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A note on the Selberg zeta function for compact quotients of hyperbolic spaces

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0. Introduction

In the previous paper [18], we have worked out a new method of the analytic continuation of the logarithmic derivative of Selberg's zeta function for the compact Riemannian surfaces X. In the present note, we consider the generalization of those results when X is a certain compact quotient of hyperbolic space $\tilde{X}$.

Now, let G be a connected component of the isometric transformation group of $\tilde{X}$. Let K be a maximal compact subgroup of G. Then $\tilde{X}$ is realized by G/K, a noncompact symmetric space of rank one. Let Γ be a discrete torsion-free subgroup of G such that $\Gamma\backslash G$ is compact. Thus our manifold X treating throughout this note is given by $\Gamma\backslash G/K$ for some such Γ. Let T be a finite-dimensional unitary representation of Γ, and let χ be its character. In the monumental work [15], A. Selberg constructed a function of complex variable $Z_\Gamma(s, \chi)$, which is called Selberg's zeta function, and showed how the location and the order of its zeroes gave us information about the spectrum of the Laplacian for X on the one hand and about the topology of X on the other hand. Furthermore, in the famous paper [4], R. Gangolli extended these results in the more general situation, that is to say, in the present case.

The zeta function of Selberg's type is given by the infinite product which converges absolutely for some half plane, say $\Re s > 2\rho_0$, where $\Re s$ stands for the real part of $s \in \boldsymbol{C}$. Here, the number ρ_0 is a positive real one depending only on $\tilde{X}$ and the product is taken over all primitive conjugacy classes in Γ and a certain semi-lattice of roots with respect to the Cartan subgroup of G. Roughly speaking, this zeta function has the following properties:

(A) $Z_\Gamma(s, \chi)$ is holomorphic in a half plane $\Re s > 2\rho_0$ and has a meromorphic continuation to the whole complex plane.

(B) $Z_\Gamma(s, \chi)$ satisfies the functional equation

$$Z_\Gamma(2\rho_0 - s, \chi) = \left\{\exp(\chi(1)\mathscr{V}(X)\int_0^{s-\rho_0} \mu(iz)dz\right\} Z_\Gamma(s, \chi).$$

Here, $\mathscr{V}(X)$ is the volume of X in a suitable normalization and $\mu(z)dz$ is the Plancherel measure of $\tilde{X}$.

(C) $Z_\Gamma(s, \chi)$ always has certain zeroes. These zeroes are called spectral because their location and order gives us the spectral information of X. Furthermore, these zeroes lie on the line $\Re s = \rho_0$ except for a finite number. Those zeroes which are off this line are all real and lie in the interval $[0, 2\rho_0]$, symmetrically about ρ_0.

(D) Apart from the spectral zeroes of $Z_\Gamma(s, \chi)$, there may exist a series of "trivial" zeroes or poles of $Z_\Gamma(s, \chi)$. These exist only when $\dim(X)$ is even. We are able to know these location and order (resp. residues) precisely.

Of course, above properties of the zeta function correspond to those of the logarithmic derivative of it. Also, these properties always derived from the Selberg trace formula. Therefore the choice of the test function which we put into the trace formula is the most important matter. The choice of the test function taken by Selberg and Gangolli, and hence the meromorphic continuation of the logarithmic derivative of the zeta function studied by them are, of course, important, but in view of the proof of the functional equation it is somewhat troublesome. More direct method is discovered by Hejhal in the special case, that is, in the case of the compact Riemannian space [8]. But the generalization with this method in higher dimensional cases seems to pose a delicate problem, because the several higher derivatives of the logarithmic derivative of the zeta function appeared at a stroke in the formula, which is expected to give us the meromorphic continuation of it.

The main purpose of this note is to prove quite another interesting formula for the logarithmic derivative of the zeta function. The properties possessed by the zeta function can be also derived from our foumula immediately. Our choice of the test function is closely related to the density function of the Plancherel measure of $\tilde{X}$. The proof of this formula, however, is more or less elementary than that of the special case discussed in [18]. Moreover, as the application, we will show a certain asymptotic formula with respect to the spectrum of Laplacian acting on the L^2-sections of the homogeneous vector bundle associated with the unitary representation T of Γ.

The author wishes to his gratitude to Professor M. Hashizume for valuable advices and comments in this presentation. In particular, the author declares that the argument to which he refers, in Section 3, is much indebted to Hashizume's idea discussed in [7]. The author also regrets that the paper [7] is written in Japanese only.

1. Preliminaries

Let G be a connected noncompact semisimple Lie group with finite center,

K a maximal compact subgroup. Let $\mathfrak{g}$, $\mathfrak{k}$ be their respective Lie algebras and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with respect to the Cartan involution θ determined by $\mathfrak{k}$. Let $\mathfrak{a}_{\mathfrak{p}}$ be a maximal abelian subspace of $\mathfrak{p}$.

We assume that the real rank of G is one, namely $\dim\mathfrak{a}_{\mathfrak{p}}=1$. Then $\tilde{X}=G/K$ is a noncompact symmetric space of rank one.

We put $A_{\mathfrak{p}}=\exp\mathfrak{a}_{\mathfrak{p}}$. Let M and M^* be the centralizer and normalizer of $A_{\mathfrak{p}}$ in K, respectively. Let $W=M^*/M$ be the Weyl group of $(\mathfrak{g},\mathfrak{a}_{\mathfrak{p}})$. It is clear that $[W]=2$. Extend $\mathfrak{a}_{\mathfrak{p}}$ to maximal abelian θ-stable subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, so that $\mathfrak{a}=\mathfrak{a}_{\mathfrak{k}}+\mathfrak{a}_{\mathfrak{p}}$, with $\mathfrak{a}_{\mathfrak{k}}=\mathfrak{a}\cap\mathfrak{k}$. Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Further, we denoe by $A=A_{\mathfrak{k}}A_{\mathfrak{p}}$ the centralizer of $\mathfrak{a}$ in G, that is, the Cartan subgroup of G.

For any subspace $\mathfrak{l}$ of $\mathfrak{g}$, we denote by $\mathfrak{l}_{\boldsymbol{C}}$ the complexification of $\mathfrak{l}$. Let $\Delta=\Delta(\mathfrak{g}_{\boldsymbol{C}},\mathfrak{a}_{\boldsymbol{C}})$ denote the set of roots of $(\mathfrak{g}_{\boldsymbol{C}},\mathfrak{a}_{\boldsymbol{C}})$, and introducing, as usual, compatible orders on the dual spaces of $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}_{\mathfrak{p}}+i\mathfrak{a}_{\mathfrak{k}}$. Let Δ_+ be a set of positive roots under this order. Let $P_+=\{\alpha\in\Delta_+;\alpha\not\equiv 0\ \mathit{on}\ \mathfrak{a}_{\mathfrak{p}}\}$. Put $\rho=\frac{1}{2}\sum_{\alpha\in P_+}\alpha$.

