HIROSHIMA MATH. J.
22 (1992), 583–590

Congruences between binomial coefficients $\binom{2f}{f}$ and Fourier coefficients of certain η-products

Tsuneo ISHIKAWA

(Received May 21, 1991)
(Revised July 16, 1991)

§ 1. Introduction

Let k and l be positive integers with $(k, l) = 1$. Let p be a prime, $p \equiv l$ *mod* k and the integer f is defined by $p = kf + l$. We consider the congruences modulo p of binomial coefficients of the form $\binom{2f}{f}$. In the classical results, for $k = 4$ and $l = 1$, Gauss proved that

$$\binom{2f}{f} \equiv 2a \ mod\ p,$$

where $p = a^2 + b^2 = 4f + 1$ and $a \equiv 1$ *mod* 4. For $k = 3$ and $l = 1$, Jacobi proved that

$$\binom{2f}{f} \equiv -a \ mod\ p,$$

where $4p = a^2 + 27b^2$ and $a \equiv 1$ *mod* 3. Moreover, the number $2a$ (resp. $-a$) can be regarded as the p-th Fourier coefficient of the cusp form of CM-type associated with the Hecke character of $\mathbb{Q}(\sqrt{-1})$ (resp. $\mathbb{Q}(\sqrt{-3})$). In the recent results, for $l = 1$ and $k \leq 24$, these were studied by Hudson and Williams [4] using Jacobi sums.

In this paper, we shall prove the congruence properties between binomial coefficients $\binom{2f}{f}$ and Fourier coefficients of certain η-products:

THEOREM 1. *Let k and l be the above and put $m = 4l/k$. Write*

$$\sum_{n=1}^{\infty} \gamma_n^{(k,l)} q^n = \eta(k\tau)^2 \eta(2k\tau)^{1+m} \eta(4k\tau)^{3-3m} \eta(8k\tau)^{2m-2}.$$

where $\eta(\tau) = q^{1/24} \prod_{n=0}^{\infty} (1 - q^n)$ is the Dedekind η-function with $q = e^{2\pi i\tau}$ and

Im $\tau > 0$. *Then, for* $p \equiv l$ *mod* k *and* $p = kf + l$,

$$\binom{2f}{f} \equiv (-1)^f \gamma_p^{(k,l)} \ mod\ p.$$

For some k and l, η-products in Theorem 1 are non-holomorphic automorphic forms of weight 2, so they were not very studied for details. But we can obtain the congruence relations like Corollary 1 for the family of these functions.

Our method is an extended analogy of method in Beukers [2] and can be applied to other numbers, too. For example, we can get the congruences of $a\left(\frac{p-l}{k}\right)$ where $a(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ are the Apéry numbers appeared in the proof of irrationality of $\zeta(3)$.

§2. Proof of Theorem 1

We consider the generating function $F(t) = \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} t^n$. Since the numbers $(-1)^n \binom{2n}{n}$ satisfy the recurrence

$$(1) \qquad (n+1)(-1)^{n+1}\binom{2(n+1)}{n+1} = -2(2n+1)(-1)^n \binom{2n}{n}, \qquad n \geq 0,$$

we have

$$F(t) = (1+4t)^{-1/2}.$$

PROPOSITION. *Let* k *and* l *be positive integers with* $(k, l) = 1$ *and* $m = 4l/k$. *Write*

$$(2) \qquad \lambda(\tau) = (\eta(2k\tau)\eta(4k\tau)^{-3}\eta(8k\tau)^2)^{4/k} = \sum_{n=1}^{\infty} A_n q^n \qquad (A_1 = 1).$$

Then

$$(3) \qquad F(\lambda^k)\, d(\lambda^l) = l\{\eta(k\tau)^2 \eta(2k\tau)^{m+1} \eta(4k\tau)^{3-3m} \eta(8k\tau)^{2m-2}\} \frac{dq}{q}.$$

REMARK 1. We may use the branch of k-th roots $x^{1/k}$ so that it takes positive real values on the positive real axis, i.e., the leading coefficients $\gamma_l^{(k,l)}$ and A_1 in the η-product of Theorem 1 and Proposition are equal to 1 respectively.

