biholomorphic mapping from D_1 onto D_2.*

This Theorem 2 was first proved by Tu-Wang in [23; Theorem 1.1]. In fact, after showing the theorem on the holomorphic extendability beyond the boundary ∂D_1 of proper holomorphic mapping $f : D_1 \to D_2$ between equidimensional Fock-Bargmann-Hartogs domains D_1, D_2 in $\mathbb{C}^N$, they proved Theorem 2 as their main result in [23] by making use of some known facts in algebraic geometry. Our proof here of Theorem 2 is completely different from theirs; we employ an elementary method in Lie group theory.

Finally it should be remarked that the assumptions $m_1 \geq 2$, $m_2 \geq 2$ in Theorem 1 and $m_1 \geq 2$ in Theorem 2 cannot be dropped. Indeed, as in Tu-Wang [23], consider the following Fock-Bargmann-Hartogs domain $D_{n,1}(\mu)$ and the mapping $\Phi : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}^n \times \mathbf{C}$ defined by

$$D_{n,1}(\mu) = \left\{(z, w) \in \mathbf{C}^n \times \mathbf{C}; |w|^2 < e^{-\mu\|z\|^2}\right\} \qquad \text{and}$$

$$\Phi(z, w) = (\sqrt{2}z, w^2) \qquad \text{for } (z, w) \in \mathbf{C}^n \times \mathbf{C}.$$

Then it is easily checked that Φ gives rise to a proper holomorphic self-mapping of $D_{n,1}(\mu)$ that is not injective on $D_{n,1}(\mu)$. Moreover, for any point $p_1 \in \partial D_{n,1}(\mu)$, one can choose open neighborhoods U_1 of p_1 and U_2 of $p_2 := \Phi(p_1) \in \partial D_{n,1}(\mu)$ in such a way that Φ defines a biholomorphic mapping $f : U_1 \to U_2$ satisfying the same condition as in (2) of Theorem 1. But $f : U_1 \to U_2$ does not extend to an automorphism of $D_{n,1}(\mu)$.

Our proof of Theorem 1 is based on three main facts: a well-known fact concerning the global extension of locally defined CR-diffeomorphisms between two strictly pseudoconvex real analytic hypersurfaces in $\mathbf{C}^N$ by Pinchuk [19, 20]; an important fact regarding the existence of CR-invariant Riemannian metrics on strictly pseudoconvex real analytic hypersurfaces without umbilical points by Webster [25, 26]; and a fact on the structure of holomorphic automorphism groups of the Fock-Bargmann-Hartogs domains by Kim-Ninh-Yamamori [10]. On the other hand, for the proof of Theorem 2, we need some lemma, which will be shown by using an elementary method in Lie group theory. Once this lemma has been verified, we obtain Theorem 2 as a direct consequence of Theorem 1.

After investigating the structure of the Fock-Bargmann-Hartogs domains closely in Section 2, we prove our theorems in Sections 3 and 4.

Notation. Throughout this paper we use the following notation: For a given $n \in \mathbf{N}$ and open subsets V, W of $\mathbf{C}^n$, we denote by

- $U(n)$ the unitary group of degree n;
- $\langle \cdot, \cdot \rangle$ (resp. $\|\cdot\|$) the standard Hermitian inner product (resp. its associated norm) on $\mathbf{C}^n$;
- $B^n = \{\zeta \in \mathbf{C}^n; \|\zeta\| < 1\}$, the unit open ball in $\mathbf{C}^n$;
- ∂V (resp. $\overline{V}$) the boundary (resp. closure) of V in $\mathbf{C}^n$;
- $V \Subset W$ if the closure $\overline{V}$ of V is a compact subset of W.

Let D be a domain in $\mathbf{C}^n$ and $F : D \to \mathbf{C}^n$ a holomorphic mapping. Then we denote by

- $\mathrm{Aut}(D)$ the group of all holomorphic automorphisms of D equipped with the compact-open topology. Thus the topology of $\mathrm{Aut}(D)$ satisfies the second axiom of countability;

- $\mathfrak{g}(D)$ the set consisting of all complete holomorphic vector fields on D;
- $F|_S : S \to \mathbf{C}^n$ the restriction of F to S, where S is a subset of D;
- $J_F(\zeta)$ the complex Jacobian determinant of F at $\zeta \in D$; and
- $V_F = \{\zeta \in D; J_F(\zeta) = 0\}$.

2. Preliminaries

For later purpose, we collect several facts on the structure of Fock-Bargmann-Hartogs domains in this section.

For a given Fock-Bargmann-Hartogs domain

$$D_{n,m}(\mu) = \left\{(z,w) \in \mathbf{C}^n \times \mathbf{C}^m; \|w\|^2 < e^{-\mu\|z\|^2}\right\}$$

in $\mathbf{C}^N = \mathbf{C}^n \times \mathbf{C}^m$, we set for a while

$$D = D_{n,m}(\mu), \qquad \Delta_D = \{(z,w) \in D; w = 0\} \cong \mathbf{C}^n \qquad \text{and} \qquad D^* = D \backslash \Delta_D.$$

First of all, we have the following:

THEOREM A (Kim-Ninh-Yamamori [10; Theorem 10]). *The automorphism group* $\mathrm{Aut}(D)$ *of the Fock-Bargmann-Hartogs domain* D *is generated by the following mappings:*

$$\varphi_A : (z,w) \mapsto (Az, w), \qquad A \in U(n);$$

$$\varphi_B : (z,w) \mapsto (z, Bw), \qquad B \in U(m);$$

$$\varphi_v : (z,w) \mapsto (z+v, e^{-\mu\langle z,v\rangle - (\mu/2)\|v\|^2} w), \qquad v \in \mathbf{C}^n.$$

Hence the following assertions are easily verified:

FACT 1. *The boundary ∂D of D is a connected, strictly pseudoconvex real analytic hypersurface in $\mathbf{C}^N$; moreover, it is simply connected if $m \geq 2$;*

FACT 2. $\mathrm{Aut}(D)$ *can be regarded as a closed subgroup of* $\mathrm{Aut}(\mathbf{C}^N)$ *and the* $\mathrm{Aut}(D)$*-action on D (resp. on ∂D) is just the restriction of the* $\mathrm{Aut}(D)$*-action on $\mathbf{C}^N$ to D (resp. to ∂D);*

FACT 3. *∂D is invariant under the* $\mathrm{Aut}(D)$*-action and moreover* $\mathrm{Aut}(D)$ *acts transitively on ∂D as a real analytic CR-automorphism group of ∂D.*

In particular, via the natural action of the product group $U(n) \times U(m)$ on $\mathbf{C}^n \times \mathbf{C}^m$, one can identify $U(n) \times U(m)$ with a compact connected subgroup of $\mathrm{Aut}(D)$. Accordingly, the compact connected Lie groups $U(n)$, $U(m)$ and $SU(m)$ can be naturally regarded as topological subgroups of $\mathrm{Aut}(D)$, where $SU(m)$ is the special unitary group of degree m.

