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On the invariant field of binary octavics

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Introduction

Katsylo and Bogomolov proved in [7, 1, 2] the rationality of the invariant field of binary n-forms over the complex numbers field C for all integers n. The purpose of this paper is to give a set of six generators explicitly along with the classical terminology, i.e. by *symbolic expressions* (cf. [5, 6]), in the case of binary octavics ($n = 8$) (Theorem B). For a general integer n, we show the following:

THEOREM A. *For each even integer* $n \geq 8$ (*see Remark* 1.12 *for odd integers* n), *there is a homogeneous invariant polynomial* M *of degree* 12 *such that the invariant field of binary n-forms over* C *is generated by* $n - 2$ (*=transcendence degree over* C) *rational functions whose denominators are certain powers of* M.

One of the culminant of the classical invariant theory is [4], in which von Gall shows that the set of 70 covariants (including 9 invariants) listed there, is a complete minimal system of the graded ring of covariants of binary octavics. About twenty years ago, Shioda[8] determined all the syzygy modules of the graded ring of invariants of binary octavics, by means of the symbolic method, generating functions and some technique due to Hilbert. In contrast to the invariant(covariant) rings, there had been, until now, little attempt to calculate algebraically independent generators of the invariant fields of binary forms. The author would like to indicate in the present article that one could apply the classical symbolic method initiated by Gordan and others [5, 6] to express the generators not only of the invariant rings, but also of the invariant fields of binary forms.

It was shown in [7] that the field in question is isomorphic to the invariant field under an action of a subgroup H of SL(2, C). Analizing this isomorphism, we give in §1 a correspondence of H-invariant polynomials to the SL(2, C)-invariant rational functions and prove Theorem A. The method used in §1 is a variation of protomorphic functions in the classical invariant theory (Remark 1.13, cf. [3]). In §2 we apply the result of §1 in the case of binary octavics and give a set of generators explicitly. After preparing four lemmas (from Lemma 2.7 to 2.10), the process of constructing the SL(2, C)-invariant rational functions from H-invariant polynomials is a formal computation and a routine of the symbolic method.

Notation. In this paper the ground field is the complex numbers field $\boldsymbol{C}$. For an integer n, we denote by $V(n)$, the representation space of the irreducible representation ρ_n of dimension $n+1$ of $\mathrm{SL}(2, \boldsymbol{C})$, or the affine space of dimension $n+1$ over $\boldsymbol{C}$, which has the regular action of $\mathrm{SL}(2, \boldsymbol{C})$ induced by ρ_n. $\boldsymbol{P}(V(n))$ is the projective space of dimension n over $\boldsymbol{C}$ associated to it. The representation ρ_n is defined as follows. For a basis $\{a_0, \dots, a_n\}$ of $V(n)$, let

$$(a_0, \dots, a_n)(x_0, x_1)^n = \textstyle\sum_{i=0}^{n} \binom{n}{i} a_i x_0^{n-i} x_1^i \quad \text{(the basic } n\text{-forms)},$$

and

$$(a_0^g, \dots, a_n^g)(x_0, x_1)^n = (a_0, \dots, a_n)(a x_0 + c x_1, b x_0 + d x_1)^n,$$

$$\text{for} \quad g = \begin{pmatrix} ab \\ cd \end{pmatrix} \in \mathrm{SL}(2, \boldsymbol{C}).$$

Then we define $\rho_n(g)$ by

$$(0.1) \qquad {}^t(a_0^g, \dots, a_n^g) = \rho_n(g) \cdot {}^t(a_0, \dots, a_n).$$

For two binary forms

$$F = (a_0, \dots, a_n)(x_0, x_1)^n, \qquad G = (b_0, \dots, b_m)(x_0, x_1)^m,$$

we denote by $(F, G)_r$ the r-th transvection of F and G (cf. [5, 6]):

$$(0.2) \qquad (F, G)_r = \frac{(n-r)!(m-r)!}{n!m!} \textstyle\sum_{k=0}^{r} (-1)^k \binom{r}{k} \dfrac{\partial^r F}{\partial x_0^{r-k} \partial x_1^k} \dfrac{\partial^r G}{\partial x_0^k \partial x_1^{r-k}},$$

$$\text{for} \quad 0 \leq r \leq \min(n, m).$$

Let

$$(F, G)_r = (c_0, \dots, c_{n+m-2r})(x_0, x_1)^{n+m-2r}.$$

Then the subspace of $\Sigma_{i,j} \boldsymbol{C} a_i b_j = V(n) \otimes V(m)$ generated by $\{c_i\}_{i=0}^{n+m-2r}$ is G-stable and equivalent to $V(n+m-2r)$.

§1. Proof of Theorem A

Let X be an algebraic variety over $\boldsymbol{C}$ which has a rational action of an algebraic group G. For a subgroup H of G and an irreducible algebraic subvariety Y (not necessarily closed) of X, Katsylo in [7] defined Y to be a (G, H)-section of X if

(0) $hy \in Y$ for any $h \in H$ and $y \in Y$,

(1) the rational map $G \times Y \to X$, $(g, y) \to gy$, is dominant,

(2) for a general point y of Y, if $g \in G$ and $gy \in Y$, then $g \in H$.