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}_{\mathfrak{p}}+\mathfrak{n}$, $G=KA_{\mathfrak{p}}N$ be the Iwasawa decompositions corresponding to these order, of course, $N=\exp\mathfrak{n}$. Let Σ be the set of restrictions to $\mathfrak{a}_{\mathfrak{p}}$ of the element of P_+. Then one can find an element $\lambda\in\Sigma$ such that 2λ is the only other possible element of Σ. Let p (resp. q) be the number of elements of P_+ which restrict to λ (resp. 2λ). Choose $H_0\in\mathfrak{a}_{\mathfrak{p}}$ so that $\lambda(H_0)=1$. We denote by ρ_0 the number $\rho(H_0)$ throughout the paper. Also, we see that the dimension of $\tilde{X}$, say d, is equal to $p+q+1$. Let $\mathfrak{a}^*_{\mathfrak{p},\boldsymbol{C}}$ be the dual space of $\mathfrak{a}_{\mathfrak{p},\boldsymbol{C}}$. Since $\dim\mathfrak{a}_{\mathfrak{p}}=1$, we have $\mathfrak{a}^*_{\mathfrak{p}}\simeq\boldsymbol{R}$, $\mathfrak{a}^*_{\mathfrak{p},\boldsymbol{C}}\simeq\boldsymbol{C}$, and we fix the following identification of $\mathfrak{a}^*_{\mathfrak{p},\boldsymbol{C}}$ with $\boldsymbol{C}$ for future use: Namely $s\in\boldsymbol{C}$ shall correspond to $s\lambda\in\mathfrak{a}^*_{\mathfrak{p},\boldsymbol{C}}$.

For every $x\in G$, let $H(x)\in\mathfrak{a}_{\mathfrak{p}}$ be defined by $x=k\exp H(x)n$, $k\in K$, $n\in N$, according to the Iwasawa decomposition.

For any $a\in A_{\mathfrak{p}}$, we put $t=\lambda(\log a)$. Then we have $a=a_t$, where we put $a_t=\exp tH_0$. We normalize the Haar measure da on $A_{\mathfrak{p}}$ by the formula $da_t=dt$, where dt stands for the Lebesgue measure on $\boldsymbol{R}$. We also fix the normalized Haar measure dk on K, and the Haar measure dn on N normalized by the condition $\int_N\exp(-2\rho(H(\theta(n^{-1}))))dn=1$. Having fixed the above measures on K, $A_{\mathfrak{p}}$, N, we normalize the Haar measure dx on G by $dx=e^{2\rho_0 t}dkdtdn$, if $x=ka_tn$. These normalization will be adhered to throughout this paper.

We denote by $C_c^\infty(K\backslash G/K)$, (resp. $C^p(K\backslash G/K)$) the space of smooth compactly supported (resp. p-times differentiable) K-biinvariant functions on G.

For each $\nu\in\mathfrak{a}^*_{\mathfrak{p},\boldsymbol{C}}$, we denote by ϕ_ν the zonal spherical function, defined by

$$\phi_\nu(x)=\int_K\exp(i\nu-\rho)(H(xk))dk,\quad x\in G.$$

This ϕ_ν is characterized by the unique (K-biinvariant) solution of the differential equation

$$\Delta\phi_\nu + (\nu^2 + \rho_0^2)\phi_\nu = 0$$

with the initial condition $\phi_\nu(1) = 1$, where Δ is the Laplacian on $\tilde{X}$.

Now we have the spherical Fourier transform $\hat{f}$ and the Abel-Selberg-Harish-Chandra transform F_f defined by

$$\hat{f}(\nu) = \int_G f(x)\phi_\nu(x)dx$$

$$F_f(a_t) = e^{\rho_0 t}\int_N f(a_t n)dn$$

for any $f \in C_c^\infty(K\backslash G/K)$. The following relation is well known:

$$\hat{f}(\nu) = F_f^*(\nu) = \int_{-\infty}^{\infty} F_f(a_t)e^{i\nu t}dt. \tag{1–1}$$

Here F_f^* is the usual Euclidean Fourier transform on the vector group $A_\mathfrak{p}$. Since $[W] = 2$, we have the inversion formula for the spherical Fourier transform

$$f(1) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{f}(\nu)\mu(\nu)d\nu, \tag{1–2}$$

where $\mu(\nu)$ is (the density function of) the Plancherel measure for $\tilde{X}$. The explicit formula of the Plancherel measure for each $\tilde{X}$ is collected in the subsequent section.

Let Γ be a discrete torsion-free subgroup of G such that $\Gamma\backslash G$ is compact. Fix a G-invariant measure $d\dot{x}$ on $\Gamma\backslash G$ by requiring that for any compactly supported function f on G we have

$$\int_G f(x)dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(\gamma x)d\dot{x}.$$

By assumption on Γ, each element $\gamma\in\Gamma$ is conjugate in G to an element of the Cartan subgroup A. Choose an element $h(\gamma)$ of A to which γ is conjugate, and let $h(\gamma) = m_\gamma a_\gamma (m_\gamma \in A_\mathfrak{k},\ a_\gamma \in A_\mathfrak{p})$. We further demand that $h(\gamma)$ be chosen so that a_γ lies in $A_\mathfrak{p}^+ = \exp\mathfrak{a}_\mathfrak{p}^+$, where $\mathfrak{a}_\mathfrak{p}^+$ is the positive Weyl chamber in $\mathfrak{a}_\mathfrak{p}$. We then define $u_\gamma = \lambda(\log a_\gamma)$. Of course, u_γ depends only on γ and it is essentially the length of the shortest geodesic in the free homotopy class associated to γ on the manifold X defined by $X = \Gamma\backslash\tilde{X}$. Also, m_γ is determined up to conjugacy in M [5]. We denote by $\{\gamma\}$ the conjugacy class corresponding to γ within itself and by $\{\Gamma\}$ the set of all Γ-conjugacy classes in Γ. Also we put $\{\Gamma\}'$

$=\{\Gamma\}-\{1\}$. An element $\gamma\in\Gamma$ $(\gamma\neq 1)$ is called primitive if it can not be expressed as δ^j for some integer $j>1$ and $\delta\in\Gamma$. Further, we denote by $\{\delta\}_p$ the primitive conjugacy class corresponding to the primitive element δ. It is well known that every $\gamma(\neq 1)$ is equal to a positive power of a unique primitive element δ. Hence, we define a positive integer $j(\gamma)$ by the relation $\gamma=\delta^{j(\gamma)}$. Thus we have $u_\gamma=j(\gamma)u_\delta$.