For later use, let us investigate the structure of $\mathrm{Aut}(D)$ more closely. Let F_D and d_D be the infinitesimal Kobayashi pseudometric and the Kobayashi pseudodistance of D, respectively, introduced by Kobayashi [12]. Then it is well-known that F_D and d_D are invariant under the $\mathrm{Aut}(D)$-action on D. Here, putting $\zeta = (\zeta_1, \dots, \zeta_N) = (z, w)$, let us define a real analytic function u on $\mathbf{C}^N$ by

$$u(\zeta) = \|w\|^2 e^{\mu\|z\|^2} \qquad \text{for } \zeta \in \mathbf{C}^N \tag{2.1}$$

and consider its complex Hessian form

$$H_u(\zeta; t) = \sum_{i,j=1}^{N} \frac{\partial^2 u(\zeta)}{\partial \zeta_i \partial \overline{\zeta_j}} t_i \overline{t_j} \qquad \text{for } t = (t_1, \dots, t_N) \in \mathbf{C}^N.$$

Then, for any point $\zeta_o = (a, b) \in \mathbf{C}^n \times \mathbf{C}^m = \mathbf{C}^N$, we have

$$\begin{aligned} H_u(\zeta_o; t) &= e^{\mu\|a\|^2}\{\mu^2\|b\|^2|\langle a, u\rangle|^2 + \mu\|b\|^2\|u\|^2 + 2\mu\,\mathrm{Re}(\langle a, u\rangle\overline{\langle b, v\rangle}) + \|v\|^2\} \\ &\geq e^{\mu\|a\|^2}\{(\mu\|b\||\langle a, u\rangle| - \|v\|)^2 + \mu\|b\|^2\|u\|^2\} \geq 0 \end{aligned}$$

for all $t = (u, v) \in \mathbf{C}^n \times \mathbf{C}^m = \mathbf{C}^N$ by Schwarz's inequality. Thus u is a plurisubharmonic function on $\mathbf{C}^N$ with $0 \leq u(\zeta) < 1$ on D and moreover it is a strictly plurisubharmonic function on D^* with $0 < u(\zeta) < 1$ on D^*. Hence, by a result of Sibony [22; Theorem 3], D is hyperbolic at every point $p \in D^*$, that is, there are an open neighborhood U of p in D and a positive constant c such that $F_D(q; \xi) \geq c\|\xi\|$ for all $q \in U$, where $\|\xi\|$ denotes the norm of the tangent vector ξ with respect to a fixed Hermitian metric on D. Therefore, d_D induces a true distance on D^* by a result of Royden [21]; accordingly, D^* is hyperbolic in the sense of Kobayashi [12], since $d_{D^*}(p, q) \geq d_D(p, q)$ for any $p, q \in D^*$. On the other hand, it is trivial that $d_D \equiv 0$ on $\Delta_D \cong \mathbf{C}^n$. Consequently, Δ_D is just the degeneracy set for the pseudodistance d_D (cf. [12; p. 68]). In particular, $\mathrm{Aut}(D^*)$ has the structure of a real Lie group. Moreover, since d_D as well as F_D is invariant under the action of $\mathrm{Aut}(D)$, we have

$$\varphi(\Delta_D) = \Delta_D, \qquad \varphi(D^*) = D^* \qquad \text{for all } \varphi \in \mathrm{Aut}(D).$$

Thus the natural restriction mapping $\Phi : \mathrm{Aut}(D) \to \mathrm{Aut}(D^*)$ gives now an injective continuous homomorphism from $\mathrm{Aut}(D)$ into $\mathrm{Aut}(D^*)$. Here we assert that the image $\Phi(\mathrm{Aut}(D))$ is closed in $\mathrm{Aut}(D^*)$; consequently, $\mathrm{Aut}(D)$ has also the structure of a real Lie group. Although, in the proof below of this assertion, there is some overlap with the recent paper by Nagata [15], we carry out the proof in detail for the sake of completeness and self-containedness. So, take a sequence $\{\varphi_\nu\}$ in $\mathrm{Aut}(D)$ arbitrarily and assume that $\{\Phi(\varphi_\nu)\}$ converges to an element $\varphi \in \mathrm{Aut}(D^*)$. Since the Kobayashi distance d_{D^*} induces the Euclidean topology on D^* by Barth [3], this assumption is equivalent to the following:

$$\lim_{\nu\to\infty} d_{D^*}(\varphi_\nu(x), \varphi(x)) = 0 \qquad \text{uniformly on compact subsets of } D^*.$$

Thus, for any compact subset K of D^*, we have

$$\begin{aligned}\lim_{\nu\to\infty} d_{D^*}(\varphi_\nu^{-1}(x), \varphi^{-1}(x)) &= \lim_{\nu\to\infty} d_{D^*}(\varphi_\nu^{-1}(\varphi(y)), y) \\ &= \lim_{\nu\to\infty} d_{D^*}(\varphi(y), \varphi_\nu(y)) = 0 \qquad \text{uniformly on } K,\end{aligned}$$

since $\mathrm{Aut}(D^*)$ is an isometry group of D^* with respect to d_{D^*}, where we have put $y = \varphi^{-1}(x)$ for $x \in K$; accordingly, $\{\Phi(\varphi_\nu^{-1})\}$ converges to φ^{-1} in $\mathrm{Aut}(D^*)$. Here we claim that φ (resp. φ^{-1}) extends to a holomorphic mapping $\hat{\varphi}$ (resp. $\widehat{\varphi^{-1}}$) from D into $\bar{D} \subset \mathbf{C}^N$ such that the sequence $\{\varphi_\nu\}$ (resp. $\{\varphi_\nu^{-1}\}$) converges to $\hat{\varphi}$ (resp. $\widehat{\varphi^{-1}}$) uniformly on compact subsets of D. To prove our claim, it suffices to show that, for any point $p \in \Delta_D$, there exists an open neighborhood U_p of p such that φ (resp. φ^{-1}) extends to a holomorphic mapping $\hat{\varphi}$ (resp. $\widehat{\varphi^{-1}}$) from U_p into $\mathbf{C}^N$ such that $\{\varphi_\nu\}$ (resp. $\{\varphi_\nu^{-1}\}$) converges to $\hat{\varphi}$ (resp. $\widehat{\varphi^{-1}}$) uniformly on U_p. For this purpose, letting $p = (z_1^o, \dots, z_n^o, 0, \dots, 0)$, we consider the polydisc

$$\Delta(p; r) = \{(z, w); |z_i^o - z_i| < r, |w_j| < r \ (1 \le i \le n, 1 \le j \le m)\}$$

in $\mathbf{C}^n \times \mathbf{C}^m$. Then, for a sufficiently small $r > 0$, we have $p \in \Delta(p; r) \Subset D$ and

$$\varphi_\nu(z, w) = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{\varphi_\nu(z, w_1, \dots, w_{m-1}, \xi)}{\xi - w_m} d\xi \qquad \text{on } \Delta(p, r),\ \nu = 1, 2, \dots,$$

by the Cauchy integral formula. Define now a holomorphic mapping $\hat{\varphi} : \Delta(p; r) \to \mathbf{C}^N$ by setting

$$\hat{\varphi}(z, w) = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{\varphi(z, w_1, \dots, w_{m-1}, \xi)}{\xi - w_m} d\xi \qquad \text{on } \Delta(p, r).$$

