

A mixed formulation of the Stokes equations with slip conditions in exterior domains and in the half-space

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ABSTRACT. We are concerned with Stokes equations in the half-space or in an exterior domain of \mathbb{R}^n when slip conditions are imposed on the boundary. We present a mixed velocity-pressure formulation and we show its well posedness. A weighted variant of Korn's inequality in unbounded domains is the cornerstone of our approach.

1. Introduction

We are interested in the Stokes system

$$\begin{cases} -\nu \Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = \rho & \text{in } \Omega, \end{cases} \quad (1)$$

where Ω is an unbounded connected open subset of \mathbb{R}^n , typically an exterior domain or a half-space, f is a body force and ρ a given function.

The Stokes system (1) is often considered with no-slip conditions which could be seen as a Dirichlet boundary condition (see, e.g., [2], [8], [11]). Nonetheless, situations can arise where slip conditions are imposed on the boundary (see, e.g., [20], [7], [19], [13], [22], [21], [24, 14] and references therein). Other kinds of boundary conditions could also be considered as in [4].

In this work, slip conditions without friction are expressed into the form

$$u \cdot n = g \quad \text{on } \partial\Omega, \quad (2)$$

$$(\sigma(u, p) \cdot n)_\tau = h_\tau \quad \text{on } \partial\Omega, \quad (3)$$

where n is the unit outward normal to boundary, g and h_τ are, respectively, a function and a tangential vector field given on $\partial\Omega$. Here (2) is a non penetration condition. In (3), $\sigma(u, p)$ designates the Cauchy stress tensor and

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$(\sigma(u, p).n)_\tau$ stands for the tangential component of $\sigma(u, p).n$ on $\partial\Omega$. It can be recalled that

$$\sigma(u, p) = -pI + 2\nu\varepsilon(u), \quad (4)$$

where $\nu > 0$ is the viscosity coefficient and $\varepsilon(u)$ is the symmetric part of the gradient, that is

$$\varepsilon_{i,j}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad 1 \leq i, j \leq n. \quad (5)$$

Another aspect of the problem is the unboundedness of the geometric region Ω . Specifically, equations (1) must be complemented by an asymptotic conditions at fareway regions, i.e. when $|x| \rightarrow +\infty$. In some sense we require that u satisfies a decay condition of the form

$$|u(x)| \rightarrow 0 \quad \text{when } |x| \rightarrow +\infty. \quad (6)$$

A precise meaning of this asymptotic condition will be given afterwards in terms of a well chosen weighted functions space to which u belongs. From a geometrical point of view, focus in this paper is on the following two cases:

- case 1: Ω is an exterior set of the form $\mathbb{R}^n \setminus \bar{\omega}$, where ω is a bounded domain of \mathbb{R}^n .
- case 2: Ω is an open upper half-space of \mathbb{R}^n :

$$\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

The key difference between these two types of geometries lies in the fact that the boundary is bounded in the former case, while it is unbounded in the latter one. We target to establish and study a mixed formulation of the Stokes system (1), when it is completed with a slip boundary conditions of the form (2)–(3) and with an asymptotic condition when $|x| \rightarrow +\infty$. When $\Omega = \mathbb{R}_+^n$, a direct approach was proposed in [8] for treating the same problem in weighted L^2 spaces, and in [8] and [5] in weighted L^p spaces. The generalized resolvent problem similar to (1), that is, when $-\nu\Delta u + \nabla p$ is replaced by $\lambda u - \nu\Delta u + \nabla p$, with $\lambda \in \mathbb{C} \setminus \{0\}$, has been studied in [23].

In this work, we first present some basics concerning the functional framework employed. Then, we lay out a mixed formulation of the problem in terms of the pair (u, p) . Well-posedness of this formulation is also shown by means of a weighted variant of Korn's inequality.