When these conditions are satisfied we see that the inclusion $Y \to X$ induces a birational map $Y/H \to X/G$, where Y/H (resp. X/G) is an algebraic variety over $\boldsymbol{C}$ whose function field is isomorphic to the invariant field $\boldsymbol{C}(Y)^H$ (resp. $\boldsymbol{C}(X)^G$). We note the existence of the (G, H)-section of X is equivalent to the following condition:

(1.1) *There is a G-equivariant rational map $\varphi: X \to G/H$ such that the fibre $\varphi^{-1}[e]$ of the class $[e]$ of H, is irreducible.*

When (1.1) is satisfied, the irreducible subvariety $\varphi^{-1}[e]$ of X is a (G, H)-section of X.

We fix an even integer $n \geq 8$ and denote $G = \mathrm{SL}(2, \boldsymbol{C})$ from now on. Let

$$H = \left\langle \sigma = \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta^3 & \zeta^5 \\ \zeta^7 & \zeta^5 \end{pmatrix}, \tau = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} \right\rangle, \qquad \zeta = \exp(2\pi\sqrt{-1}/8),$$

$$Y = \{(a, 0, 6b, 0, a) | a, b \in \boldsymbol{C}\} \subset V(4).$$

Then the subgroup H of $\mathrm{SL}(2, \boldsymbol{C})$ is isomorphic to the central extension of $\boldsymbol{Z}/2\boldsymbol{Z}$ by the symmetric group of degree 4 and Y is a (G, H)-section of $V(4)$ (cf. [7]). Let ψ: $V(4) \to \boldsymbol{P}(V(6))$ be a G-equivariant rational map defined by

$$\psi(Q_0, \ldots, Q_4) = (t_0 : \ldots : t_6), \tag{1.2}$$

where $Q = \Sigma_{i=0}^{4}\binom{4}{i}Q_i x_0^{4-i}x_1^i$ and $((QQ)_2Q)_1 = \Sigma_{i=0}^{6}\binom{6}{i}t_i x_0^{6-i}x_1^i$. Then we see by direct calculations that the (G, H)-section Y is mapped to the one point $P_0 = (0:1:0:0:0:-1:0)$ by ψ. Hence, in view of (1.1), the image of ψ is birational to G/H and the class of H corresponds to the point P_0.

We denote the orbit map $\hat{\pi}: G \to V(6)$ at the point $\hat{P}_0 = (0, 1/6, 0, 0, 0, -1/6, 0)$ by

$$\hat{\pi}(g) = (\pi_0, \ldots, \pi_6), \tag{1.3}$$

and by $\pi: G \to \boldsymbol{P}(V(6))$, the composition of $\hat{\pi}$ and the natural projection $V(6) \to \boldsymbol{P}(V(6))$. Let φ: $V(n) \to V(4)$ be the G-equivariant morphism defined by the $(n-2)$-th transvection:

$$\varphi(a) = (Q_0, \ldots, Q_4), \tag{1.4}$$

where $f = \Sigma_{i=0}^{n}\binom{n}{i}a_i x_0^{n-i}x_1^i$ and $(f, f)_{n-2} = \Sigma_{i=0}^{4}\binom{4}{i}Q_i x_0^{4-i}x_1^i$. Then we can find an irreducible component Z of $\varphi^{-1}(Y)$ which is a (G, H)-section of

$V(6)$([7]). Let $G\cdot Z$ be the image of the morphism $\gamma\colon G\times Z\to V(n)$, $\gamma(g,z)=g\cdot z$ and let U be the G-invariant open set $V(n)\backslash\overline{V(n)\backslash G\cdot Z}$, which is non-empty because $G\cdot Z$ is dense in $V(n)$ by the definition of the (G,H)-sections.

Now we consider the fibre product X of G and U over G/H:

$$X=\{(g,a)\in G\times U\,|\,\pi(g)=\psi\varphi(a)\}\,.$$

Then the induced action of $H\times G$ on X is given by

(1.5)
$$^{\sigma}\!\left(\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix},{}^t(a_0,\dots,a_n)\right)=\left(\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix}\begin{pmatrix}ab\\ cd\end{pmatrix},{}^t(a_0,\dots,a_n)\right)\quad\text{for}\quad\sigma=\begin{pmatrix}ab\\ cd\end{pmatrix}\in H\,,$$
$$\left(\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix},{}^t(a_0,\dots,a_n)\right)^{g}=\left(\begin{pmatrix}ab\\ cd\end{pmatrix}\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix},\rho_n(g)\,{}^t(a_0,\dots,a_n)\right)\quad\text{for}\quad g=\begin{pmatrix}ab\\ cd\end{pmatrix}\in G\,.$$

Since Z is birational to the image of the morphism $\phi\colon X\to V(n)$ defined by $\phi(g,a)=g^{-1}a$, *the generic point* $(b)=(b_0,\dots,b_n)$ *of* Z is written by

$$\text{(1.6)}\qquad {}^t(b_0,\dots,b_n)=\rho_n\begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}{}^t(a_0,\dots,a_n)\,.$$

If we define the action of $H\times G$ on $Z=\phi(X)$ so as to the morphism ϕ is $H\times G$-equivariant, then (1.5) implies the G-invariance of the elements $b_j(0\le j\le n)$ and the action of H on $\{b_j\}$ is given by

$$\text{(1.7)}\qquad {}^t({}^{\sigma}b_0,\dots,{}^{\sigma}b_n)=\rho_n(\sigma)^{-1}\,{}^t(b_0,\dots,b_n)\quad\text{for}\quad\sigma=\begin{pmatrix}ab\\ cd\end{pmatrix}\in H\,.$$

In fact

$$\begin{aligned}
{}^t({}^{\sigma}b_0,\dots,{}^{\sigma}b_n)&=\rho_n\begin{pmatrix}{}^{\sigma}\delta & -{}^{\sigma}\beta\\ -{}^{\sigma}\gamma & {}^{\sigma}\alpha\end{pmatrix}{}^t(a_0,\dots,a_n)\\
&=\rho_n\left(\begin{pmatrix}\gamma b+\delta d & -\alpha b-\beta d\\ -\gamma a-\delta c & \alpha a+\beta c\end{pmatrix}\right){}^t(a_0,\dots,a_n)\quad\text{by}\quad(1.5)\\
&=\rho_n\left(\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}\right){}^t(a_0,\dots,a_n)\\
&=\rho_n(\sigma^{-1})\,{}^t(b_0,\dots,b_n)\,.
\end{aligned}$$

Next we show the following.