Since $\{\varphi_\nu\}$ converges to φ uniformly on compact subsets of D^*, it then follows that $\{\varphi_\nu\}$ converges to $\hat{\varphi}$ uniformly on any connected open neighborhood U_p of p with $U_p \Subset \Delta(p; r)$ and $\hat{\varphi} = \varphi$ on $U_p \backslash \Delta_D$. Analogously we have the same conclusion for φ^{-1}, as claimed. Moreover, note that, if $h : D \to \mathbf{C}^N$ is a non-constant holomorphic mapping with $h(D) \subset \bar{D}$, then $h(D) \subset D$. Indeed, assume that $h(\zeta_o) =: p \in \partial D$ for some point $\zeta_o \in D$. Then $u(h(\zeta_o)) = 1$, $u(h(\zeta)) \le 1$ on D and hence $u(h(\zeta)) = 1$ for all $\zeta \in D$ by the maximum principle for the plurisubharmonic function $u \circ h$ defined on D, where u is the function appearing in (2.1). In view of the strict plurisubharmonicity of u

on D^*, this implies that $h(\zeta) = p$ on D, a contradiction. Therefore we conclude that

$$\hat{\varphi}(D) \subset D, \qquad \widehat{\varphi^{-1}}(D) \subset D \qquad \text{and} \qquad \hat{\varphi} \circ \widehat{\varphi^{-1}} = \mathrm{id}_D = \widehat{\varphi^{-1}} \circ \hat{\varphi} \qquad \text{on } D.$$

Thus $\hat{\varphi} \in \mathrm{Aut}(D)$ and $\Phi(\hat{\varphi}) = \varphi$; proving the closedness of $\Phi(\mathrm{Aut}(D))$ in $\mathrm{Aut}(D^*)$, as asserted.

Now, denoting by Π the subgroup of $\mathrm{Aut}(D)$ generated by all elements of $\{\varphi_v; v \in \mathbf{C}^n\}$, we assert that Π is a connected closed subgroup of $\mathrm{Aut}(D)$ of $\dim_{\mathbf{R}} \Pi = 2n+1$. For this, we introduce the one-parameter subgroup $\mathscr{R}$ of $\mathrm{Aut}(D)$ consisting of all transformations $R_\theta : (z,w) \mapsto (z, e^{i\theta}w)$, $\theta \in \mathbf{R}$. Then $\mathscr{R}$ is the center of the subgroup $U(m)$ of $\mathrm{Aut}(D)$ with $U(m) = \mathscr{R} \cdot SU(m)$ and, for any two elements $v, v' \in \mathbf{C}^n$, we have

$$\begin{aligned} \varphi_v \circ \varphi_{v'}(z,w) &= (z + v + v', e^{-\mu\langle z, v+v'\rangle - (\mu/2)\|v+v'\|^2} e^{(-\mu \operatorname{Im}\langle v', v\rangle)i} w) \\ &= \varphi_{v+v'} \circ R_\theta(z,w) \qquad \text{with } \theta = -\mu \operatorname{Im}\langle v', v\rangle. \end{aligned}$$

Thus $\varphi_0 = \mathrm{id}_D$ and $\varphi_v^{-1} = \varphi_{-v}$ for every $v \in \mathbf{C}^n$. In addition to this, note that $\varphi_v \circ R_\theta = R_\theta \circ \varphi_v$ for all $v \in \mathbf{C}^n$ and all $\theta \in \mathbf{R}$. Then it is not difficult to check that the set $\Pi' := \{\varphi_v \circ R_\theta; v \in \mathbf{C}^n, \theta \in \mathbf{R}\}$ becomes a connected closed subgroup of $\mathrm{Aut}(D)$ of $\dim_{\mathbf{R}} \Pi' = 2n+1$ and $\Pi \subset \Pi'$. Once it is shown that $\mathscr{R} \subset \Pi$, we conclude that $\Pi' \subset \Pi$ and hence $\Pi = \Pi'$ satisfies all the requirements in our assertion. Therefore we have only to show that $\mathscr{R} \subset \Pi$. To this end, take two elements φ_v, $\varphi_{v'}$ arbitrarily and compute their commutator $[\varphi_v, \varphi_{v'}] := \varphi_v^{-1} \circ \varphi_{v'}^{-1} \circ \varphi_v \circ \varphi_{v'}$. Then we have

$$[\varphi_v, \varphi_{v'}] = R_\theta \qquad \text{with } \theta = -2\mu \operatorname{Im}\langle v', v\rangle;$$

accordingly, for any $v_o \in \mathbf{C}^n$ with $\|v_o\| = 1$,

$$[\varphi_{tv_o}, \varphi_{tiv_o}] = R_{-2\mu t^2}, \qquad [\varphi_{tiv_o}, \varphi_{tv_o}] = R_{2\mu t^2} \qquad \text{for all } t \in \mathbf{R}.$$

Clearly this implies that $\mathscr{R} \subset \Pi$, as desired.

Next we consider the centralizer of $SU(m)$ in $\mathrm{Aut}(D)$ and denote it by $C_{\mathrm{Aut}(D)}(SU(m))$. Then it is obvious by Theorem A that $C_{\mathrm{Aut}(D)}(SU(m))$ is generated by all the elements of the set $\{\varphi_v; v \in \mathbf{C}^n\} \cup U(n) \cup \mathscr{R}$; so that $\mathrm{Aut}(D) = C_{\mathrm{Aut}(D)}(SU(m)) \cdot SU(m)$ and Π is a subgroup of $C_{\mathrm{Aut}(D)}(SU(m))$. More precisely, since $\varphi_A \circ \varphi_v \circ \varphi_A^{-1} = \varphi_{Av}$ for any $A \in U(n)$ and $v \in \mathbf{C}^n$, Π is a normal subgroup of $C_{\mathrm{Aut}(D)}(SU(m))$ and, in fact, $C_{\mathrm{Aut}(D)}(SU(m)) = \Pi \cdot U(n)$ with $\Pi \cap U(n) = \{\mathrm{id}_D\}$. Notice that $C_{\mathrm{Aut}(D)}(SU(m)) \cap SU(m) = \mathscr{R} \cap SU(m)$ is a finite subgroup of $\mathrm{Aut}(D)$ of order m. Hence $\mathrm{Aut}(D) = \Pi \cdot U(n) \cdot SU(m)$ and $\dim_{\mathbf{R}} \mathrm{Aut}(D) = (2n+1) + n^2 + (m^2 - 1) = n^2 + m^2 + 2n$. As a result, we have obtained the following:

FACT 4. $\mathrm{Aut}(D)$ *is a connected Lie group of* $\dim_{\mathbb{R}} \mathrm{Aut}(D) = n^2 + m^2 + 2n$.