2. Preliminaries

Let Ω designate the exterior of a bounded domain ω or the upper half-space of \mathbb{R}^n . In the former case, ω is supposed to be of class \mathcal{C}^2 . In both

cases, $L^2(\Omega)$ designates the usual Lebesgue space of real square integrable functions over Ω , equipped with the norm

$$\|u\|_{L^2(\Omega)} = \left(\int_{\Omega} |u(x)|^2 dx \right)^{1/2}.$$

For all $m \in \mathbb{N}$ and $\ell \in \mathbb{R}$, define $W_{\ell}^m(\Omega)$ as the space of all the functions satisfying

$$(|x|^2 + 1)^{(\ell-m+|\lambda|)/2} D^{\lambda} u \in L^2(\Omega) \quad \text{for all } |\lambda| \leq m.$$

We may notice that $W_{\ell}^m(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{W_{\ell}^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_{\Omega} (|x|^2 + 1)^{(\ell-m+|\alpha|)} |D^{\alpha} u(x)|^2 dx \right)^{1/2}.$$

The following property can easily be proved by means of spherical coordinates: if $n \geq 2(m - \ell)$ then

$$(P \in W_{\ell}^m(\Omega) \text{ and } P \text{ polynomial}) \Rightarrow P = 0. \quad (7)$$

When Ω is an exterior domain, the first trace operator can be defined from $W_{\ell}^1(\Omega)$ onto the usual boundary space $H^{1/2}(\partial\Omega)$.

LEMMA 1. *Let $\Omega = \mathbb{R}^n \setminus \bar{\omega}$, where ω is a bounded domain of \mathbb{R}^n . The trace operator $\gamma_0 : v \in \mathcal{D}(\bar{\Omega}) \rightarrow v|_{\partial\Omega}$ can be extended to a linear and continuous operator from $W_{\ell}^1(\Omega)$ into $H^{1/2}(\partial\Omega)$.*

PROOF. Let \mathcal{O} be a *bounded* open subset of Ω containing a neighborhood of $\partial\Omega = \partial\omega$ (that is there exists $\epsilon_0 > 0$ such that $\{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \epsilon_0\} \subset \mathcal{O}$). For every function $v \in W_{\ell}^1(\Omega)$, its restriction to \mathcal{O} (still denoted v) belongs to $H^1(\mathcal{O})$ where

$$H^1(\mathcal{O}) = \{v \in L^2(\mathcal{O}) \mid \nabla v \in L^2(\mathcal{O})^n\}.$$

Moreover,

$$\forall v \in W_{\ell}^1(\Omega), \quad \|v\|_{H^1(\mathcal{O})} \leq C \|v\|_{W_{\ell}^1(\Omega)},$$

where C is a constant not depending on v . It follows that the trace operator $\gamma_0 : v \in \mathcal{D}(\bar{\Omega}) \rightarrow v|_{\partial\Omega}$ can be extended by continuity to a linear and continuous operator from $W_{\ell}^1(\Omega)$ into $H^{1/2}(\partial\Omega)$.

The situation is slightly different when Ω is the upper half-space since the boundary $\partial\Omega = \{x_n = 0\}$ is not compact. In the latter case, Hanouzet [17] showed that the operator $v \in \mathcal{D}(\bar{\Omega}) \mapsto v(., 0)$ can be extended to a continuous trace operator from $W_{\ell}^1(\Omega)$ into $W_0^{1/2}(\partial\Omega)$, where $W_0^{1/2}(\partial\Omega)$ denotes the space

of all (generalized) functions $u \in \mathcal{D}'(\partial\Omega) = \mathcal{D}'(\mathbb{R}^{n-1})$ such that $(1 + |x|^2)^{-1/4}u \in L^2(\mathbb{R}^{n-1})$ and

$$\int_0^\infty t^{-2} \int_{\mathbb{R}^{n-1}} |u(x + te_i) - u(x)|^2 dx dt < \infty, \quad \forall i = 1, 2, \dots, n-1.$$

Throughout this paper, we set

$$W^{1/2}(\partial\Omega) = \begin{cases} H^{1/2}(\partial\Omega) & \text{if } \Omega \text{ is an exterior domain,} \\ W_0^{1/2}(\partial\Omega) & \text{if } \Omega \text{ is the upper half-space } \mathbb{R}_+^n. \end{cases}$$