Our chief tool is the Selberg trace formula. Let T be a finite dimensional unitary representation of Γ on a vector space V, with the character χ. We denote by π the unitary representation of G induced by T. Thus π acts on the Hilbert space H consisting of functions $f\colon G\to V$ which satisfy (i) $f(\gamma x)=T(\gamma)f(x)$ and (ii) $\int_{\Gamma\backslash G}\langle f(x),f(x)\rangle\,d\dot{x}<\infty$, where $\langle\,,\,\rangle$ is the inner product on V. The action on G on H is given by the right translation. Since $\Gamma\backslash G$ is compact, π is a discrete direct sum of irreducible representations of G, occurring with finite multiplicities. Let $\{\pi_j; j\geq 0\}$ be the spherical (namely, class 1 with respect to K) representations that occur in π, and let $m_j(\chi)$ denote their multiplicities. Each π_j is completely determined by its zonal spherical function, say $\phi_{\nu_j}(\nu_j\in\boldsymbol{C})$. Since π_j is unitary, ϕ_{ν_j} is positive definite function. Therefore one knowns that $\pi_j(\Delta)=-(\nu_j^2+\rho_0^2)\leq 0$, where Δ denotes the Laplacian on $\tilde{X}$. By means of this fact, we see that ν_j is either real or purely imaginary. We choose and fix ν_j so that when it is real, we have $\nu_j\geq 0$, and when it is purely imaginary, we have $i\nu_j<0$. For technical reasons, we always let π_0 be the trivial representation of G, namely $\nu_0=i\rho_0$. Its multiplicity $m_0(\chi)$ is equal to the multiplicity of the trivial representation of Γ in T. Thus $m_0(\chi)$ may be zero.

Let $L^1(K\backslash G/K)$ be the convolution algebra of K-biinvariant integrable functions on G. For $f\in L^1(K\backslash G/K)$, the operator $\pi(f)=\int_G f(x)\pi(x)dx$ is a bounded operator on H. As in [15] [6], we say that f is admissible if (i) the series $\sum_{\gamma\in\Gamma}f(y^{-1}\gamma x)T(\gamma)$ converges absolutely, uniformly on any compact subset of $G\times G$, to a continuous End(V)-valued function $K_f(x,y,T)$ and (ii) the operator $\pi(f)$ is of trace class. When f is admissible, we have the trace formula

$$\sum_{j=0}^{\infty}m_j(\chi)\hat{f}(\nu_j)=\int_{\Gamma\backslash G}\operatorname{Trace}K_f(x,x,T)d\dot{x}. \tag{1–3}$$

Since the total mass of K is equal to 1, it is clear that the volume of $\Gamma\backslash G$ is equal to $\mathscr{V}(X)$. Thus, as in [15], we can rewrite the right side of (1–3) to get the final form of the trace formula:

$$\sum_{j=0}^{\infty}m_j(\chi)\hat{f}(\nu_j)=\chi(1)\mathscr{V}(X)f(1)+\sum_{\gamma\in\{\Gamma\}'}\chi(\gamma)u_\gamma j(\gamma)^{-1}C(h(\gamma))F_f(a_\gamma). \tag{1–4}$$

Here the number $C(h(\gamma))$ is the positive one and is explicitly give by

$$C(h(\gamma))=e^{-\rho_0 u_\gamma}\prod_{\alpha\in P_+}(1-\xi_\alpha(h(\gamma))^{-1})^{-1},$$

where ξ_α stands for the character of A defined by $\xi_\alpha(h) = \exp \alpha(\log h)$ for any $\alpha \in P_+$. It is clear that the number $C(h(\gamma))F_f(a_\gamma)$ depends only on the G-conjugacy class of each γ.

2. Main results

At the first place, we prepare several notations. We enumerate the roots in P_+ as $\alpha_1, \dots, \alpha_t$. Let Λ be the semi lattice in $\mathfrak{a}_C^*$ defined by $\Lambda = \{\sum_{i=1}^t m_i\alpha_i;\ m_i \geq 0, m_i \in \boldsymbol{Z}\}$. For $\lambda \in \Lambda$, define m_λ to be the number of distinct ordered t-tuples $(m_1, \dots, m_t)$ such that $\lambda = \sum_{i=1}^t m_i\alpha_i$. For such $\lambda \in \Lambda$, we denote by ξ_λ the character of Cartan subgroup A defined by $\xi_\lambda = \prod_{t=1}^t \xi_{\alpha_i}^{m_i}$.

With these notations, the Selberg zeta function is given by

$$Z_\Gamma(s, \chi) = \prod_{\{\delta\}} \prod_{\lambda \in \Lambda} (\det(I - T(\delta)\xi_\lambda(h(\delta))^{-1} e^{-su_\delta}))^{m_\lambda}$$

where the outer product is taken over all primitive conjugacy classes in Γ. Moreover the product converges absolutely, if $\Re s > 2\rho_0$. The reason for denoting the infinite product by $Z_\Gamma(s, \chi)$ instead of $Z_\Gamma(s, T)$ is that we can, in fact, easily show that this product depends only on the character χ.

Let $\Psi_\Gamma(s, \chi)$ be the logarithmic derivative of the Selberg zeta function. Namely,

$$\Psi_\Gamma(s, \chi) = \frac{d}{ds} \log Z_\Gamma(s, \chi).$$

Then, by the straightforward computation we obtain

$$\begin{aligned}\Psi_\Gamma(s, \chi) &= \sum_{\{\delta\}} \sum_{\lambda \in \Lambda} \sum_{j \geq 1} u_\delta m_\lambda \chi(\delta^j)\xi_\lambda(h(\delta))^{-j} e^{-sju_\delta} \\ &= \sum_{\gamma \in \{\Gamma\}'} \chi(\gamma)u_\gamma j(\gamma)^{-1} C(h(\gamma)) e^{(\rho_0 - s)u_\gamma}.\end{aligned}$$

It is clear that this infinite series converges absolutely and uniformly in any half plane $\Re s > 2\rho_0 + \varepsilon (\varepsilon > 0)$.