(1.8) $\boldsymbol{C}[G]^H=\boldsymbol{C}[\pi_i\pi_j(0\le i,j\le 6)]$, *where* π_i *is the polynomial defined by* (1.3).

For the centralizer H_+ of G at the point $\hat{P}_0=(0,1/6,0,0,0,-1/6,0)$ of $V(6)$ is the subgroup of H such that H/H_+ is isomorphic to $\boldsymbol{Z}/2\boldsymbol{Z}$. Hence $\hat{P}_0$ is a

G-stable point of $V(6)$ and the image $\hat{\pi}(G)$ of $\hat{\pi}$ is closed in $V(6)$. Therefore G/H_+ is isomorphic to $\pi(G)$ by Zariski Main Theorem: $\boldsymbol{C}[G]^{H_+} = \boldsymbol{C}[\pi_0, \dots, \pi_6]$. The assertion (1.8) follows because $H = \langle H_+, \zeta \rangle$ and ζ transforms π_i to $-\pi_i$.

Let t, π be the 6-forms corresponding to the rational maps ψ and π:

$$t = (t_0, \dots, t_6)(x_0, x_1)^6, \qquad \pi = (\pi_0, \dots, \pi_6)(x_0, x_1)^6.$$

Since X is the fibre product of G and U over G/H, we put

$$t = \lambda\pi \quad \text{i.e.,} \quad t_i = \lambda\pi_i \quad (0 \le i \le 6). \tag{1.9}$$

Apply the 6-th transvection to (1.9):

$$\begin{aligned} (t, t)_6 &= \lambda^2(\pi, \pi)_6 \\ &= \lambda^2(\pi, \pi)_6|_{g = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)} \quad \text{because } (\pi, \pi)_6 \text{ is } G\text{-invariant}, \\ &= \lambda^2 2(\pi_0\pi_6 - 6\pi_1\pi_5 + 15\pi_2\pi_4 - 10\pi_3^2)|_{g = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)} \\ &= \lambda^2 2(-6)(1/6)(-1/6) = \lambda^2/3. \end{aligned} \tag{1.10}$$

Summing up, we have the following equalities:

$$\begin{aligned} \boldsymbol{C}[b_0, \dots, b_n]^H &= \boldsymbol{C}[\alpha, \beta, \gamma, \delta, b_0, \dots, b_n]^{H \times G} && \text{because } b_i \text{ is } G\text{-invariant} \\ &= \boldsymbol{C}[\alpha, \beta, \gamma, \delta, a_0, \dots, a_n]^{H \times G} && \text{by (1.6)} \\ &= \boldsymbol{C}[\pi_i\pi_j(0 \le i, j \le 6), a_0, \dots, a_n]^G && \text{by (1.8)} \\ &= \boldsymbol{C}[t_it_j/\lambda^2(0 \le i, j \le 6), a_0, \dots, a_n]^G && \text{by (1.9)} \\ &= \boldsymbol{C}[t_it_j/(t, t)_6(0 \le i, j \le 6), a_0, \dots, a_n]^G && \text{by (1.10)} \\ &\subset \boldsymbol{C}[a_0, \dots, a_n]^G[1/M], \end{aligned} \tag{1.11}$$

where $M = (t, t)_6$ is a G-invariant polynomial of degree 12.

Hence, in order to complete the proof of Theorem A, we have only to show that there are $(n-2)$ polynomials in $\boldsymbol{C}[b_0, \dots, b_n]^H$ which generates $\boldsymbol{C}(b_0, \dots, b_n)^H = \boldsymbol{C}(a_0, \dots, a_n)^G$. This follows from the existence of an H-subspace W_1 of dimension $n-2$ of the vector space generated by $b_0, \dots, b_n$ such that $\boldsymbol{C}(W_1) = \boldsymbol{C}(b_0, \dots, b_n)$ and that W_1 contains an irreducible H-subspace of dimension 3. We omit the detailes (cf. §2, [7]).

REMARK 1.12. In the case of an odd integer $n \ge 5$, Katsylo proved the rationality using a (G, N)-section, where N is the normalizer of a maximal torus of G. By the above observation, we find a G-invariant polynomial M of degree 4 for which the same statement in Theorem A holds. In fact, M can be taken as $(q, q)_2$, where $q = (f, f)_2$ and f is the basic n-forms.

REMARK 1.13. As pointed out in the introduction, the above method is not different in principle from protomorphic functions. Let U be the upper triangular unipotent subgroup of G and let $Y = \{(a_0, 0, a_2, \dots, a_n) | a_i \in \boldsymbol{C}\}$ be the subvariety of $V(n)$. Then we see easily that Y is a $(U, \{1\})$-section of $V(n)$ and the U-invariant a_0 only appears as the donominator. Hence we get a system of protomorphic functions as in [3].