Let $W^{-1/2}(\partial\Omega)$ designate the dual of $W^{1/2}(\partial\Omega)$. As consequence, $W^{-1/2}(\partial\Omega) = H^{-1/2}(\partial\Omega)$ when Ω is an exterior domain, and $W^{-1/2}(\partial\Omega) = W_0^{-1/2}(\partial\Omega)$ when $\Omega = \mathbb{R}_+^n$ (see, e.g., [17] or [9]). The symbol $\langle \cdot, \cdot \rangle$ will be used to designate the duality pairings between $W^{-1/2}(\partial\Omega)$ and $W^{1/2}(\partial\Omega)$. We also introduce the space $H_*(\text{div}; \Omega)$ of all vector fields $v \in L^2(\Omega)^n$ satisfying $(|x|^2 + 1)^{1/2} \text{div } v \in L^2(\Omega)$. This space is equipped with the norm

$$\|v\|_{H_*(\text{div}; \Omega)} = (\|v\|_{L^2(\Omega)^n}^2 + \|(|x|^2 + 1)^{1/2} \text{div } v\|_{L^2(\Omega)}^2)^{1/2}.$$

It is well known that $\mathcal{D}(\bar{\Omega})^n$ is dense in $H_*(\text{div}; \Omega)$ (see [9] Lemma 5 when $\Omega = \mathbb{R}_+^n$). The proof can be adapted to the case of an exterior domain. By sake of completeness, we give a proof in appendix A for the latter case). On the basis of the argument mentioned herein above, the normal trace operator $v \in \mathcal{D}(\bar{\Omega})^n \rightarrow v \cdot n$ can be extended by continuity to a linear continuous operator from $H_*(\text{div}; \Omega)$ into $W^{-1/2}(\partial\Omega)$. As consequence, the following Green's formula can be easily established by density: for all $v \in H_*(\text{div}; \Omega)$ and $\psi \in W_0^1(\Omega)$,

$$\int_\Omega \text{div } v \cdot \psi \, dx = - \int_\Omega v \cdot \nabla \psi \, dx + \langle v \cdot n, \psi \rangle_{W^{-1/2}(\partial\Omega), W^{1/2}(\partial\Omega)}. \quad (8)$$

Consider also the following space of vector fields which are tangential on the boundary

$$X_\tau(\Omega) = \{v \in W_0^1(\Omega)^n \mid v \cdot n = 0 \text{ on } \partial\Omega\},$$

and

$$W_\tau^{1/2}(\partial\Omega) = \{v \in W^{1/2}(\partial\Omega)^n \mid v \cdot n = 0 \text{ on } \partial\Omega\}.$$

It is obvious that $X_\tau(\Omega)$ is a closed subspace of $W_0^1(\Omega)^n$ and

$$\gamma_0(X_\tau(\Omega)) = W_\tau^{1/2}(\partial\Omega). \quad (9)$$

Supposing that h_τ is the tangential component of a vector function $h \in W^{-1/2}(\partial\Omega)^n$ makes the slip boundary condition (3) meaningful since it can be

understood in the following weak sense: for every $v \in W_\tau^{1/2}(\partial\Omega)$

$$\langle \sigma(u).n, v \rangle_{W^{-1/2}(\partial\Omega)^n, W^{1/2}(\partial\Omega)^n} = \langle h, v \rangle_{W^{-1/2}(\partial\Omega)^n, W^{1/2}(\partial\Omega)^n}. \quad (10)$$

3. The mixed formulation

We need the following lemma:

LEMMA 2. *Assume that $n \geq 3$ and let $\rho \in L^2(\Omega)$ and $g \in W^{1/2}(\partial\Omega)$. Then, there exists a unique $\psi \in W_0^2(\Omega)/\mathbb{R}$ if $n = 3$ and $\psi \in W_0^2(\Omega)$ if $n \geq 4$ such that*

$$\Delta\psi = \rho \quad \text{in } \Omega, \quad \frac{\partial\psi}{\partial n} = g \quad \text{on } \partial\Omega, \quad (11)$$

and there exists a constant C_0 depending only on Ω such that

$$\|\psi\|_{W_0^2(\Omega)} \leq C_0(\|g\|_{W^{1/2}(\partial\Omega)} + \|\rho\|_{L^2(\Omega)}). \quad (12)$$

The proof of this lemma can be found in [3] when Ω is an exterior domain and in [9] when Ω is the half-space.

One may observe here that unlike in bounded domains no compatibility condition on the data ρ and g is required for the Neumann problem (11). This can be construed as a consequence of the fact that the asymptotic condition on ψ when $|x| \rightarrow +\infty$ is slightly released. Compatibility conditions on ρ and g could appear when a stronger decay of ψ at remote distances is requested (see [3] and [9]).