In the paper [4], $\Psi_\Gamma(s, \chi)$ was studied with the trace formula by Gangolli. He found its pole in terms of the spectrum of the Laplacian and he proved the residues are all rational with common denominator, say κ. But, in [3], Fried pointed out that one may take $\kappa = 1$. This is the reason for the absence of κ in our definition of the Selberg zeta function.

Now we will go to our situation, that is to say, we will describe hyperbolic spaces which we shall treat. So, we need the following complete list of the irreducible noncompact Riemannian symmetric spaces $\tilde{X} = H(\boldsymbol{F})$ of rank one and their Plancherel measures $\mu(r)$, [11][21]:

$H(\boldsymbol{R})_o = SO_0(2n+1, 1)/SO(2n+1)$, $(n \geq 1, \rho_0 = n)$

$(2n+1)$-dimensional real hyperbolic space

$$\mu(r) = \frac{\pi}{2^{4n-2}\Gamma(n+\frac{1}{2})^2}\prod_{k=0}^{n-1}[r^2+k^2],$$

$H(\boldsymbol{R})_e = SO_0(2n, 1)/SO(2n)$, $\left(n \geq 1, \rho_0 = n - \dfrac{1}{2}\right)$

$2n$ – dimensional real hyperbolic space

$$\mu(r) = \frac{\pi}{2^{4n-4}\Gamma(n)^2}r\prod_{k=0}^{n-2}\left[r^2 + \left(k + \frac{1}{2}\right)^2\right]\tanh \pi r,$$

$H(\boldsymbol{C}) = SU(n, 1)/S(U(n) \times U(1))$, $(n \geq 2, \rho_0 = n)$

$2n$ – dimensional complex hyperbolic space

$$\mu(r) = \frac{\pi}{2^{8m-3}\Gamma(2m)^2}r^3\prod_{k=1}^{m-1}[r^2+(2k)^2]^2\coth\frac{\pi r}{2} \qquad (n = 2m)$$

$$\mu(r) = \frac{\pi}{2^{8m+1}\Gamma(2m+1)^2}r\prod_{k=1}^{m}[r^2+(2k-1)^2]^2\tanh\frac{\pi r}{2} \qquad (n = 2m+1),$$

$H(\boldsymbol{H}) = Sp(n, 1)/Sp(n) \times Sp(1)$, $(n \geq 2, \rho_0 = 2n+1)$

$4n$ – dimensional quaternion hyperbolic space

$$\mu(r) = \frac{\pi}{2^{8n-1}\Gamma(2n)^2}r\prod_{k=1}^{n-1}[r^2+(2k-1)^2]^2[r^2+(2n-1)^2]\tanh\frac{\pi r}{2},$$

$H(\boldsymbol{O}) = (F_{4(-20)}, \mathfrak{so}(9))$, $(\rho_0 = 11)$

16 – dimensional Cayley (or, octonion) hyperbolic space

$$\mu(r) = \frac{\pi}{2^{35}\Gamma(8)^2}r(r^2+1)(r^2+3^2)\prod_{k=1}^{5}[r^2+(2k-1)^2]\tanh\frac{\pi r}{2}.$$

Moreover, we define the rational function which is closely related to the polynomial part of the Plancherel measure. Let $d = \dim \tilde{X}$. For any distinct $[\frac{d}{2}]$-tuples of numbers $a = (a_0, a_1, \dots, a_{[\frac{d}{2}]-1})$, we define them by the following formulas:

$$D(r, a) = \prod_{k=0}^{[\frac{d}{2}]-1}\frac{r^2+a_k^2}{r^2+k^2} \quad \text{if} \quad \tilde{X} = H(\boldsymbol{R})_o,\ H(\boldsymbol{R})_e \text{ and } H(\boldsymbol{C}),$$

$$D(r, a) = \frac{\prod_{k=0}^{\frac{d}{2}-2}[r^2+a_k^2]}{\prod_{k=0}^{\frac{d}{2}-1}[r^2+k^2][r^2+(\frac{d}{2})^2]} \quad \text{if} \quad \tilde{X} = H(\boldsymbol{H})$$

and

$$D(r, a)=\frac{\prod_{k=0}^{7}[r^2+a_k^2]}{\prod_{k=0}^{4}[r^2+k^2]\prod_{k=3}^{5}[r^2+(2k)^2]} \quad \text{if} \quad \tilde{X}=H(\boldsymbol{O}).$$

Also, we put

$$D^k(r, a)=\frac{D(r, a)}{r^2+a_k^2} \quad \left(k=0, 1, \dots, \left[\frac{d}{2}\right]-1\right).$$

Further, we will denote by $[\mu(r)]_P$ the polynomial part of $\mu(r)$. We now state the main theorem of this paper.

THEOREM. *The logarithmic derivative* $\Psi_\Gamma(s, \chi)=\frac{d}{ds}\log Z_\Gamma(s, \chi)$ *of the Selberg zeta function is holomorphic in the half plane* $\Re s>2\rho_0$ *and has the following analytic continuation as a meromorphic function on the whole complex plane*:

$$\Psi_\Gamma(s+\rho_0, \chi)+\frac{1}{2}\chi(1)\mathscr{V}(X)\mu(is)=s\mu(is)A_\Gamma(s, a, \chi).$$

Here, the even meromorphic function $A_\Gamma(s, a, \chi)$ *is given by the following way*:

$\tilde{X}=H(\boldsymbol{C}), H(\boldsymbol{H}), H(\boldsymbol{O})$:

$$\begin{aligned} A_\Gamma(s, a, \chi) &= 2D(is, a)\sum_{j=0}^{\infty}\frac{m_j(\chi)}{\mu(v_j)D(v_j, a)(s^2+v_j^2)} \\ &+\frac{4}{\pi}(-1)^{\rho_0-1}D(is, a)\sum_{N>\rho_0, N\not\equiv\rho_0 (\mathrm{mod}\, 2)}\frac{\Psi_\Gamma(N+\rho_0, \chi)}{[\mu(iN)]_P D(iN, a)(s^2-N^2)} \\ &+\frac{1}{2}\chi(1)\mathscr{V}(X)\sum_{k=0}^{\frac{d}{2}-1}\frac{D^k(is, a)}{a_k D^k(ia_k, a)}, \end{aligned}$$

where $a=(a_0, a_1, \dots, a_{\frac{d}{2}-1})$ *satisfies the conditions*:

$$a_0=\rho_0, \quad a_k\in\rho_0+2\boldsymbol{N} \quad \left(k=1, \dots, \frac{d}{2}-1\right).$$