In this case, it is well-known that the Lie algebra $\mathfrak{g}$ of $\mathrm{Aut}(D)$ can be canonically identified with some Lie subalgebra $\mathfrak{g}^*$ of $\mathfrak{X}(D)$, the Lie algebra consisting of all differentiable vector fields on D (cf. [14; pp. 236–237]). More precisely, we here assert that $\mathfrak{g}$ can be identified with $\mathfrak{g}(D)$, that is, the set $\mathfrak{g}(D)$ of all complete holomorphic vector fields on D becomes a Lie subalgebra of $\mathfrak{X}(D)$ and $\mathfrak{g}^*$ coincides with $\mathfrak{g}(D)$. Indeed, the Lie group $\mathrm{Aut}(D)$ endowed with the compact-open topology acts continuously on D. Hence, the action is real analytic by [6]. Moreover, we know that $\mathrm{Aut}(D)$ satisfies the second axiom of countability. Then, by Theorem VI in [17; p. 101], the group $\mathrm{Aut}(D)$ is a Lie transformation group of D in the sense of Definition V in [17; p. 101]; consequently, the Lie algebra $\mathfrak{g}$ can be identified with $\mathfrak{g}(D)$ (cf. [17; p. 103, Theorem VII]), as asserted. Anyway, this fact will be used in Section 4.

Next, let D be the Fock-Bargmann-Hartogs domain in $\mathbb{C}^N$ and let K_D be the Bergman kernel function for D. Then, by making use of an explicit formula for K_D in terms of the polylogarithm function by Yamamori [27], Tu-Wang [23; Theorem 2.3] verified that $K_D(\zeta, \eta)$ extends holomorphically in ζ to some open neighborhood of the closure $\overline{D}$ of D. Thanks to this extension theorem together with Bell's transformation rule for Bergman kernels under proper holomorphic mappings, they obtained the following:

THEOREM B (Tu-Wang [23; Theorem 2.5]). *Let D_1, D_2 be two equidimensional Fock-Bargmann-Hartogs domains in $\mathbb{C}^N$ and $f : D_1 \to D_2$ a proper holomorphic mapping. Then f extends holomorphically to an open neighborhood W of $\overline{D_1}$.*

We finish this section by the following fact which is an immediate consequence of the invariance of degeneracy sets for Kobayashi pseudodistances under biholomorphic mappings (cf. [23; Theorem 1.2]):

FACT 5. *Let $D_1 = D_{n_1,m_1}(\mu_1)$ and $D_2 = D_{n_2,m_2}(\mu_2)$ be two Fock-Bargmann-Hartogs domains in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively, where $N_j = n_j + m_j$ for $j = 1, 2$. Then D_1 is biholomorphically equivalent to D_2 if and only if D_1 is linearly equivalent to D_2, that is, there exists a non-singular linear mapping $L : \mathbb{C}^{N_1} \to \mathbb{C}^{N_2}$ such that $L(D_1) = D_2$. Moreover, this can only happen when $(n_1, m_1) = (n_2, m_2)$; and every biholomorphic mapping $f : D_1 \to D_2$ can be written in the form*

$$f(z, w) = \varphi(\sqrt{\mu_1/\mu_2}z, w), \qquad (z, w) \in D_1, \qquad \textit{with some } \varphi \in \mathrm{Aut}(D_2).$$

In fact, it is clear that D_1 is biholomorphically equivalent to D_2, if D_1 is linearly equivalent to D_2. Conversely, assume that there exists a biholomorphic mapping $g : D_1 \to D_2$. Then we have that $N_1 = N_2$ and $d_{D_2}(g(p), g(q)) =$

$d_{D_1}(p,q)$ for any points $p,q \in D_1$. On the other hand, we know that Δ_{D_1}, Δ_{D_2} are the degeneracy sets for d_{D_1}, d_{D_2}, respectively. Thus it follows at once that $g(\Delta_{D_1}) = \Delta_{D_2}$ and g induces a biholomorphic mapping from $\Delta_{D_1} \cong \mathbf{C}^{n_1}$ onto $\Delta_{D_2} \cong \mathbf{C}^{n_2}$. Consequently, we have $n_1 = n_2$ and so $m_1 = m_2$. In the case where $(n_1, m_1) = (n_2, m_2)$, it is easy to see that the non-singular linear mapping $L : \mathbf{C}^{n_1} \times \mathbf{C}^{m_1} \to \mathbf{C}^{n_2} \times \mathbf{C}^{m_2}$ defined by $L(z,w) = (\sqrt{\mu_1/\mu_2}z, w)$ for $(z,w) \in \mathbf{C}^{n_1} \times \mathbf{C}^{m_1}$ gives a linear equivalence between D_1 and D_2. In particular, for every biholomorphic mapping $f : D_1 \to D_2$, we obtain that $\varphi := f \circ L^{-1} \in \mathrm{Aut}(D_2)$ and hence $f = \varphi \circ L$ on D_1; proving our assertion.

3. Proof of Theorem 1

Our proof of Theorem 1 will be carried out along the same line as in the previous paper [13]. Before undertaking the proof, we need to introduce one terminology. Let D be a domain in $\mathbf{C}^N$ and let $p \in \partial D$. Then the boundary point p is said to be *spherical* if the following condition (‡) is fulfilled:

(‡) There are an open neighborhood U of p in $\mathbf{C}^N$ and a biholomorphic mapping f from U into $\mathbf{C}^N$ such that $f(U \cap D) = f(U) \cap B^N$ and $f(U \cap \partial D) = f(U) \cap \partial B^N$.

The following lemma will play a crucial role in our proof of Theorem 1.

Lemma 1. *Let $D = D_{n,m}(\mu)$ be the Fock-Bargmann-Hartogs domain in $\mathbf{C}^N$. Assume that $m \geq 2$. Then there is not a spherical boundary point of D.*

Proof. To derive a contradiction, assume to the contrary that there exists a spherical boundary point p of D, so that the condition (‡) is fulfilled for some connected open neighborhood U of p and a biholomorphic mapping $f : U \to f(U) \subset \mathbf{C}^N$. Since ∂D is a connected strictly pseudoconvex real analytic hypersurface in $\mathbf{C}^N$, it follows from a result of Pinchuk [19; Proposition 1.2], [20; p. 193] that f can be continued along any path lying in ∂D as a locally biholomorphic mapping. Since ∂D is now simply connected by our assumption $m \geq 2$, the monodromy theorem guarantees that f extends to a locally biholomorphic mapping F defined on some connected open neighborhood V of ∂D in $\mathbf{C}^N$ such that $F(\partial D) \subset \partial B^N$ and $F(V \cap D) \subset B^N$. Now we will proceed in steps.