Throughout this paper we set

$$u_0 = \nabla\psi \in W_0^1(\Omega)^n, \quad (13)$$

where $\psi \in W_0^2(\Omega)$ is the unique solution of (11). It follows that

$$\begin{aligned} \operatorname{div} u_0 &= \rho, \quad \Delta u_0 = \nabla\rho, \quad \operatorname{div} \varepsilon(u_0) = \nabla\rho, \quad \text{in } \Omega, \\ u_0.n &= g \quad \text{on } \partial\Omega. \end{aligned}$$

Now, we are able to give a mixed formulation of the main problem:

PROPOSITION 1 (mixed formulation). *Let $f \in W_1^0(\Omega)^n$, $\rho \in W_1^1(\Omega)$, $g \in W^{1/2}(\partial\Omega)$ and $h \in W^{-1/2}(\partial\Omega)^n$. Then, the pair $(u, p) \in W_0^1(\Omega)^n \times L^2(\Omega)$ is a solution of (1)-(2)-(3) if and only if the pair $(w = u - u_0, p)$ is solution in $X_\tau(\Omega) \times L^2(\Omega)$ of the problem:*

$$(\mathcal{P}') \quad \begin{cases} 2\nu \int_{\Omega} \varepsilon(w) : \varepsilon(v) dx - \int_{\Omega} p \operatorname{div} v \, dx \\ \quad = \int_{\Omega} f_0 \cdot v \, dx + \langle \chi_0, v \rangle, \quad \text{for all } v \in X_{\tau}(\Omega), \\ \int_{\Omega} (\operatorname{div} w) q \, dx = 0 \quad \text{in } \Omega, \quad \text{for all } q \in L^2(\Omega), \end{cases} \quad (14)$$

where $f_0 = f + \nu \nabla p \in W_1^0(\Omega)^n$ and $\chi_0 = h - 2\nu \varepsilon(u_0) \cdot n \in W^{-1/2}(\partial\Omega)^n$.

PROOF. Let $(u, p) \in W_0^1(\Omega)^n \times L^2(\Omega)$ be solution of (1)-(2)-(3). Then $\sigma(u, p) \in L^2(\Omega)^{n^2}$, and $w = u - u_0 \in X_{\tau}(\Omega)$. Combining identity

$$2 \operatorname{div} \varepsilon(u) = \nabla(\operatorname{div} u) + \Delta u, \quad (15)$$

with equations (1) yields

$$-\operatorname{div} \sigma(u, p) = f - \nu \nabla p. \quad (16)$$

Since

$$\sigma(w, p) = \sigma(u, p) - 2\nu \varepsilon(u_0), \quad (17)$$

we obtain that

$$-\operatorname{div} \sigma(w, p) = f + \nu \nabla p = f_0. \quad (18)$$

Hence, $\sigma(w, p) \in H_*(\operatorname{div}; \Omega)^n$. Since $\nabla p \in W_1^0(\Omega)$ it follows that $\varepsilon(u_0) \in H_*(\operatorname{div}; \Omega)^n$ and $\varepsilon(u_0) \cdot n \in W^{-1/2}(\partial\Omega)^n$. The boundary condition (3) can be written as follows:

$$\langle \sigma(w, p) \cdot n, \eta \rangle = \langle h, \eta \rangle - 2\nu \langle \varepsilon(u_0) \cdot n, \eta \rangle.$$

Multiplying (18) by $v \in X_{\tau}(\Omega)$ and using formula (8) we deduce that u is solution of (\mathcal{P}') .

The converse is obtained by a standard argument.

REMARK 1. *In the case of the half-space, we can write*

$$\begin{aligned} X_{\tau}(\mathbb{R}_+^n) &= X_0^T(\mathbb{R}_+^n) = \{v \in W_0^1(\mathbb{R}_+^n)^n \mid v_n = 0 \text{ at } x_n = 0\} \\ &= W_0^1(\mathbb{R}_+^n)^{n-1} \times \dot{W}_0^1(\mathbb{R}_+^n), \end{aligned}$$

with $\dot{W}_0^1(\mathbb{R}_+^n) = \{w \in W_0^1(\mathbb{R}_+^n) \mid w = 0 \text{ at } x_n = 0\}$. It can easily be proved that assertion of Proposition 1 holds true when $f \in W_1^0(\mathbb{R}_+^n)^{n-1} \times W_0^{-1}(\mathbb{R}_+^n)$, where $W_0^{-1}(\mathbb{R}_+^n)$ is the dual space of $\dot{W}_0^1(\mathbb{R}_+^n)$.