$\tilde{X}=H(\boldsymbol{R})_e$:

$$\begin{aligned} A_\Gamma(s, a, \chi) &= 2D(is, a)\sum_{j=0}^{\infty}\frac{m_j(\chi)}{\mu(v_j)D(v_j, a)(s^2+v_j^2)} \\ &+\frac{2}{\pi}D(is, a)\sum_{N>\rho_0, N\in\boldsymbol{N}}\frac{\Psi_\Gamma(N+\rho_0, \chi)}{[\mu(iN)]_P D(iN, a)(s^2-N^2)} \end{aligned}$$

$$+\frac{1}{2}\chi(1)\mathscr{V}(X)\sum_{k=0}^{\frac{d}{2}-1}\frac{D^k(is,a)}{a_k D^k(ia_k,a)},$$

where $a=(a_0,a_1,\dots,a_{\frac{d}{2}-1})$ *satisfies the conditions*:

$$a_0=\rho_0,\quad a_k\in\rho_0+\boldsymbol{N}\quad\left(k=1,\dots,\frac{d}{2}-1\right).$$

$\tilde{X}=H(\boldsymbol{R})_o$:

$$A_\Gamma(s,a,\chi)=2D(is,a)\sum_{j=0}^{\infty}\frac{m_j(\chi)}{\mu(v_j)D(v_j,a)(s^2+v_j^2)}$$

$$-D(is,a)\sum_{k=0}^{[\frac{d}{2}]-1}\frac{\Psi_\Gamma(a_k+\rho_0,\chi)}{a_kD^k(ia_k,a)\mu(ia_k)(s^2-a_k^2)}+\frac{1}{2}\chi(1)\mathscr{V}(X)\sum_{k=0}^{[\frac{d}{2}]-1}\frac{D^k(is,a)}{a_kD^k(ia_k,a)},$$

where $a=(a_0,a_1,\dots,a_{[\frac{d}{2}]-1})$ *satisfies the conditions*:

$$a_0=\rho_0,\quad a_k>\rho_0\quad\left(k=1,\dots,\left[\frac{d}{2}\right]-1\right).$$

The following assertion is derived immediately from this theorem.

COROLLARY 1. *The function* $\Psi_\Gamma(s,\chi)$ *satisfies the functional equation*

$$\Psi_\Gamma(s+\rho_0,\chi)+\Psi_\Gamma(-s+\rho_0,\chi)+\chi(1)\mathscr{V}(X)\mu(is)=0\quad(s\in\boldsymbol{C}).$$

Moreover, the poles of $\Psi_\Gamma(s,\chi)$ *are all simple, and are as follows*:

Pole	*Residue*
$\rho_0\pm iv_j$	$m_j(\chi)\quad j\geq 1$
ρ_0+ir_k	$i\chi(1)\mathscr{V}(X)d_k\quad k\geq 1$
0	$m_0(\chi)-i\chi(1)\mathscr{V}(X)d_0$.

If 0 *occures in the spectrum, that is, if for some j, we have* $v_j=0$, *then for that j, the residue at this pole is* $2m_j(\chi)$. *Here* r_k, $(k\geq 0)$ *are the poles of* $\mu(r)$ *in the upper half plane, if any, and* d_k *stands for the residue of* $\mu(r)$ *at the pole* r_k, *and of course, these are explicitly computable numbers.*

The properties possessed by the Selberg zeta function which we have described in the introduction can be easily shown by the above results.

Moreover, we find that the following more or less interesting formula.

COROLLARY 2. *The following formula holds*

$$A_{\Gamma}(m, a, \chi) = \frac{1}{|2m|}\chi(1)\mathscr{V}(X)$$

under the conditions that m $(|m| > \rho_0)$ *satisfies* $m \equiv \rho_0 (\mathrm{mod}\, 2)$ *if* $\tilde{X} = H(\boldsymbol{C})$, $H(\boldsymbol{H})$, $H(\boldsymbol{O})$ *and if* $\tilde{X} = H(\boldsymbol{R})_e$ *then* m *is the integer.*

In their paper [5] and [19], Gangolli and Wallach respectively showed the following asymptotic formula for the spectrum of X:

$$\lim_{t\downarrow 0} t^{\frac{d}{2}} \sum_{j=0}^{\infty} m_j(\chi)e^{-t(\nu_j^2+\rho_0^2)} = c\chi(1)\mathscr{V}(X)$$

for some constant c depending only on $\tilde{X}$. Now, if we let m to ∞ in the formula described in the preceding corollary, then we have the one kind of the asymptotic formula different from the above one.

COROLLARY 3.

$$\lim_{t\downarrow 0} t \sum_{j=0}^{\infty} \frac{m_j(\chi)}{(1 + t^2\nu_j^2)\mu(\nu_j)D(\nu_j, a)} = \frac{1}{4}\chi(1)\mathscr{V}(X),$$

where $a = (a_0, a_1, \dots, a_{[\frac{d}{2}]-1})$ *satisfies the conditions described in the theorem.*

In the following sections, we devote ourselves to the proof of the theorem.

3. The general principle of the proof

We have to make clear that the kind of functions which we can put into the trace formula. For the sake of the proof, it is necessary to find that the wide class of admissible functions rather than that of [5], [6].

The following result was proved by Hashizume [7], and it is the extension of the result in [15].

PROPOSITION. *Suppose that the integer* N *satisfies the condition*: $N \geq [\frac{d}{2}] + 1$. *For any positive number* ε *we put*

$$D_\varepsilon = \{r \in \boldsymbol{C};\, |\Im r| < \rho_0 + \varepsilon\},$$

where $\Im r$ *denotes the imaginary part of* $r \in \boldsymbol{C}$. *Let* $\mathscr{A}^{N,\varepsilon}$ *be a set of all even holomorphic functions* h *on* D_ε *which satisfy the following growth condition*:

$$\sup_{r\in D_\varepsilon} |h(r)|(1 + |r|)^{-N-\varepsilon} < \infty.$$

Then there exist an inverse spherical Fourier transform f *of* h *for each* $h \in \mathscr{A}^{N,\varepsilon}$ *such that*

$$f \in C^{N-[\frac{d}{2}]-1}(K\backslash G/K) \tag{3–1}$$

and

$$|f^{(j)}(a_t)| \leq C_j(\cosh t)^{-(2\rho_0+\frac{\varepsilon}{2}+j)} \quad \left(0 \leq j \leq N - \left[\frac{d}{2}\right] - 1\right) \tag{3–2}$$

for some constants $C_j > 0$.