(1) *F extends to a holomorphic mapping $\tilde{F}$ from D into B^N.* To prove this, take an arbitrary $r \in \mathbf{R}$ with $r > 1$ and put

$$K_r = \left\{(z,w) \in \mathbf{C}^n \times \mathbf{C}^m;\ \|z\| \leq r,\ \|w\| = e^{-(\mu/2)\|z\|^2}\right\}.$$

Since $K_r \subset \partial D \subset V$ and K_r is compact in V, one can choose a small $\varepsilon = \varepsilon(r) > 0$ in such a way that

$$U_{r,\varepsilon} := \{(z,w) \in \mathbf{C}^n \times \mathbf{C}^m; \|z\| < r,$$
$$e^{-(\mu/2)\|z\|^2} - \varepsilon < \|w\| < e^{-(\mu/2)\|z\|^2} + \varepsilon\} \subset V.$$

Clearly, $U_{r,\varepsilon}$ is a bounded Reinhardt domain in $\mathbf{C}^N$. Moreover, since $m \geq 2$, we have that

$$U_{r,\varepsilon} \cap \{\zeta \in \mathbf{C}^N; \zeta_k = 0\} \neq \varnothing \qquad \text{for } k = 1, \dots, N,$$

where we have set $(z,w) = (\zeta_1, \dots, \zeta_N) = \zeta$. Hence, by a well-known fact [16; p. 15], every component function F_k of F has a holomorphic extension F_k^r to the domain

$$\hat{U}_{r,\varepsilon} := \{(z,w) \in \mathbf{C}^n \times \mathbf{C}^m; \ \|z\| < r, \ \|w\| < e^{-(\mu/2)\|z\|^2} + \varepsilon\},$$

the smallest complete Reinhardt domain in $\mathbf{C}^N$ containing $U_{r,\varepsilon}$. In particular, putting

$$D_r = \{(z,w) \in \mathbf{C}^n \times \mathbf{C}^m; \ \|z\| < r, \ \|w\| < e^{-(\mu/2)\|z\|^2}\},$$

we see that $F = (F_1, \dots, F_N)$ has a holomorphic extension $F^r := (F_1^r, \dots, F_N^r)$ to $D_r \cup V$. Note that $D_r \subset D_s$ for $1 < r < s$, $\bigcup_{1<r<\infty} D_r = D$ and that the holomorphic extensions F^r are uniquely determined by the values of F on a small neighborhood of an arbitrarily given point $(0, w_o) \in \partial D$. Then, by standard argument, one can define a holomorphic extension $\tilde{F} : D \cup V \to \mathbf{C}^N$ of $F : V \to \mathbf{C}^N$.

Now we wish to show that $\tilde{F}(D) \subset B^N$. To this end, let us fix an arbitrary point $(z_o, w_o) \in D$ and define an open ball

$$D(z_o) = \{w \in \mathbf{C}^m; \|w\|^2 < e^{-\mu\|z_o\|^2}\} \qquad \text{in } \mathbf{C}^m.$$

Clearly $w_o \in D(z_o)$. Consider here the non-constant, real analytic plurisubharmonic function $\psi : w \mapsto -1 + \|\tilde{F}(z_o, w)\|^2$ defined on some open neighborhood of the closure $\overline{D(z_o)}$ in $\mathbf{C}^m$. Then $\psi(\partial D(z_o)) = 0$ and $\psi(w) < 0$ on $D(z_o) \cap V$ (regarding $D(z_o)$ as a subset of D in the canonical manner). This, combined with the maximum principle for plurisubharmonic functions, guarantees that $\psi(w_o) < 0$, i.e., $\tilde{F}(z_o, w_o) \in B^N$ and accordingly $\tilde{F}(D) \subset B^N$, as desired.

(2) *There is a locally injective, real analytic homomorphism* $\Phi : \mathrm{Aut}(D) \to \mathrm{Aut}(B^N)$ *such that* $\Phi(\varphi) \circ \tilde{F} = \tilde{F} \circ \varphi$ *on* D *for all* $\varphi \in \mathrm{Aut}(D)$. Indeed, fix a point $p \in \partial D$ and take an arbitrary element $\varphi \in \mathrm{Aut}(D)$. Then one can choose a connected, small open neighborhood W of p in such a way that $W \cup \varphi(W) \subset V$ and $\tilde{F}$ is injective on W and on $\varphi(W)$. Let us consider the biholomorphic mapping

$$\hat{\varphi} := \tilde{F} \circ \varphi \circ (\tilde{F}|_W)^{-1} : \tilde{F}(W) \to \tilde{F}(\varphi(W)).$$

Clearly $\hat{\varphi}$ satisfies the following:

$$\hat{\varphi}(\tilde{F}(W) \cap B^N) \subset B^N \qquad \text{and} \qquad \hat{\varphi}(\tilde{F}(W) \cap \partial B^N) \subset \partial B^N.$$

Hence, by the main result of Alexander [1], $\hat{\varphi}$ extends to a holomorphic automorphism, say again $\hat{\varphi}$, of B^N. Note that $W \cap D$ and $\tilde{F}(W \cap D)$ are non-empty open subsets of D and B^N, respectively. Then, by the principle of analytic continuation, we have that $\hat{\varphi} \circ \tilde{F} = \tilde{F} \circ \varphi$ on D and $\hat{\varphi} \in \mathrm{Aut}(B^N)$ is uniquely determined by φ. Accordingly, one can define a mapping

$$\Phi : \mathrm{Aut}(D) \to \mathrm{Aut}(B^N) \qquad \text{by setting } \Phi(\varphi) = \hat{\varphi},$$

so that $\Phi(\varphi) \circ \tilde{F} = \tilde{F} \circ \varphi$ on D for all $\varphi \in \mathrm{Aut}(D)$.

It is easy to check that Φ is a group homomorphism. Once it is shown that Φ is continuous at the identity element id_D of $\mathrm{Aut}(D)$, it follows that Φ is real analytic on the Lie group $\mathrm{Aut}(D)$ (cf. [9; p. 117]). Since the topology of $\mathrm{Aut}(D)$ satisfies the second axiom of countability, we have only to show that Φ is sequentially continuous at id_D. For this, let us take an arbitrary sequence $\{\varphi_\nu\}$ in $\mathrm{Aut}(D)$ which converges to id_D and assume that $\{\Phi(\varphi_\nu)\}$ does not converge to the identity element id_{B^N} of $\mathrm{Aut}(B^N)$. Passing to a subsequence, if necessary, we may assume that there is an open neighborhood O of id_{B^N} in $\mathrm{Aut}(B^N)$ such that $\Phi(\varphi_\nu) \notin O$ for all ν. Pick an arbitrary point $\zeta \in D$. Then

$$\lim_{\nu \to \infty} \Phi(\varphi_\nu)(\tilde{F}(\zeta)) = \lim_{\nu \to \infty} \tilde{F}(\varphi_\nu(\zeta)) = \tilde{F}(\zeta) \in B^N,$$

which implies that $\{\Phi(\varphi_\nu)(\tilde{F}(\zeta))\}$ lies in a compact subset of B^N. Hence, after taking a subsequence if necessary, we may assume that $\{\Phi(\varphi_\nu)\}$ converges to some element $g \in \mathrm{Aut}(B^N)$ (cf. [16; p. 82]). Since $g \notin O$, we see that $g \neq \mathrm{id}_{B^N}$. On the other hand, we have

$$g(\tilde{F}(\zeta)) = \lim_{\nu \to \infty} \Phi(\varphi_\nu)(\tilde{F}(\zeta)) = \lim_{\nu \to \infty} \tilde{F}(\varphi_\nu(\zeta)) = \tilde{F}(\zeta) \qquad \text{for all } \zeta \in W \cap D;$$

consequently, $g = \mathrm{id}_{B^N}$ by analytic continuation. This is a contradiction. Therefore Φ is continuous at id_D, as desired.