§2. Proof of Theorem B

In this section we apply the result of §1 to the case of binary octavics($n = 8$.) First we give a set of generators of the H-invariant field $\boldsymbol{C}(b_0, \dots, b_8)^H$ which is the function field of the (G, H)-section Z of $V(8)$.

2.1. Generators of $\boldsymbol{C}(b_0, \dots, b_8)^H$

Recall that the set $\{b_i\}_{i=0}^{8}$ is a basis of $V(8)$ on which the subgroup H of $G = \mathrm{SL}(2, \boldsymbol{C})$ acts as defined in (0.1). The decomposition of the H-module $V(8)$ into simple components is given by

$$V(8) = \varepsilon \oplus \theta_2 \oplus \theta_3 \oplus (\theta_3 \otimes \varepsilon^-),$$

where ε(resp. ε^-) is the identity (resp. the alternating) representation and θ_d is an irreducible representation of H of dimension d. We denote $\theta_3^+ = \theta_3$ and $\theta_3^- = \theta_3 \otimes \varepsilon$. We set basis of the representations as follows:

$$\begin{array}{ll}
\varepsilon & c_0 = b_0 + 14b_4 + b_8 , \\
\theta_2 & \begin{cases} c_1 = b_0 - 10b_4 + b_8 , \\ c_2 = b_2 + b_6 , \end{cases} \\
\theta_3^+ & \begin{cases} c_3 = b_1 - b_5 , \\ c_4 = b_2 - b_6 , \\ c_5 = b_3 - b_7 , \end{cases} \\
\theta_3^- & \begin{cases} c_6 = b_1 + 7b_5 , \\ c_7 = b_0 - b_8 , \\ c_8 = 7b_3 + b_7 . \end{cases}
\end{array} \tag{2.1}$$

We defined in (1.4) the G-equivariant morphism φ: $V(8) \to V(4)$ as the 6-th transvection (cf. (0.2)):

$$\varphi(a) = (Q_0 : \dots : Q_4),$$

where

$$Q_0 = 2(a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2),$$
$$Q_1 = 4(a_0a_7 - 5a_1a_6 + 9a_2a_5 - 5a_3a_4),$$
$$Q_2 = 2(a_0a_8 - 2a_1a_7 - 8a_2a_6 + 34a_3a_5 - 25a_4^2),$$
$$Q_3 = 4(a_1a_8 - 5a_2a_7 + 9a_3a_6 - 5a_4a_5),$$
$$Q_4 = 2(a_2a_8 - 6a_3a_7 + 15a_4a_6 - 10a_5^2).$$

Since $(b) = (b_0, \dots, b_8)$ is a generic point of Z (see (1.6)), we have $Q_1(b) = Q_3(b) = Q_0(b) - Q_4(b) = 0$. Substituting (2.1) into them, we have

$$12Q_1 = -10c_5c_0 + (-11c_5 + 3c_8)c_1 + 12(-11c_3 + c_6)c_2 + F_1(c_3, \dots, c_8),$$
$$12Q_3 = 10c_3c_0 + (11c_3 + 3c_6)c_1 + 12(11c_5 + c_8)c_2 + F_2(c_3, \dots, c_8),$$
$$24(Q_0 - Q_4) = 20c_4c_0 - 44c_4c_1 + 24c_7c_2 + F_3(c_3, \dots, c_8),$$

where F_1, F_2, F_3 are integral quadratic forms on $c_3, c_4, \dots, c_8$.

In these relations the determinant formed by the coefficients of c_0, c_1, c_2 is non-zero. Hence the remaining six elements $c_3, c_4, \dots, c_8$ generate $\boldsymbol{C}(c_0, \dots, c_8) = \boldsymbol{C}(b_0, \dots, b_8)$. The action of H on them is given by

$$^{\sigma}\begin{pmatrix} c_3 \\ c_4 \\ c_5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i & -2i & i \\ 1 & 0 & -1 \\ -i & -2i & -i \end{pmatrix}\begin{pmatrix} c_3 \\ c_4 \\ c_5 \end{pmatrix}, \quad {}^{\tau}\begin{pmatrix} c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} -i & & \\ & -1 & \\ & & i \end{pmatrix}\begin{pmatrix} c_3 \\ c_4 \\ c_5 \end{pmatrix} (i = \sqrt{-1}),$$

$$^{\sigma}\begin{pmatrix} c_6 \\ c_7 \\ c_8 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & -i & i \\ -2 & 0 & -2 \\ -i & i & i \end{pmatrix}\begin{pmatrix} c_6 \\ c_7 \\ c_8 \end{pmatrix}, \quad {}^{\tau}\begin{pmatrix} c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} -i & & \\ & 1 & \\ & & i \end{pmatrix}\begin{pmatrix} c_6 \\ c_7 \\ c_8 \end{pmatrix}.$$

Now we set six polynomials as follows:

(2.2)
$$I_2 = c_3c_5 - c_4^2,$$
$$I_3 = c_4(c_3^2 - c_5^2),$$
$$I_4 = c_3^4 + c_5^4 + 2c_4^2(6c_3c_5 + c_4^2),$$
$$J_2 = \nabla \cdot (-c_3, c_4, c_5)\, {}^t(c_6, c_7, c_8),$$
$$J_3 = \nabla \cdot (4c_4c_5, -c_3^2 + c_5^2, 4c_3c_4)\, {}^t(c_6, c_7, c_8),$$
$$J_4 = (A_1, A_2, A_3)\, {}^t(c_6, c_7, c_8),$$

where

$$
(2.3)\qquad
\begin{aligned}
\nabla &= (c_3^2+c_5^2)\{(c_3^2-c_5^2)^2-16c_4^2(c_4^2+c_3c_5)\}\,,\\
A_1 &= -c_3(c_3^2+c_5^2)+2c_5(c_3c_5+2c_4^2)\,,\\
A_2 &= -2c_4(c_3^2+c_5^2)\,,\\
A_3 &= c_5(c_3^2+c_5^2)-2c_3(c_3c_5+2c_4^2)\,.
\end{aligned}
$$

Then we see by the following that these polynomials are H-invariant.