The proof of this proposition is follows from the explicit inversion formula of Abel-Selberg-Harish-Chandra transform. Also, using the integral formula relative to the Cartan decomposition of G [11] and the estimate (3–2), one can easily verify that f is, in fact, belong to $L^1(K\backslash G/K)$. By means of the proposition, the following assertion can be proved in the similar way that are discussed in [7].

LEMMA. *For any* $h \in \mathscr{A}^{N,\varepsilon}$, *let* f *be the* K*-biinvariant function on* G *such that* $\hat{f}(\nu) = h(\nu)$. *Then the series*

$$\sum_{\gamma \in \Gamma} f(y^{-1}\gamma x) T(\gamma)$$

converges absolutely, uniformly on any compact subset of $G \times G$, *to a continuous* $\mathrm{End}(V)$*-valued function.*

On the other hand, it is well known that (see, for example, [4][19]) the series

$$\sum_{\nu_j \neq 0} \frac{m_j(\chi)}{|\nu_j|^s} \tag{3–3}$$

converges if $\Re s > d$. Therefore, if the integer N satisfies $N > d$ then the inverse spherical Fourier transform of the elements in $\mathscr{A}^{N,\varepsilon}$ are admissible functions.

Thus, it is easy to see that the following function $F_s^*(r)$ ($\Re s > \rho_0$) is a spherical Fourier image of certain admissible function:

$H(\boldsymbol{R})_o$, $d = 2n+1$ $(n \geq 1)$:

$$F_s^*(r) = \frac{1}{\prod_{k=0}^{n}[r^2 + a_k^2][r^2 + s^2]},$$

$H(\boldsymbol{R})_e$, $d = 2n$ $(n \geq 1)$:

$$F_s^*(r) = \frac{r\prod_{k=1}^{n-1}[r^2 + k^2]}{\prod_{k=1}^{n-1}[r^2 + (k-\frac{1}{2})^2]\prod_{k=0}^{n-1}[r^2 + a_k^2][r^2 + s^2]} \coth \pi r,$$

$H(\boldsymbol{C})$, $d = 4n$ $(n \geq 1)$:

$$F_s^*(r) = \frac{r\prod_{k=0}^{n-1}[r^2 + (2k+1)^2]}{\prod_{k=0}^{n-1}[r^2 + (2k)^2]\prod_{k=0}^{2n-1}[r^2 + a_k^2][r^2 + s^2]} \tanh \frac{\pi r}{2},$$

$H(\boldsymbol{C})$, $d = 4n + 2$ $(n \geq 1)$:

$$F_s^*(r) = \frac{r\prod_{k=1}^{n}[r^2 + (2k)^2]}{\prod_{k=0}^{n-1}[r^2 + (2k+1)^2]\prod_{k=0}^{2n}[r^2 + a_k^2][r^2 + s^2]}\coth\frac{\pi r}{2},$$

$H(\boldsymbol{H})$, $d = 4n$ $(n \geq 2)$:

$$F_s^*(r) = \frac{r\prod_{k=1}^{n}[r^2 + (2k)^2]}{\prod_{k=0}^{n-1}[r^2 + (2k+1)^2]\prod_{k=0}^{2n-1}[r^2 + a_k^2][r^2 + s^2]}\coth\frac{\pi r}{2},$$

$H(\boldsymbol{O})$, $d = 16$:

$$F_s^*(r) = \frac{r\prod_{k=1}^{5}[r^2 + (2k)^2]}{\prod_{k=0}^{4}[r^2 + (2k+1)^2]\prod_{k=0}^{7}[r^2 + a_k^2][r^2 + s^2]}\coth\frac{\pi r}{2},$$

where the distinct $[\frac{d}{2}]$-tuple of numbers $a = (a_0, a_1, \dots, a_{[\frac{d}{2}]-1})$ satisfies the condition described in the theorem for all cases. If we choose the above F_s^* as the test function $\hat{f}$ of the trace formula in each case, using the well known representation of $\tanh \pi h$ (resp. $\coth \pi r$) by partial fractions, we are able to show the analytic continuation of $\Psi_\Gamma(s, \chi)$. Since each case is proved in the same manner, we will only prove the case $X = \Gamma\backslash H(\boldsymbol{C})$ $(d = 4n)$ precisely in the ensuing section.

4. The compact quotient of the complex hyperbolic space

In the first place, we review some notations. Let $G = SU(2n, 1)$, $K = S(U(2n) \times U(1))$. Then the space we now consider is of the form $X = \Gamma\backslash G/K$. In this case, we have $d = 4n$, $\rho_0 = 2n$. If we put $c_G = \pi/\{2^{8n-3}\Gamma(2n)^2\}$, then

$$\mu(r) = c_G r^3 \prod_{k=1}^{n-1}[r^2 + (2k)^2]^2 \coth\frac{\pi r}{2}.$$

By the formula (1–1) and (1–2), in terms of F_s^*, the trace formula takes the form

$$\sum_{j=0}^{\infty} m_j(\chi)F_s^*(v_j) = \frac{\chi(1)\mathscr{V}(X)}{4\pi}\int_{-\infty}^{\infty} F_s^*(r)\mu(r)dr$$
$$+ \sum_{\gamma\in\{\Gamma\}'} \varepsilon_{\chi,\gamma}\frac{1}{2\pi}\int_{-\infty}^{\infty} F_s^*(r)e^{iru_\gamma}dr, \tag{4–1}$$

where we put

$$\varepsilon_{\chi,\gamma} = \chi(\gamma)u_\gamma j(\gamma)^{-1}C(h(\gamma)).$$

Of course, for $\Re s > 2n$, both side of series converge absolutely.