Next we claim that Φ is locally injective. It is sufficient to prove that Φ is injective on some open neighborhood O of id_D. To this end, choose two open sets W_1, W_2 in $\mathbf{C}^N$ with $\emptyset \neq W_1 \Subset W_2 \subset W \cap D$. We claim that

$$O := \{\varphi \in \mathrm{Aut}(D); \varphi(\overline{W_1}) \subset W_2\}$$

is what is required. Indeed, it is clear that O is an open neighborhood of id_D in $\mathrm{Aut}(D)$. Moreover, assume that $\Phi(\varphi_1) = \Phi(\varphi_2)$ for $\varphi_1, \varphi_2 \in O$. It then follows that

$$\tilde{F}(\varphi_1(\zeta)) = \Phi(\varphi_1)(\tilde{F}(\zeta)) = \Phi(\varphi_2)(\tilde{F}(\zeta)) = \tilde{F}(\varphi_2(\zeta)) \qquad \text{for all } \zeta \in D.$$

Since $\tilde{F}$ is injective on $W_2 \subset W$ and since $\varphi_1(\zeta), \varphi_2(\zeta) \in W_2$ for all $\zeta \in W_1$, this says that $\varphi_1 = \varphi_2$ on W_1; and hence, $\varphi_1 = \varphi_2$ on D by analytic continuation. Therefore Φ is locally injective on $\mathrm{Aut}(D)$.

(3) *$\tilde{F} : D \to B^N$ is a locally biholomorphic mapping from D into B^N. However this is absurd.* We first prove that the set $V_{\tilde{F}} = \{\zeta \in D; J_{\tilde{F}}(\zeta) = 0\}$ is empty. To derive a contradiction, assume to the contrary that $V_{\tilde{F}} \neq \emptyset$. Then $V_{\tilde{F}}$ is a complex analytic subset of D of $\dim_{\mathbf{C}} V_{\tilde{F}} = N - 1 = n + m - 1$. If $V_{\tilde{F}} \subset \Delta_D \cong \mathbf{C}^n$, then we obtain a contradiction, since $\dim_{\mathbf{C}} V_{\tilde{F}} > n = \dim_{\mathbf{C}} \Delta_D$ by our assumption $m \geq 2$. Hence $V_{\tilde{F}} \not\subseteq \Delta_D$ and there exists a point $\zeta_o = (z_o, w_o) \in V_{\tilde{F}}$ with $w_o \neq 0$. Let $\mathrm{Aut}(D) \cdot \zeta_o$ be the $\mathrm{Aut}(D)$-orbit passing through the point ζ_o. This is a real analytic submanifold of D. Here we assert that $\mathrm{Aut}(D) \cdot \zeta_o$ is contained in $V_{\tilde{F}}$. To this end, take an arbitrary element $\varphi \in \mathrm{Aut}(D)$. Then, since

$$J_{\tilde{F}}(\varphi(\zeta_o)) \cdot J_\varphi(\zeta_o) = J_{\Phi(\varphi)}(\tilde{F}(\zeta_o)) \cdot J_{\tilde{F}}(\zeta_o) \qquad \text{and} \qquad J_{\tilde{F}}(\zeta_o) = 0,\ J_\varphi(\zeta_o) \neq 0,$$

we have that $J_{\tilde{F}}(\varphi(\zeta_o)) = 0$ or equivalently $\varphi(\zeta_o) \in V_{\tilde{F}}$; hence, $\mathrm{Aut}(D) \cdot \zeta_o \subset V_{\tilde{F}}$, as asserted. Therefore we have $\dim_{\mathbf{R}}(\mathrm{Aut}(D) \cdot \zeta_o) \leq 2(N-1)$. On the other hand, by using the explicit description of the generators of $\mathrm{Aut}(D)$ given in Theorem A, it is easily checked that $\dim_{\mathbf{R}}(\mathrm{Aut}(D) \cdot \zeta_o) = 2n + 2m - 1 > 2(N-1)$. This is a contradiction. Thus we conclude that $V_{\tilde{F}} = \emptyset$ and $\tilde{F} : D \to B^N$ is, in fact, a locally biholomorphic mapping. However, this is absurd. Indeed, consider the holomorphic mapping $h : \mathbf{C}^n \to B^N$ given by $h(z) = \tilde{F}(z, 0)$ for $(z, 0) \in \Delta_D \cong \mathbf{C}^n$. Then it follows at once from the classical Liouville theorem on bounded entire functions that h is a constant mapping on $\mathbf{C}^n$. Consequently, for any point $\zeta_o \in \Delta_D$, $\tilde{F}$ is never injective on any open neighborhood of ζ_o, a contradiction.

Therefore we have proved that there is not a spherical boundary point of D; completing the proof of Lemma 1. □

Remark. Let D be a strictly pseudoconvex domain in $\mathbf{C}^N$ with simply connected and real analytic boundary ∂D. Assume that D is bounded in $\mathbf{C}^N$ and there exists a spherical boundary point $p \in \partial D$, so that there are open neighborhood U of p and a biholomorphic mapping $f : U \to \mathbf{C}^N$ satisfying the condition (‡). Then D is biholomorphically equivalent to B^N by a result of Pinchuk [18; Theorem 2]. Here the assumption that D is a bounded domain in $\mathbf{C}^N$ cannot be avoided. Indeed, in the proof of this assertion, he first proved that $f : U \to \mathbf{C}^N$ extends to a locally biholomorphic mapping $F : V \to \mathbf{C}^N$ from some open neighborhood V of ∂D into $\mathbf{C}^N$. After that, he used the Osgood-Brown theorem (Hartogs extension theorem) to obtain a holomorphic mapping $\tilde{F} : \overline{D} \to \overline{B^N} \subset \mathbf{C}^N$ that is an extension of $F : V \to \mathbf{C}^N$ (see [18; p. 390], also [19; p. 518]). Thus D has to be a bounded domain enclosed by the connected compact hypersurface ∂D imbedded in $\mathbf{C}^N$. On the other hand, the Fock-Bargmann-Hartogs domain $D = D_{n,m}(\mu)$ is not bounded and ∂D is not compact in $\mathbf{C}^N$. Therefore, our Lemma 1 is not an immediate consequence of Pinchuk [18; Theorem 2].