4. Well posedness of the mixed formulation

The next theorem concerns well posedness of the problem considered here:

THEOREM 1. *Assume that Ω is a half-space of \mathbb{R}^n or $\Omega = \mathbb{R}^n \setminus \bar{\omega}$, where ω is a bounded and convex open subset of \mathbb{R}^n with $\partial\omega$ of class \mathcal{C}^2 . Suppose also that $n \geq 3$, $f \in W_1^0(\Omega)^n$, $\rho \in W_1^1(\Omega)$, $g \in W^{1/2}(\partial\Omega)$ and $h \in W^{-1/2}(\partial\Omega)^n$. Then, problem (1)-(2)-(3) has one and only one solution $(u, p) \in X_\tau(\Omega) \times L^2(\Omega)$. Moreover,*

$$\begin{aligned} \|u\|_{W_1^1(\Omega)^n} + \|p\|_{L^2(\Omega)} &\leq C(\|f\|_{W_1^0(\Omega)^n} + \|\rho\|_{W_1^1(\Omega)} \\ &\quad + \|g\|_{W^{1/2}(\partial\Omega)} + \|h\|_{W^{-1/2}(\partial\Omega)^n}), \end{aligned} \quad (19)$$

for a constant $C > 0$ depending only on Ω .

For proving Theorem 1 we need the following Korn's inequality:

LEMMA 3. *Under assumptions of Theorem 1 on n and Ω , there exists two constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\|(1 + |x|^2)^{-1/2}v\|_{L^2(\Omega)^n} \leq C_1 \|\varepsilon(v)\|_{L^2(\Omega)^{n^2}}, \quad (20)$$

$$\|\nabla v\|_{L^2(\Omega)^{n^2}} \leq C_2 \|\varepsilon(v)\|_{L^2(\Omega)^{n^2}}, \quad (21)$$

for all $v \in X_\tau(\Omega)$.

PROOF. This proof is inspired by the reference [18] which contains a proof when Ω is the exterior of a bounded convex domain (see also [10]). For the sake of completeness we give a proof of this theorem only when $\Omega = \mathbb{R}_+^n$. Let $\varphi \in \mathcal{D}(\overline{\mathbb{R}_+^n})^{n-1} \times \mathcal{D}(\mathbb{R}_+^n)$. Then,

$$|\nabla \varphi|^2 = 2|\varepsilon(\varphi)|^2 + |\operatorname{div} \varphi|^2 + 2\varphi \cdot \nabla(\operatorname{div} \varphi) - \sum_{i,j=1}^n \partial_i \partial_j (\varphi_i \varphi_j). \quad (22)$$

Integrating over \mathbb{R}_+^n yields

$$\begin{aligned} \int_{\mathbb{R}_+^n} |\nabla \varphi|^2 dx &= \int_{\mathbb{R}_+^n} (2|\varepsilon(\varphi)|^2 + |\operatorname{div} \varphi|^2 + 2\varphi \cdot \nabla(\operatorname{div} \varphi)) dx \\ &\quad + \sum_{j=1}^n \int_{\partial\omega} \partial_j (\varphi_n \varphi_j) d\sigma \end{aligned}$$

Since $\varphi_n = \varphi \cdot e_n = 0$ on $\partial\mathbb{R}_+^n = \{x_n = 0\}$, we get

$$\sum_{j=1}^n \int_{\partial\omega} \partial_j (\varphi_n \varphi_j) d\sigma = \int_{\partial\omega} (\operatorname{div} \varphi) \varphi_n d\sigma + \int_{\partial\omega} \varphi \cdot \nabla(\varphi_n) d\sigma = 0,$$

and

$$\int_{\mathbb{R}_+^n} \varphi \nabla(\operatorname{div} \varphi) dx = - \int_{\mathbb{R}_+^n} |\operatorname{div} \varphi|^2 dx.$$

We obtain

$$\int_{\mathbb{R}_+^n} |\nabla \varphi|^2 dx = \int_{\mathbb{R}_+^n} (2|\varepsilon(\varphi)|^2 - |\operatorname{div} \varphi|^2) dx. \quad (23)$$

Since $\mathcal{D}(\overline{\mathbb{R}_+^n})^{n-1} \times \mathcal{D}(\mathbb{R}_+^n)$ is dense in $X_\tau(\mathbb{R}_+^n)$, identity (23) is still valid for every $\varphi \in W_0^1(\mathbb{R}_+^n)^{n-1} \times \dot{W}_0^1(\mathbb{R}_+^n)$. This completes the proof of (21).