For the sake of simplicity we put $s = a_{2n}$ for a while. Then it is easy to see that

$$F_s^*(r)\mu(r) = c_G \frac{\prod_{k=0}^{2n-1}[r^2+k^2]}{\prod_{k=0}^{2n}[r^2+a_k^2]}$$
$$= c_G \sum_{k=0}^{2n} \frac{\prod_{k=0}^{2n-1}[k^2-a_i^2]}{\prod_{j=0,j\neq i}^{2n}[a_j^2-a_i^2][r^2+a_i^2]}$$

Hence we have

$$\begin{aligned}\frac{1}{4\pi}&\int_{-\infty}^{\infty} F_s^*(r)\mu(r)dr \\ &= \frac{c_G}{4}\left\{\sum_{i=0}^{2n-1} \frac{\prod_{k=0}^{2n-1}[k^2-a_i^2]}{a_i\prod_{j=0,j\neq i}^{2n-1}[a_j^2-a_i^2][s^2-a_i^2]} + \frac{\prod_{k=0}^{2n-1}[k^2-s^2]}{s\prod_{j=0}^{2n-1}[a_j^2-s^2]}\right\}.\end{aligned} \tag{4–2}$$

On the other hand, using the representation of $\tanh \frac{\pi r}{2}$ by partial fractions, we obtain

$$F_s^*(r) = \frac{4}{\pi} \frac{\prod_{k=0}^{n-1}[r^2+(2k+1)^2]}{\prod_{k=1}^{n-1}[r^2+(2k)^2]\prod_{j=0}^{2n}[r^2+a_j^2]} \sum_{m=1}^{\infty} \frac{1}{r^2+(2m-1)^2}$$
$$= \frac{4}{\pi}\sum_{m=1}^{\infty}\left\{\sum_{k=1}^{n-1}\frac{A_k(m)}{r^2+(2k)^2} + \sum_{k=0}^{2n}\frac{B_k(m)}{r^2+a_k^2} + \frac{C(m)}{r^2+(2m-1)^2}\right\}.$$

Here we put

$$A_k(m) = \frac{\prod_{i=0}^{n-1}[(2i+1)^2-(2k)^2]}{\prod_{i=1,i\neq k}^{n-1}[(2i)^2-(2k)^2]\prod_{j=0}^{2n}[a_j^2-(2k)^2][(2m-1)^2-(2k)^2]},$$

$$B_k(m) = \frac{\prod_{i=0}^{n-1}[(2i-1)^2-a_k^2]}{\prod_{i=1}^{n-1}[(2i)^2-a_k^2]\prod_{j=0,j\neq k}^{2n}[a_j^2-a_k^2][(2m-1)^2-a_k^2]}$$

and

$$C(m) = \frac{\prod_{i=0}^{n-1}[(2i+1)^2-(2m-1)^2]}{\prod_{i=0}^{n-1}[(2i)^2-(2m-1)^2]\prod_{j=0}^{2n-1}[a_j^2-(2m-1)^2][s^2-(2m-1)^2]}.$$

It should be noted that $C(m) = 0$ for $m = 1, 2, \dots, n$.

Since

$$\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2-(2k)^2} = \frac{1}{2k}\tan k\pi = 0$$

we have

$$\sum_{m=1}^{\infty} A_k(m) = 0 \qquad (k = 1, 2, \dots, n-1).$$

Also, since $a_0 = \rho_0 = 2n$ and $a_k \equiv 2n \pmod 2$, $a_k > 2n$, we have

$$\sum_{m=1}^{\infty} B_k(m) = 0 \quad (k = 0, 1, 2, \dots, 2n-1).$$

Furthermore, since $a_{2n} = s$ we see that

$$\frac{4}{\pi}\sum_{m=1}^{\infty} B_{2n}(m) = \frac{\prod_{k=0}^{n-1}[(2k+1)^2 - s^2]}{s\prod_{k=1}^{n-1}[(2k)^2 - s^2]\prod_{j=0}^{2n-1}[a_j^2 - s^2]} \tan\frac{\pi s}{2}.$$

Since the series $\sum_{m=1}^{\infty} C(m)$ converges absolutely, uniformly for s in any compact set disjoint from the numbers $\{\pm(2m-1); m \geq n+1\}$, thanks to the Lebesgue dominated convergence theorem, we observe

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} F_s^*(r)e^{iru}dr \\
&= \frac{\prod_{k=0}^{n-1}[(2k+1)^2 - s^2]}{2s^2\prod_{k=1}^{n-1}[(2k)^2 - s^2]\prod_{j=0}^{2n-1}[a_j^2 - s^2]} \tan\frac{\pi s}{2} e^{-us} \\
&\quad + \frac{2}{\pi}\sum_{m \geq n+1} \frac{\prod_{k=1}^{n}[(2k-1)^2 - (2m-1)^2]}{(2m-1)\prod_{k=1}^{n-1}[(2k)^2 - (2m-1)^2]} \\
&\quad \times \frac{e^{-(2m-1)u}}{\prod_{k=0}^{2n-1}[a_k^2 - (2m-1)^2][s^2 - (2m-1)^2]},
\end{aligned} \tag{4-3}$$

for positive u. Here we use the elementary integral formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ixu}}{x^2 + s^2} dx = \frac{1}{2|s|} e^{-su}.$$

These manipulation are valid if s is equal to neither odd integers nor $a_j (j = 0, 1, \dots, 2n-1)$ and satisfies the condition $\Re s > 2n$.

Since

$$\Psi_\Gamma(s + \rho_0, \chi) = \sum_{\gamma \in \{\Gamma\}'} \varepsilon_{\chi,\gamma} e^{-su_\gamma}$$

for $\Re s > \rho_0 = 2n$, by the trace formula (4–1), (4–2) and (4–3) we get

$$\begin{aligned}
&c_G \sum_{j=0}^{\infty} \frac{m_j(\chi)}{\mu(v_j)} \frac{\prod_{k=0}^{2n-1}[v_j^2 + k^2]}{\prod_{k=0}^{2n-1}[v_j^2 + a_k^2][v_j^2 + s^2]} \\
&= \frac{\chi(1)\mathscr{V}(X)c_G}{4}\left\{\sum_{i=0}^{2n-1} \frac{\prod_{k=0}^{2n-1}[k^2 - a_i^2]}{a_i\prod_{j=0, j\neq i}^{2n-1}[a_j^2 - a_i^2][s^2 - a_i^2]} + \frac{1}{s}\prod_{k=0}^{2n-1}\frac{k^2 - s^2}{a_j^2 - s^2}\right\}
\end{aligned}$$

$$+\frac{\prod_{k=0}^{n-1}[(2k+1)^2-s^2]\tan\frac{\pi s}{2}\,\Psi_\Gamma(s+\rho_0,\chi)}{2s^2\prod_{k=1}^{n-1}[(2k)^2-s^2]\prod_{j=0}^{2n-1}[a_j^2-s^2]}$$

$$+\frac{2}{\pi}\sum_{m\geq n+1}\frac{\prod_{k=1}^{n}[(2k-1)^2-(2m-1)^2]\,\Psi_\Gamma(2m-1+\rho_0,\chi)}{(2m-1)\prod_{k=1}^{n-1}[(2k)^2-(2m-1)^2]\prod_{k=0}^{2n-1}[a_k^2-(2m-1)^2]}$$

$$\times\frac{1}{[s^2-(2m-1)^2]}$$

The procedure can be justified by the following argument: It is known that the numbers $\{u_\gamma;\gamma\in\{\Gamma\}'\}$ are bounded away from zero. If we choose a positive number ε_0 so small that it is smaller than those numbers, then it is easy to see that

$$|\Psi_\Gamma(2m-1+\rho_0,\chi)|\leq\chi(1)\Psi_\Gamma(2\rho_0+\varepsilon_0,\hat{1})$$

for $m\geq n+1$, where $\hat{1}$ denotes the trivial character of Γ. Hence the last term of the series, in fact, converges absolutely and uniformly for s in any compact set disjoint from the numbers $\{\pm(2m-1);m\geq n+1\}$.