We can now prove our theorem as follows. First we claim that, for each $i = 1, 2$, the strictly pseudoconvex real analytic hypersurface ∂D_i has no umbilical points in the sense of CR-geometry; hence, Webster's CR-invariant Riemannian metric g_i can be defined on the whole space ∂D_i. (For the notion of umbilical points and Webster's CR-invariant metrics in CR-geometry, see [25, 26] and also [7], [24].) To prove our claim, assume that there exists an umbilical point on ∂D_i. Then, all the points of ∂D_i are umbilical, since $\mathrm{Aut}(D_i)$ acts transitively on ∂D_i by Fact 3. Hence, ∂D_i must be locally biholomorphically equivalent to the sphere ∂B^N (see, for example, [7; p. 153], [24; p. 213]). However this is impossible by Lemma 1; proving our claim. Moreover, we see that $(\partial D_i, g_i)$ is complete as a Riemannian manifold, because ∂D_i is homogeneous under the CR-automorphism group $\mathrm{Aut}(D_i)$. As a result, each $(\partial D_i, g_i)$ is a connected and simply connected, complete real analytic Riemannian manifold. On the other hand, $f : U_1 \cap \partial D_1 \to U_2 \cap \partial D_2$ is a local isometry with respect to the CR-invariant metrics g_1 and g_2. Hence, by a well-known fact in Riemannian geometry [11; p. 256], f can be uniquely extended to a global isometry $F : (\partial D_1, g_1) \to (\partial D_2, g_2)$. From the fact that F is induced by the biholomorphic mapping $f : U_1 \to U_2$ and from the construction of Webster's CR-invariant metric, it follows at once that $F : \partial D_1 \to \partial D_2$ is a real analytic CR-diffeomorphism. Accordingly, as an immediate consequence of Bell [5; Theorem 2], one can find open neighborhoods V_1 of ∂D_1 and V_2 of ∂D_2 in $\mathbf{C}^N$ such that $F : \partial D_1 \to \partial D_2$ and its inverse $G := F^{-1} : \partial D_2 \to \partial D_1$ extend to locally biholomorphic mappings written in the same notation $F : V_1 \to \mathbf{C}^N$ and $G : V_2 \to \mathbf{C}^N$ satisfying $F(V_1 \cap D_1) \subset D_2$ and $G(V_2 \cap D_2) \subset D_1$. Hence, in exactly the same way as in (1) of the proof of Lemma 1, it can be shown that F and G extend to holomorphic mappings $\tilde{F} : D_1 \to \mathbf{C}^N$ and $\tilde{G} : D_2 \to \mathbf{C}^N$. Moreover, replacing $\psi(w)$ by $\psi_1(w) := \rho_2(\tilde{F}(z_o, w))$ in (1) of the proof of Lemma 1, we can prove that $\tilde{F}(D_1) \subset D_2$, where ρ_2 is the real analytic plurisubharmonic function on $\mathbf{C}^N$ defined by

$$\rho_2(z, w) = -1 + \|w\|^2 e^{\mu_2 \|z\|^2} \qquad \text{for } (z, w) \in \mathbf{C}^{n_2} \times \mathbf{C}^{m_2} = \mathbf{C}^N.$$

Analogously, we see that $\tilde{G}(D_2) \subset D_1$. Since $\tilde{G} \circ \tilde{F} = \mathrm{id}_{D_1}$ near ∂D_1 and $\tilde{F} \circ \tilde{G} = \mathrm{id}_{D_2}$ near ∂D_2, we conclude by analytic continuation that $\tilde{G} \circ \tilde{F} = \mathrm{id}_{D_1}$ and $\tilde{F} \circ \tilde{G} = \mathrm{id}_{D_2}$; consequently, $\tilde{F} : D_1 \to D_2$ is a biholomorphic mapping. Therefore the proof of Theorem 1 is completed.

4. Proof of Theorem 2

By Theorem B there exists an open neighborhood W of $\overline{D_1}$ such that f extends to a holomorphic mapping, say again, $f : W \to \mathbf{C}^N$. Since each ∂D_i for $i = 1, 2$ is strictly pseudoconvex real analytic hypersurface in $\mathbf{C}^N$, it follows from the same method as in the proof of [4; Theorem 2] or [18; Lemma 1.3] that $J_f(\zeta) \neq 0$ for every point $\zeta \in \partial D_1$. Thus, for an arbitrarily given point $p_1 \in \partial D_1$, there exists an open neighborhood U_1 of p_1 in $\mathbf{C}^N$ such that f gives rise to a biholomorphic mapping $F : U_1 \to U_2 := f(U_1) \subset \mathbf{C}^N$ with

$$F(U_1 \cap D_1) = U_2 \cap D_2 \qquad \text{and} \qquad F(U_1 \cap \partial D_1) = U_2 \cap \partial D_2. \tag{4.1}$$

Consequently, if $m_2 \geq 2$, then F extends to a biholomorphic mapping $\hat{F} : D_1 \to D_2$ by Theorem 1; and moreover, in such a case, it is clear that $f = \hat{F}$ on D_1. Hence the proof of Theorem 2 is now reduced to showing the following:

Lemma 2. *Under the same situation as in Theorem* 2, *we have* $m_2 \geq 2$.

Proof. Once it is shown that

$$n_1^2 + m_1^2 + 2n_1 = \dim \mathrm{Aut}(D_1) \geq \dim \mathrm{Aut}(D_2) = n_2^2 + m_2^2 + 2n_2,$$

then we conclude that $m_2 \geq 2$, since $n_1 + m_1 = n_2 + m_2$ and $m_1 \geq 2$ by our assumption. Thus it suffices to show that there exists an injective linear mapping $L : \mathfrak{g}(D_2) \to \mathfrak{g}(D_1)$ from the Lie algebra $\mathfrak{g}(D_2)$ of $\mathrm{Aut}(D_2)$ into the Lie algebra $\mathfrak{g}(D_1)$ of $\mathrm{Aut}(D_1)$. To this end, we shall construct a mapping $\Phi : O_2 \to \mathrm{Aut}(D_1)$ from some open neighborhood O_2 of the identity element id_{D_2} of $\mathrm{Aut}(D_2)$ into $\mathrm{Aut}(D_1)$ that induces such a mapping $L : \mathfrak{g}(D_2) \to \mathfrak{g}(D_1)$. We will carry out this by two steps as follows:

(1) *A construction of a mapping* $\Phi : O_2 \to \mathrm{Aut}(D_1)$: We fix two connected open neighborhoods W_2, V_2 of $p_2 := F(p_1)$ in $\mathbf{C}^N$ with $W_2 \Subset V_2 \Subset U_2$ and put $W_1 = F^{-1}(W_2)$, $V_1 = F^{-1}(V_2) \subset U_1$ respectively, where $F : U_1 \to U_2$ is the biholomorphic mapping appearing in (4.1). Then W_1, V_1 are open neighborhoods of p_1 with $W_1 \Subset V_1 \Subset U_1$. Here, recalling that $\mathrm{Aut}(D_2)$ can be regarded as a topological subgroup of $\mathrm{Aut}(\mathbf{C}^N)$ by Fact 2, we define a subset O_2 of $\mathrm{Aut}(D_2)$ by setting

$$O_2 = \{\varphi \in \mathrm{Aut}(D_2);\ \varphi(\overline{W_2}) \subset V_2,\ \varphi(\overline{V_2}) \subset U_2\}.$$