(1) The polynomial ring $C[I_2, I_3, I_4]$ is equal to the H-invariant ring $C[c_3, c_4, c_5]^H$. Note that $\Pi_{i=2}^4 \deg I_i = 4! = |H|/2$ and $-$(identity) acts trivially on $V(8)$.

(2) In J_2 and J_3, ∇ is a basis of the alternating representation ε^- of H.

(3) $\{-c_3, c_4, c_5\}$ and $\{-4c_4c_5, -c_3^2+c_5^2, 4c_3c_4\}$ are basis of the contragradient representation of θ_3^+ which is equivalent to θ_3^+.

(4) $\{A_1, A_2, A_3\}$ is a basis of the contragradient representation of θ_3^- which is equivalent to θ_3^-.

Since the determinant

$$
\begin{vmatrix}
-c_3 & 4c_4c_5 & A_1\\
c_4 & -c_3^2+c_5^2 & A_2\\
c_5 & 4c_3c_5 & A_3
\end{vmatrix}
$$

formed from J_2, J_3, J_4 is non-zero, the six polynomials (2.2) generate the H-invariant field $C(c_3, \ldots, c_8)^H = C(b_0, \ldots, b_8)^H$.

2.2. G-invariant rational functions corresponding to (2.2)

In this subsection we give the symbolic expressions of the H-invariant polynomials (2.2).

We use the following notations in the rest of the paper:

$$
(2.4)\qquad
\begin{aligned}
&f = (a_0, \ldots, a_8)(x_0, x_1)^8\text{: the basic 8-form}\,,\\
&Q = (f, f)_6\,, \qquad t = (Q, (Q, Q)_2)_1\,, \qquad \theta = (f, t)_6\,,\\
&\square = (\theta, \theta)_2\,, \qquad j = ((t, t)_2, t)_1\,, \qquad \Delta = \alpha\delta - \beta\gamma\ (=1)\,,\\
&p = \alpha x_0 + \gamma x_1\,, \qquad q = \beta x_0 + \delta x_1\,.
\end{aligned}
$$

Then by the definition (1.6), b_j is expressed by the transvection:

$$
(2.5)\qquad b_j = (-1)^j (f, p^j q^{8-j})_8 \qquad \text{for } 0 \le j \le 8\,.
$$

By (2.5) we get the following relations.

LEMMA 2.7. *With the notation of* (2.1) *and* (2.4), *we have*

$$\lambda c_3 = (\theta, q^2)_2\,, \qquad \lambda c_4 = -(\theta, pq)_2\,, \qquad \lambda c_5 = (\theta, p^2)_2\,.$$

PROOF. Since $t = \lambda\pi = \lambda pq(p^4 - q^4)$ by (1.9), we have

$$\begin{aligned}\lambda c_3 &= \lambda(b_1 - b_5) = \lambda\{-(f, pq^7)_8 + (f, p^5q^3)_8\} \qquad \text{by (2.5)}\\ &= (f, \lambda pq^3(p^4 - q^4))_8 = (f, q^2t)_8\\ &= ((f, t)_6, q^2)_2 = (\theta, q^2)_2\,.\end{aligned}$$

The rests are proved similarly. Q.E.D.

In the construction we use the following classical formula.

LEMMA 2.8 (*Gordan series* [5, 6]). *For n-forms* F_1, F_2 *and m-forms* H_1, H_2 ($n \geq m$), *we have*

$$(F_1, F_2)_n(H_1, H_2)_m = \textstyle\sum_{i=0}^{m} \lambda_i((F_1, H_1)_i, (F_2, H_2)_i)_{n+m-2i}\,,$$

where

$$\lambda_i = \binom{n}{i}\binom{m}{i}\Big/\binom{n+m-i+1}{i}\,.$$

Apart from the explicit non-zero constants λ_i, the representation theoretic meaning of the above formula is as follows. Both of $V(n) \otimes V(n)$ and $V(m) \otimes V(m)$ contain $V(0)$ with multiplicity one and their tensor product corresponds to the left hand side. On the other hand, since

$$V(n) \otimes V(m) = V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m)\,,$$

$\{V(n) \oplus V(m)\} \otimes \{V(n) \otimes V(m)\}$ contains $V(0)$ with multiplicity $m+1$, which corresponds to $((F_1, H_1)_i, (F_2, H_2)_i)_{n+m-2i}$ $(0 \leq i \leq m)$ of the right hand side.

The following two equalities follows easily from the definition (0.2), so we shall omit the proof.

LEMMA 2.9. (1) $(p^iq^j, p^kq^h)_r = \mu\, \Delta^r p^{i+k-r}q^{j+h-r}$.