Inequality (20) is a consequence of the usual Hardy inequality which is valid for $n \geq 3$ (see, e.g., [16], [3], [9]):

$$\forall v \in W_0^1(\mathbb{R}_+^n), \quad \int_{\mathbb{R}_+^n} \frac{|v|^2}{|x|^2 + 1} dx \leq C \int_{\mathbb{R}_+^n} |\nabla v|^2 dx. \quad (24)$$

REMARK 2. *The reader may be surprised by inequality (24) since it is valid without any boundary condition on v . Qualitatively speaking, this may be interpreted as follows: for $n \geq 3$, belonging to $W_0^1(\Omega)$ is in some way like vanishing at infinity (in other words, infinity acts, in some sense, as a boundary). The proof can be found in [16] and [3] for the whole space and exterior domains, and in [9] for the half-space. In the case of the whole space and the exterior of a ball B_R ($R \geq 0$), the proof is straightforward since it can be deduced from the identity*

$$\int_R^{+\infty} |\varphi(r\sigma)|^2 dr = -R|\varphi(R\sigma)|^2 - \int_R^{+\infty} r\varphi(r\sigma) \frac{\partial}{\partial r} \varphi(r\sigma) dr,$$

which is valid for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\sigma \in \mathbb{S}^2$ and $R > 0$. Observing that the term $-R|\varphi(R\sigma)|^2$ is nonpositive and using Cauchy-Schwarz inequality gives

$$\int_R^{+\infty} |\varphi(r\sigma)|^2 dr \leq \int_R^{+\infty} r^2 \left(\frac{\partial}{\partial r} \varphi(r\sigma) \right)^2 dr.$$

Integrating with respect to σ and using the density of $\mathcal{D}(\overline{\Omega})$ in $W_0^1(\Omega)$, we obtain (24) when $\Omega = \mathbb{R}^n \setminus \overline{B_R}$.

PROOF OF THEOREM 1. The proof uses the well known theorem due to Brezzi [12] (Theorem 1.1) and Babuska [6] (see also [15] Theorem 4.1 and Corollary 4.1):

THEOREM 2. *Let X and M be two Hilbert spaces and consider the abstract problem: find $(w, p) \in X \times M$ such that*

$$\begin{aligned} a(w, v) + b(v, p) &= \ell(v), & \forall v \in X, \\ b(w, q) &= 0, & \forall q \in M, \end{aligned} \quad (25)$$

where a (resp. b) is a continuous bilinear form defined on $X \times X$ (resp. $X \times M$) and $\ell \in X'$ (the dual of X). Suppose that there exists two constants $\alpha > 0$ and $\beta > 0$ such that

$$\forall v \in V, \quad a(v, v) \geq \alpha \|v\|_X^2, \quad \inf_{q \in M} \sup_{v \in X} \frac{b(v, q)}{\|v\|_X \|q\|_M} \geq \beta,$$

where $V = \{v \in X \mid b(v, q) = 0, \forall q \in M\}$. Then, problem (25) has one and only one solution $(w, q) \in X \times M$. Moreover, there exists a constant $C > 0$ not depending on ℓ , w and p such that

$$\|w\|_X + \|p\|_M \leq C \|\ell\|_{X'}.$$

Formulation (\mathcal{P}') can be written into the form

$$a_0(w, v) + b_0(v, p) = \ell_0(v), \quad \text{for all } v \in X_\tau(\Omega),$$

$$b_0(w, q) = 0, \quad \text{for all } q \in L^2(\Omega),$$

with

$$a_0(w, v) = 2\nu \int_{\Omega} \varepsilon(w) : \varepsilon(v) dx,$$

$$b_0(w, q) = - \int_{\Omega} (\operatorname{div} w) q \, dx,$$

$$\ell_0(v) = \int_{\Omega} f_0 \cdot v \, dx + \langle \chi_0, v \rangle.$$

It is obvious that the bilinear form $a_0(\cdot, \cdot)$ (resp. $b_0(\cdot, \cdot)$) is continuous on $X_\tau(\Omega)^2$ (resp. on $X_\tau(\Omega) \times L^2(\Omega)$). The linear form ℓ_0 is also continuous over $X_\tau(\Omega)$ and $a_0(\cdot, \cdot)$ is $X_\tau(\Omega)$ -elliptic, thanks to Korn's inequalities (20) and (21). It remains to prove the inf-sup condition on b_0 . Let $q \in L^2(\Omega)$, with $q \neq 0$, and let ψ the unique solution in $W_0^2(\Omega)/\mathbb{R}$ if $n = 3$ and in $W_0^2(\Omega)$ if $n \geq 4$ of the Poisson equation with a Neumann boundary data