Moreover, since

$$D(is,a)=\prod_{k=0}^{2n-1}\frac{a_k^2-s^2}{k^2-s^2},\quad D^k(is,a)=\frac{D(is,a)}{a_k^2-s^2}$$

$$\mu(is)=-c_G s^3\prod_{k=1}^{n-1}[(2k)^2-s^2]^2\cot\frac{\pi s}{2},$$

we have

$$\sum_{i=0}^{\infty}\frac{m_j(\chi)}{\mu(v_j)D(v_j,a)(s^2+v_j^2)}$$

$$=\frac{\chi(1)\mathscr{V}(X)}{4}\sum_{k=0}^{2n-1}\frac{1}{a_kD^k(ia_k,a)(s^2-a_k^2)}+\frac{\chi(1)\mathscr{V}(X)}{4sD(is,a)}+\frac{\Psi_\Gamma(s+\rho_0,\chi)}{2s\mu(is)D(is,a)}$$

$$+\frac{2}{\pi}\sum_{m\geq n+1}\frac{\Psi_\Gamma(2m-1+\rho_0,\chi)}{[\mu(i(2m-1))]_PD(i(2m-1),a)[s^2-(2m-1)^2]}.$$

Now multiply by $2s\mu(is)D(is,a)$ to both sides and transpose the terms we have the desired formula:

$$\Psi_\Gamma(s+\rho_0,\chi)+\frac{1}{2}\chi(1)\mathscr{V}(X)\mu(is)$$

$$=s\mu(is)\{2D(is,a)\sum_{j=0}^{\infty}\frac{m_j(\chi)}{\mu(v_j)D(v_j,a)(s^2+v_j^2)}$$

$$-\frac{4D(is, a)}{\pi}\sum_{m\geq n+1}\frac{\Psi_\Gamma(2m-1+\rho_0, \chi)}{[\mu(i(2m-1))]_P D(i(2m-1), a)[s^2-(2m-1)^2]}$$
$$+\frac{\chi(1)\mathscr{V}(X)}{2}\sum_{k=0}^{2n-1}\frac{D^k(is, a)}{a_k D^k(ia_k, a)}\Bigg\}.$$

This completes the proof of the theorem in the case of the compact quotient of $4n$-dimensional complex hyperbolic space. In the event of other spaces, we can do their proofs quite similar way.

References

[1] E. D'Hoker and D. H. Phong, *On determinants of Laplacians on Riemann surfaces*, Comm. Math. Phys. **104** (1986), 537–545.

[2] D. De George, *Length spectrum for compact locally symmetric spaces of strictly negative curvature*, Ann. Sci. École Norm. Sup. **10** (1977), 133–152.

[3] D. Fried, *The zeta functions of Ruelle and Selberg I*, Ann. Sci. École Norm. Sup. **19** (1986), 491–517.

[4] R. Gangolli, *Zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one*, Illinois J. Math. **21** (1977), 1–41.

[5] ———, *On the length spactra of certain compact manifolds of negative curvature*, J. Diff. Geom. **12** (1977), 403–423.

[6] R. Gangolli and G. Warner, *On Selberg's trace formular*, J. Math. Soc. Japan **27** (1975), 328–343.

[7] M. Hashizume, *Harmonic analysis on noncompact symmetric spaces of rank one and its applications (Japanese)*, RIMS Kôkyûroku Kyoto Univ. **481** (1983), 155–177.

[8] D. Hejhal, *The Sellberg trace formula and the Riemann zeta function*, Duke Math. J. **43** (1976), 441–482.

[9] ———, "The Selberg Trace Formula for $PSL(2, \boldsymbol{R})$, Vol. 1," Lecture Notes in Math. **548**, Springer-Verlag, New York, 1976.

[10] S. Helgason, "Differential Geometry, Lie Groups, and Symmetric Spaces," Academic Press, New York, 1978.

[11] ———, "Group and Geometric Analysis," Academic Press, New York, 1984.

[12] M. Kuga, *Topological analysis and its applications in weakly symmetric Riemannian spaces (Japanese)*, *Sûgaku* **9** (1958), 166–185.

[13] R. Miatello, *The Minakshisundaram-Pleijel coefficients for the vector valued heat kernel on compact locally symmetric spaces of negative curvature*, Trans. Amer. Math. Soc. **260** (1980), 1–33.

[14] P. Sarnak, *Determinants of Laplacians*, Comm. Math. Phys. **110** (1987), 113–120.

[15] A. Selberg, *Harmonic analysis and discontinuous subgroups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.

[16] A. Voros, *Spectral functions, special functions and the Selberg zeta function*, Comm. Math. Phys. **110** (1987), 439–465.

[17] M. Wakayama, *Zeta functions of Selberg's type associated with homogeneous vector bundles*, Hiroshima Math. J. **15** (1985), 235–295.

[18] ———, *A formula for the logarithmic derivative of Selberg's zeta function*, J. Math. Soc. Japan **41** (1989), 463–471.

[19] ———, *The relation between the η-invariant and the spin representation in terms of the Selberg zeta function*, to appear in Advanced Studies in Pure Math. **21** (1991).

[20] N. Wallach, *An asymptotic formula of Gelfand and Gangolli for the spectrum of $\Gamma\backslash G$*, J. Diff. Geom. **11** (1976), 91–101.

[21] G. Warner, "Harmonic analysis on Semi-Simple Lie Groups, I, II." Springer-Verlag, New York, 1972.

[22] J. A. Wolf, "Spaces of Constant Curvature, Third Edition," Publish or Perish, Boston, 1974.

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