(2) $(\theta^i, \theta^j)_r = 0$ *for any odd integer r*,

$$(\theta^i, \theta^j)_{2r} = \nu \,\square^r\, \theta^{i+j-2r}\,, \quad \textit{where}$$

$$\mu = \binom{i+j}{j}^{-1}\binom{k+h}{k}^{-1} \textstyle\sum_{s=0}^{r}(-1)^s\binom{r}{s}\binom{i+j-r}{j-s}\binom{k+h-r}{k-s}\,,$$

$$\nu = (-1)^r 2^r\binom{2i}{i}^{-1}\binom{2j}{j}^{-1} \textstyle\sum_{s=0}^{2r}(-1)^s\binom{2r}{s}\binom{2i-2r}{i-s}\binom{2j-2r}{j-s}\,.$$

Since the action of H on $\{p, q\}$ is the same as on $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ of (1.5), it is known classically that the H-invariant ring $\boldsymbol{C}[p, q]^H$ is equal to $\boldsymbol{C}[A_6^2, A_8, A_6 A_{12}]$, where

$$A_6 = pq(p^4 - q^4),$$

$$A_8 = -18(A_6 A_6)_2/\Delta^2 = p^8 + 14p^4q^4 + q^8,$$

$$A_{12} = -6(A_6 A_8)_1/\Delta = p^{12} - 33p^4q^4(p^4 + q^4) + q^{12},$$

with a relation $108A_6^4 - A_8^3 + A_{12}^2 = 0$. Recall that we have put $t = \lambda A_6$ in (1.9).

LEMMA 2.10. (1) $(t, t)_6 = \Delta^6\lambda^2/3$. (2) $(t, t)_2 = -\Delta^2\lambda^2 A_8/18$. (3) $((t, t)_2, t)_1$ $(=j) = \Delta^3\lambda^3 A_{12}/108$.

PROOF. (1) follows from (1.10). By Lemma 2.9(1), we have $(p^5q, p^5q)_2 = -\Delta^2p^8/18$, $(pq^5, pq^5) = -\Delta^2q^8/18$ and $(p^5q, pq^5)_2 = (pq^5, p^5q)_2 = 7\Delta^2p^4q^4/18$. Hence $(tt)_2$ is equal to

$$\begin{aligned}\lambda^2(pq(p^4 - q^4), pq(p^4 - q^4))_2 &= -\lambda^2\Delta^2(p^8 + 14p^4q^4 + q^8)/18 \\ &= -\lambda^2\Delta^2 A_8/18\end{aligned}$$

and (2) is proved. (3) is proved similarly. Q.E.D.

Now we shall give the symbolic expressions of the H-invariant polynomials (2.2) by use of Lemmas 2.7, 2.8, 2.9 and 2.10. The G-invariants corresponding to I_2, I_3, I_4 are as follows.

LEMMA 2.11. (1) $\lambda^2 I_2 = \Delta^2\square/2$. (2) $\lambda^4 I_3 = (\theta^3, t)_6$.
(3) $\lambda^4 I_4 = -18(\theta^4, (tt)_2)_8/\Delta^2\lambda^2 - 3\Delta^4\square^2/10$.

PROOF. (1) We have

$$\begin{aligned}\lambda^2 I_2 = \lambda^2(c_3c_5 - c_4^2) &= (\theta, q^2)_2(\theta, p^2)_2 - (\theta, pq)_2^2 \\ &= ((\theta, \theta)_2, (q^2, p^2)_2)_0/3 - ((\theta, \theta)_2, (pq, pq)_2)_0/3\,.\end{aligned}$$

Since $(p^2, q^2)_2 = \Delta^2$ and $(pq, pq)_2 = -\Delta^2/2$ by Lemma 2.9(1), we get the desired equalities: $\lambda^2 I_2 = (\square/3)(3\Delta^2/2) = \Delta^2\square/2$.

(2) We have

$$\begin{aligned}\lambda^3 I_3 = \lambda^3 c_4(c_3^2 - c_5^2) &= -(\theta, pq)_2\{(\theta, q^2)_2^2 - (\theta, p^2)_2^2\} \\ &= (\theta^3, p^5q - pq^5)_6\,.\end{aligned}$$

Since $t = \lambda pq(p^4 - q^4)$, the assertion (2) holds.

(3) We eliminate c_4^2 in I_4 by $I_2 = c_3c_5 - c_4^2$;

$$I_4 = (c_3^4 + c_5^4) + 2(c_3c_5 - I_2)(7c_3c_5 - I_2)$$
$$= (c_3^4 + c_5^4 + 14c_3^2c_5^2) - 16c_3c_5I_2 + 2I_2^2 .$$

Hence

$$(*)\ \lambda^4 I_4 = (\theta^4, q^8 + p^8)_8 + 14(\theta^2, q^4)_4(\theta^2, p^4)_4 - 16(\theta, q^2)_2(\theta, p^2)_2\lambda^2 I_2 + 2(\lambda^2 I_2)^2 .$$

We calculate $(\theta^2, q^4)_4(\theta^2, p^4)_4$ and $(\theta, q^2)_2(\theta, p^2)_2$:

$$(\theta^2, q^4)_4(\theta^2, p^4)_4 = (\theta^4, q^4p^4)_8 + 12((\theta^2, \theta^2)_2(q^4, p^4)_2)_4/7 + ((\theta^2, \theta^2)_4, (q^4, p^4)_4)_0/5$$
$$= (\theta^4, q^4p^4)_8 + 12(\Box\theta^2/3, \Delta^2p^2q^2)_4/7 + (2\Box^2/3, \Delta^4)_0/5.$$