$$\Delta \psi = q \quad \text{in } \Omega, \quad \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial \Omega.$$

Let $v_0 = \nabla \psi \neq 0$. It's obvious that $v_0 \in X_\tau(\Omega)$ and we have

$$\|v_0\|_{W_0^1(\Omega)^n} = \|\nabla \psi\|_{W_0^1(\Omega)^n} \leq \|\psi\|_{W_0^2(\Omega)} \leq C \|q\|_{L^2(\Omega)},$$

for some constant $C > 0$. In addition, $\operatorname{div} v_0 = q$ and

$$b_0(v_0, q) = \|q\|_{L^2(\Omega)}^2 \geq \frac{1}{C} \|q\|_{L^2(\Omega)} \|v_0\|_{W_0^1(\Omega)^n}.$$

We conclude that

$$\sup_{v \in X_\tau(\Omega)} \frac{b_0(v, q)}{\|q\|_{L^2(\Omega)} \|v\|_{W_0^1(\Omega)^n}} \geq \frac{1}{C}.$$

This completes the proof of Theorem 1.

REMARK 3. *It is worth noting that the solution $u \in W_0^1(\Omega)$ satisfies*

$$\|u(r.\cdot)\|_{L^2(\mathbb{S}^n)} = o\left(\frac{1}{r^{n-2}}\right) \quad \text{when } r \rightarrow +\infty, \quad (26)$$

where \mathbb{S}^n denotes the unit sphere of \mathbb{R}^n if Ω is an exterior domain or the upper half of the unit sphere of \mathbb{R}^n if Ω is a half-space (see, e.g., [1]). It follows in particular that

$$\lim_{r \rightarrow +\infty} \|u(r.\cdot)\|_{L^2(\mathbb{S}^n)} = 0.$$

REMARK 4. *In [8], the author proposed a reflection approach for solving the system (1)-(2)-(3) in the half-space.*

Discussion

The approach we propose here can be extended for many other situations. On the one hand, other kinds of unbounded domains can be considered. In this regard, weighted Korn's inequality, which is the corner stone in proving well posedness, could be very useful (see, e.g., [18]). On the other hand, one can also consider slip boundary conditions with friction instead of (3), that is

$$(\sigma(u, p).n)_\tau + ku = 0,$$

with k a constant. Finally, it should be noted that a mixed formulation of the form (\mathcal{P}') can also be done in the two-dimensional case ($n = 2$), provided that the underlying functional spaces are slightly adapted by considering logarithmic weights (see, e.g., [3]).

Appendix A. Density of $\mathcal{D}(\bar{\Omega})^n$ in $H_*(\text{div}; \Omega)$ when Ω is an exterior domain

The proof is inspired by [17]. Assume that $\Omega = \mathbb{R}^n \setminus \bar{\omega}$, where ω is a bounded domain of \mathbb{R}^n . Throughout this section, for every real number $r > 0$ B_r designates the open ball of radius r centered at the origin and $\Omega_r = \Omega \cap B_r$. Let $v \in H_*(\text{div}; \Omega)$ and $\epsilon > 0$. Consider a function $\theta \in \mathcal{D}(\mathbb{R}^n)$ such that

$$\begin{aligned} 0 \leq \theta(x) \leq 1 & \quad \text{for all } x \in \mathbb{R}^n, \\ \theta(x) = 1 & \quad \text{when } |x| \leq 1 \quad \text{and} \quad \theta(x) = 0 \quad \text{when } |x| \geq 2. \end{aligned}$$

Since ω is bounded, there exists an integer $k_0 \geq 1$ such that $\omega \subset B_{k_0}$, where B_{k_0} is the ball of radius k_0 centered at the origin. Set

$$v_k = \theta_k(x)v(x) \quad \text{for } k \geq 1 \text{ with } \theta_k(x) = \theta\left(\frac{x}{k}\right).$$