Then O_2 is an open neighborhood of $\mathrm{id}_{D_2} \in \mathrm{Aut}(D_2)$ and, for any element $\varphi \in O_2$, we obtain a biholomorphic mapping

$$\hat{\varphi} := F^{-1} \circ \varphi \circ F : V_1 \to \hat{V}_1 := F^{-1}(\varphi(V_2)) \subset U_1 \tag{4.2}$$

with $\hat{\varphi}(V_1 \cap D_1) = \hat{V}_1 \cap D_1$ and $\hat{\varphi}(V_1 \cap \partial D_1) = \hat{V}_1 \cap \partial D_1$. Recall that $m_1 \geq 2$. Then, as an immediate consequence of Theorem 1, $\hat{\varphi}$ extends to a holomorphic automorphism written in the same notation $\hat{\varphi} : D_1 \to D_1$. Thus

$$\varphi(f(\zeta)) = f(\hat{\varphi}(\zeta)) \qquad \text{for all } \zeta \in D_1$$

by analytic continuation; and moreover, it is obvious that this $\hat{\varphi} \in \mathrm{Aut}(D_1)$ is uniquely determined by φ. Accordingly, one can define a mapping

$$\Phi : O_2 \to \mathrm{Aut}(D_1) \qquad \text{by setting } \Phi(\varphi) = \hat{\varphi}, \tag{4.3}$$

so that $\varphi \circ f = f \circ \Phi(\varphi)$ on D_1 for all $\varphi \in O_2$.

(2) *There exists an injective linear mapping* $L : \mathfrak{g}(D_2) \to \mathfrak{g}(D_1)$: We would like to induce such a mapping L from the mapping Φ in (4.3). For this, let us take an arbitrary element $X \in \mathfrak{g}(D_2)$ and consider the one-parameter subgroup $\{\varphi_t = \exp tX\}_{t \in \mathbb{R}}$ of $\mathrm{Aut}(D_2)$ generated by X. Then one can choose a constant $\epsilon_o > 0$ such that $\varphi_t \in O_2$ for all $t \in \mathbb{R}$ with $|t| < \epsilon_o$; and moreover, it is easy to check that

$$\Phi(\varphi_s)(\Phi(\varphi_t)(\zeta)) = \Phi(\varphi_{s+t})(\zeta), \qquad \zeta \in W_1 \cap D_1, \text{ whenever } |s|, |t|, |s+t| < \epsilon_o;$$

consequently, $\Phi(\varphi_s) \circ \Phi(\varphi_t) = \Phi(\varphi_{s+t})$ on D_1 by analytic continuation. Thus $\{\Phi(\varphi_t)\}_{|t|<\epsilon_o}$ is a local one-parameter group of local holomorphic transformations of D_1. Let $\hat{X}$ be the holomorphic vector field on D_1 induced by this local one-parameter group $\{\Phi(\varphi_t)\}_{|t|<\epsilon_o}$. Then $\hat{X}$ is also a complete holomorphic vector field on D_1, that is, $\hat{X} \in \mathfrak{g}(D_1)$ (cf. [14; p. 83]) and $\{\Phi(\varphi_t)\}_{|t|<\epsilon_o}$ is the restriction of the global one-parameter subgroup $\{\hat{\varphi}_t = \exp t\hat{X}\}_{t \in \mathbb{R}}$ of $\mathrm{Aut}(D_1)$ to $|t| < \epsilon_o$. Clearly this $\hat{X}$ is uniquely determined by the given X; accordingly, one can define a mapping

$$L : \mathfrak{g}(D_2) \to \mathfrak{g}(D_1) \qquad \text{by setting } L(X) = \hat{X}$$

for every $X \in \mathfrak{g}(D_2)$. Since $F : U_1 \to U_2$ is a biholomorphic mapping, the differential $(dF^{-1})_{F(\zeta)}$ of F^{-1} at $F(\zeta)$ is a linear isomorphism for every point $\zeta \in U_1$. Moreover, it follows from (4.2) that

$$\hat{X}_\zeta = (dF^{-1})_{F(\zeta)}(X_{F(\zeta)}) \qquad \text{for all } \zeta \in V_1 \cap D_1,\ X \in \mathfrak{g}(D_2).$$

Thus, by analytic continuation, we conclude that $L : \mathfrak{g}(D_2) \to \mathfrak{g}(D_1)$ is, in fact, an injective linear mapping, as desired.

More precisely, we assert that $\Phi : O_2 \to \mathrm{Aut}(D_1)$ is a real analytic imbedding of O_2 into $\mathrm{Aut}(D_1)$ and so $\dim \mathrm{Aut}(D_2) \leq \dim \mathrm{Aut}(D_1)$. Indeed, let $\{X_1, \dots, X_{d_2}\}$ be a basis of $\mathfrak{g}(D_2)$, where $d_2 = \dim \mathrm{Aut}(D_2)$. Then, for each $j = 1, \dots, d_2$, there is a small constant $\epsilon_j > 0$ such that $\exp tX_j \in O_2$ for all $t \in \mathbb{R}$ with $|t| < \epsilon_j$; consequently we have

$$\Phi(\exp t_j X_j) = \exp t_j L(X_j) \qquad \text{for all } t_j \in \mathbb{R} \text{ with } |t_j| < \epsilon_j.$$

On the other hand, by just the definition of Φ, one can choose a constant $\delta_o > 0$ so small that

$$\Phi(\exp t_1 X_1 \cdots \exp t_{d_2} X_{d_2}) = \exp t_1 L(X_1) \cdots \exp t_{d_2} L(X_{d_2})$$

for all $t_j \in \mathbb{R}$ with $|t_j| < \delta_o$ $(1 \leq j \leq d_2)$. Hence, taking a basis $\{\hat{X}_1, \ldots, \hat{X}_{d_1}\}$ of $\mathfrak{g}(D_1)$ in such a way that $\hat{X}_j = L(X_j)$ for $1 \leq j \leq d_2$, we obtain the following: With respect to the canonical coordinate systems of the second kind

$$\psi_1 : \exp x_1 \hat{X}_1 \cdots \exp x_{d_1} \hat{X}_{d_1} \mapsto (x_1, \ldots, x_{d_1}),$$

$$\psi_2 : \exp y_1 X_1 \cdots \exp y_{d_2} X_{d_2} \mapsto (y_1, \ldots, y_{d_2})$$

defined on some open neighborhoods of $\mathrm{id}_{D_1} \in \mathrm{Aut}(D_1)$, $\mathrm{id}_{D_2} \in \mathrm{Aut}(D_2)$ respectively, Φ has the expression

$$\psi_1 \circ \Phi \circ \psi_2^{-1} : (t_1, \ldots, t_{d_2}) \mapsto (t_1, \ldots, t_{d_2}, 0, \ldots, 0) \qquad \text{on } \psi_2(O_2)$$

(after shrinking O_2 sufficiently small, if necessary). Clearly this means that $\Phi : O_2 \to \mathrm{Aut}(D_1)$ is a real analytic imbedding of O_2 into $\mathrm{Aut}(D_1)$, as asserted. $\square$

Therefore the proof of Theorem 2 is completed.

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