The last equality follows from Lemma 2.9: $(\theta^2, \theta^2)_2 = \Box\theta^2/3$, $(\theta^2, \theta^2)_4 = 2\Box^2/3$, $(q^4, p^4)_2 = \Delta^2p^2q^2$, $(q^4, p^4)_4 = \Delta^4$. Similarly

$$(\theta, q^2)_2(\theta, p^2)_2 = (\theta^2, q^2p^2)_4 + ((\theta, \theta)_2, (q^2, p^2)_2)_0$$
$$= (\theta^2, q^2p^2)_4 + \Box\Delta^2 .$$

Substituting them and $\lambda^2 I_2 = \Delta^2\Box/2$ into $(*)$, we get

$$\lambda^4 I_4 = (\theta^4, q^8 + p^8)_8 + 14\{(\theta^4, q^4p^4)_8 + 4\Box\Delta^2(\theta^2, p^2q^2)_4/7 + 2\Box^2\Delta^4\}$$
$$- 16\{(\theta^2, q^2p^2)_4 + \Box\Delta^2/3\}\Delta^2\Box/2 + \Delta^4\Box^2/2$$
$$= (\theta^4, q^8 + p^8 + 14p^4q^4)_8 - 3\Delta^4\Box^2/10$$
$$= -18(\theta^4, (tt)_2)_8/\Delta^2\lambda^2 - 3\Delta^4\Box^2/10 \qquad \text{by Lemma 2.10 (2)}. \qquad \text{Q.E.D.}$$

Next we calculate ∇ in (2.3).

LEMMA 2.12. $\lambda^6\nabla = 108(\theta^6, j)_{12}/\Delta^3\lambda^3$.

PROOF. We eliminate c_4^2 in ∇ by $I_2 = c_3c_5 - c_4^2$:

$$\nabla = (c_3^2 + c_5^2)\{(c_3^2 - c_5^2)^2 - 16(c_3c_5 - I_2)(2c_3c_5 - I_2)\}$$
$$(*) \qquad = (c_3^2 + c_5^2)\{(c_3^4 - 34c_3^2c_5^2 + c_5^4) + 16I_2(3c_3c_5 - I_2)\}$$
$$= \{(c_3^6 + c_5^6 - 33c_3^2c_5^2(c_3^2 + c_5^2)\} + 16I_2(c_3^2 + c_5^2)(3c_3c_5 - I_2) .$$

The following equalities hold by use of Lemmas 2.7, 2.8, 2.9 and 2.10:

$$\lambda^4 c_3^3 c_5 = (\theta^4, q^6p^2)_8 + 3\Delta^2\Box(\theta^2, q^4)_4/7 .$$

$$\lambda^4 c_3 c_5^3 = (\theta^4, q^2p^6)_8 + 3\Delta^2\Box(\theta^2, p^4)_4/7 .$$

$$\lambda^6 c_3^4 c_5^2 = (\theta^6, q^8p^4)_{12} + 8\Delta^2\Box(\theta^4, q^6p^2)_8/11 + 4\Delta^4\Box^2(\theta^2, q^4)_4/21 .$$

$$\lambda^6 c_3^2 c_5^4 = (\theta^6, q^4p^8)_{12} + 8\Delta^2\Box(\theta^4, q^2p^6)_8/11 + 4\Delta^4\Box^2(\theta^2, p^4)_4/21 .$$

Hence we have

$$\begin{aligned}\lambda^6\{(c_3^6 + c_5^6) - 33c_3^2c_5^2(c_3^2 + c_5^2)\} = &(\theta^6, q^{12} + p^{12} - 33q^4p^4(q^4 + p^4))_{12}\\ &- 24\Delta^2\Box(\theta^4, p^2q^2(p^4 + q^4))_8\\ &- 33\cdot 4\Delta^4\Box^2(\theta^2, p^4 + q^4)_4/21 ,\end{aligned}$$

and

$$\begin{aligned}&16\lambda^6 I_2(c_3^2 + c_5^2)(3c_3c_5 - I_2)\\ &= 8\Delta^2\Box\{3\lambda^4c_3c_5(c_3^2 + c_5^2) - \lambda^2(c_3^2 + c_5^2)\Delta^2\Box/2\}\\ &= 8\Delta^2\Box\{3(\theta^4, p^2q^2(p^4 + q^4))_8 + 9\Delta^2\Box(\theta^2, p^4 + q^4)_4/7\\ &\quad - \Delta^2\Box(\theta^2, p^4 + q^4)_4/2\}\\ &= 8\Delta^2\Box\{3(\theta^4, p^2q^2(p^4 + q^4))_8 + 11\Delta^2\Box(\theta^2, p^4 + q^4)_4/14\} .\end{aligned}$$

Substituting them into (*), we get

$$\lambda^6\nabla = (\theta^6, q^{12} + p^{12} - 33q^4p^4(p^4 + q^4))_{12} .$$

Hence we have the desired result by Lemma 2.10(3). Q.E.D.

LEMMA 2.13. (1) $\lambda J_2/\nabla = 18((\theta, f)_1, (t, t)_2)_8/\Delta\lambda^2$. (2) $J_3/\nabla = -108(\theta^2 f, j)_{12}/\lambda^5\Delta^3 - 14\Delta^3((\theta^2, f)_3, t)_6/3\lambda^3$. (3) $-J_4 = -54(f\theta^3, j)_{14}/\lambda^6\Delta^2 + 135((f, \theta^3)_1, j)_{12}/7\lambda^6\Delta^2 + 140\Delta^4((f, \theta^3)_4, t)_6/33\lambda^4$.