For $k \geq k_0$ we have

$$\begin{aligned} \|v_k - v\|_{H_*(\text{div}; \Omega)}^2 &= \int_{|x| \geq k} |\theta_k - 1|^2 |v|^2 dx + \int_{k \leq |x| \leq 2k} (|x|^2 + 1) |\text{div } v - \text{div } v_k|^2 dx \\ &\quad + \int_{|x| \geq 2k} (|x|^2 + 1) |\text{div } v|^2 dx \\ &= \int_{|x| \geq k} |\theta_k - 1|^2 |v|^2 dx + 2 \int_{k \leq |x| \leq 2k} (|x|^2 + 1) (1 - \theta_k)^2 |\text{div } v|^2 dx \\ &\quad + 2 \int_{k \leq |x| \leq 2k} (|x|^2 + 1) |\nabla \theta_k \cdot v|^2 + \int_{|x| \geq 2k} (|x|^2 + 1) |\text{div } v|^2 dx. \\ &= C_1 \int_{|x| \geq k} |v|^2 dx + 2 \int_{k \leq |x| \leq 2k} (|x|^2 + 1) |\text{div } v|^2 dx \\ &\quad + \frac{4k^2 + 1}{k^2} \int_{k \leq |x| \leq 2k} |v|^2 + \int_{|x| \geq 2k} (|x|^2 + 1) |\text{div } v|^2 dx. \end{aligned}$$

Obviously, $\|v_k - v\|_{H_*(\text{div}; \Omega)} \rightarrow 0$, and there exists an integer $\ell \geq k_0$ such that

$$\|v_\ell - v\|_{H_*(\text{div}; \Omega)} < \epsilon$$

Observe now that v_ℓ has a compact support included in $\overline{\Omega_{2\ell}}$. Since $\mathcal{D}(\overline{\Omega_{4\ell}})$ is dense in the usual space $H(\text{div}; \Omega_{4\ell})$, there exists a function $\psi_\ell \in \mathcal{D}(\mathbb{R}^n)^n$ such that

$$\|\psi_\ell - v_\ell\|_{H(\text{div}; \Omega_{4\ell})} < \epsilon.$$

Set

$$\varphi_\ell(x) = \theta\left(\frac{x}{2\ell}\right)\psi_\ell(x).$$

Hence, $\varphi_\ell \in \mathcal{D}(\mathbb{R}^n)^n$ and $\varphi_\ell = \psi_\ell$ in $\Omega_{2\ell}$. Thus,

$$\begin{aligned} \|\varphi_\ell - v_\ell\|_{H_*(\text{div}; \Omega)}^2 &= \|\varphi_\ell - v_\ell\|_{H_*(\text{div}; \Omega_{4\ell})}^2 \\ &= \|\psi_\ell - v_\ell\|_{\Omega_{2\ell}}^2 + \|\varphi_\ell\|_{H_*(\text{div}; \Omega_{4\ell} \setminus \overline{\Omega_{2\ell}})}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|\psi_\ell - v_\ell\|_{\Omega_{2\ell}}^2 + \int_{2\ell \leq |x| \leq 4\ell} |\theta_{2\ell}|^2 |\psi_\ell|^2 dx \\
&\quad + 2 \int_{2\ell \leq |x| \leq 4\ell} (|x|^2 + 1)(|\nabla \theta_{2\ell}|^2 |\psi_\ell|^2 + |\theta_{2\ell}|^2 |\operatorname{div} \psi_\ell|^2) dx \\
&\leq \|\psi_\ell - v_\ell\|_{\Omega_{2\ell}}^2 + 2\|\psi_\ell\|_{H_*(\operatorname{div}; \Omega_{4\ell} \setminus \overline{\Omega_{2\ell}})}^2 \\
&\quad + 2\|\theta'\|_\infty \frac{(4\ell)^2 + 1}{4\ell^2} \int_{2\ell \leq |x| \leq 4\ell} |\psi_\ell|^2 dx \\
&\leq C\|\psi_\ell - v_\ell\|_{H(\operatorname{div}; \Omega_{4\ell})}^2 \\
&\leq C'\epsilon^2,
\end{aligned}$$

where the constants C and C' are not depending on ϵ . This completes the proof.

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