PROOF. Since the calculations of (2) and (3) are similar to that of (1), we shall prove only the assertion (1). By the definition (2.2), the left hand side of (1) is equal to

$$\begin{aligned}&\lambda(-c_3c_6 + c_4c_7 + c_5c_8)\\ &= -(\theta, q^2)_2(b_1 + 7b_5) - (\theta, pq)_2(b_0 - b_8) + (\theta, p^2)_2(7b_3 + b_7)\\ (*)\quad &= (\theta, q^2)_2(f, pq^3(7p^4 + q^4))_8 - (\theta, pq)_2(f, q^8 - p^8)_8\\ &\quad - (\theta, p^2)_2(f, p^3q(p^4 + 7q^4))_8\\ &= (\theta f, \Xi_{10})_{10} + 8((\theta f)_1, \Xi_8)_8/5 + 7((\theta f)_2, \Xi_6)_6/9 ,\end{aligned}$$

where

$$\Xi_{10-2k} = (q^2, pq^3(7p^4 + q^4))_k - (pq, q^8 - p^8)_k - (p^2, p^3q(p^4 + 7q^4))_k$$

is a covariant of order $10 - 2k$ for $k = 0, 1, 2$. It is easy to see that $\Xi_{10} = 0$. By Lemma 2.9(1) we have $(q^2, p^5q^3)_1 = -5\Delta p^4q^4/8$, $(q^2, pq^7)_1 = -\Delta q^8/8$, $(pq, q^8)_1 = \Delta q^8/2$ etc. and hence

$$\begin{aligned}\Xi_8 &= (q^2, 7p^5q^3 + pq^7)_1 + (pq, q^8 - p^8)_1 - (p^2, p^7q + 7p^3q^5)_1 \\ &= -5\Delta(q^8 + 14p^4q^4 + p^8)/8 = -5\Delta/8 - 18(tt)_2/\Delta^2\lambda^2 , \\ \Xi_6 &= (q^2, 7p^5q^3 + pq^7)_2 + (pq, q^8 - p^8)_2 - (p^2, p^7q + 7p^3q^5)_2 \\ &= 7\cdot 10\Delta^2p^3q^3/28 - 7\cdot 10\Delta^2p^3q^3/28 = 0.\end{aligned}$$

Hence the assertion follows from the substitution of them into (*). Q.E.D.

By the identity (1) in Lemma 2.10, we eliminate λ in the relations obtained in Lemmas 2.11, 2.12 and 2.13. Then we have finally

THEOREM B. *The invariant field of binary octavics over* $\boldsymbol{C}$ *is generated by the following six algebraically independent rational functions* $I_2, \dots, J_4$:

$$I_2 = (\theta\theta)_2/M , \qquad I_3 = (\theta^3, t)_6/M^2 , \qquad I_4 = (\theta^4, (tt)_2)_8/M^3 ,$$

$$J_2 = ((\theta f)_1(tt)_2)_8(\theta^6, j)_{12}/M^6 ,$$

$$J_3 = \{36(\theta^2 f, j)_{12}/M^7 + 14((\theta^2, f)_3 t)_6/9M\}(\theta^6, j)_{12}/M^5 ,$$

$$J_4 = -2(f\theta^3, t(tt)_2)_{14}/M^3 + 5((f, \theta^3)_1 j)_{12}/21M^3 + 140((f, \theta^3)_4 t)_6/297M^2 ,$$

where $f = a_x^8$ (*the basic form*), $Q = (ff)_6$ (deg 2, ord 4), $t = ((QQ)_2Q)_1$ (deg 6, ord 6), $\theta = (f, t)_6$ (deg 7, ord 2), $M = (tt)_6$ (deg 12, *invariant*) $j = ((tt)_2t)_1$ (deg 18, ord 12).

REMARK. The denominator $M = (tt)_6$ in Theorem B coincides with the discriminant of the binary quartic $Q = (ff)_6$. Hence, in the notation in [8], the denominator $M = (tt)_6$ is equal to

$$(I_k^3 - 6J_k^2)/24 = (J_4^3 - 6J_6^2)/24 .$$

But it seems to be so complicated to express the numerators in Theorem B by the nine fundamental invariant polynomials listed in [8].

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References

[1] F. A. Bogomolov, Rationality of the moduli of hyperelliptic curves of arbitrary genus, Preprint.

[2] F. A. Bogomolov and P. I. Katsylo, Rationality of some quotient varieties, Math. USSR. Sbornik **54**(1986), 571–576.

[3] E. B. Elliott, *An introduction to the algebra of quantics*, 2nd ed. Oxford Press, Oxford, 1913.

[4] von. Gall, Das vollständige Formensystem einer binären Form achter ordnung, Math. Ann. **17**(1880), 31–51, 139–152, 456.

[5] J. H. Grace and A. Young, *The algebra of invariants*, Cambridge Univ. Press, Cambridge, 1903.

[6] P. Gordan, *Vorlesungen über Invariantentheorie*, (Binäre Formen), Leipzig, 1885.

[7] P. I. Katsylo, The rationality of moduli spaces of hyperelliptic curves, Math. USSR. Izv. **25** (1985), 45–50.

[8] T. Shioda, On the graded ring of invariants of binary octavics, Amer. J. Math. **89**(1967), 1022–1046.

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