Proof of Proposition. First we prove in the case of $k = 4$ and $l = 1$. We consider the following congruence modular subgroup

$$\Gamma_0(8) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \ mod\ 8 \right\}.$$

It has no elliptic elements, and a set of representatives of inequivalent cusps is $\left\{ i\infty,\ 0,\ \frac{1}{4},\ \frac{1}{2} \right\}$. $\mathbb{H}^*/\Gamma_0(8)$ is a curve of genus 0. Putting

$$t(\tau) = \eta(2\tau)^4 \eta(4\tau^{-12} \eta(8\tau)^8,$$

it is a modular function with respect to $\Gamma_0(8)$, and the values at the cusps are given by $t(i\infty) = 0$ (simple), $t(0) = \frac{1}{4}$, $t\left(\frac{1}{4}\right) = \infty$ (simple), and $t\left(\frac{1}{2}\right) = -\frac{1}{4}$. Hence $t(\tau)$ generates the function field of modular functions with respect to $\Gamma_0(8)$. Therefore we see that $F^2(t(\tau)) = \dfrac{1}{1 + 4t(\tau)}$ has a simple pole at $\tau = \frac{1}{2}$ and a simple zero at $\tau = \frac{1}{4}$. $M_k(\Gamma_0(8))$ (resp. $S_k(\Gamma_0(8))$) denote the space of modular forms (resp. cusp forms) of weight k. It is not hard to check that $t^{-1}\dfrac{dt}{d\tau}$ is in $M_2(\Gamma_0(8))$ and it has a simple zero at $\tau = 0, \frac{1}{2}$. Hence the function

$$\begin{aligned} \Psi(\tau) &= \left(\frac{1}{2\pi i}\right)^4 F^4(t(\tau)) \left(t^{-1}\frac{dt}{d\tau}\right)^4 t(\tau) \\ &= q - 8q^2 + 12q^3 - 64q^4 + 210q^5 - 96q^6 + \cdots \end{aligned} \tag{4}$$

is an element of $S_8(\Gamma_0(8))$. We choose

$$\eta(\tau)^8 \eta(2\tau)^8 = q - 8q^2 + 12q^3 - 64q^4 + 210q^5 - \cdots$$

as another form (this is an old form) in $S_8(\Gamma_0(8))$. Since $dim\ S_8(\Gamma_0(8)) = 5$, comparing with the coefficients, we have

$$\Psi(\tau) = \eta(\tau)^8 \eta(2\tau)^8. \tag{5}$$

Taking 4-th roots with Remark 1 and replacing τ by 4τ, we have

$$F(\lambda^4) d\lambda = \eta(4\tau)^2 \eta(8\tau)^2\, dq/q. \tag{6}$$

In the general case, from (4) and (5), we see

$$\begin{aligned} \Psi_{k,l}(\tau) &= \left(\frac{1}{2\pi i}\right)^k F(t(\tau))^k \left(t^{-1}\frac{dt}{d\tau}\right)^k t(\tau)^l \\ &= \eta(\tau)^{2k} \eta(2\tau)^{4l+k} \eta(4\tau)^{3k-12l} \eta(8\tau)^{8l-2k}. \end{aligned}$$

Hence our proposition forllows from taking k-th roots and replacing τ by $k\tau$.

REMARK 2. When $k=4$ and $l=1$, since the function

$$\sum_{n=1}^{\infty} \gamma_n q^n = \eta(4\tau)^2\eta(8\tau)^2$$

is the unique cusp form in $S_2(\Gamma_0(32))$, applying Beukers [2, Prop. 3] to (3), for any $m, r \in \mathbb{N}$, $m \equiv 1$ *mod* 4 and any prime $p \equiv 1$ *mod* 4, we have

$$\begin{aligned}\binom{(mp^r-1)/2}{(mp^r-1)/4)}(-1)^{(mp^r-1)/4} - \gamma_p\binom{(mp^{r-1}-1)/2}{(mp^{r-1}-1)/4}(-1)^{(mp^{r-1}-1)/4} \\ + p\binom{(mp^{r-2}-1)/2}{(mp^{r-2}-1)/4}(-1)^{(mp^{r-2}-1)/4} \equiv 0 \qquad mod\ p^r.\end{aligned}$$

These congruences are quite Atkin-Swinnerton-Dyer type associated to the elliptic curve: $y^2 = x^3 + 2x$ (see Atkin-Swinnerton-Dyer [1]).

In our case, we can not use directly the method of Beukers [2] or Stienstra-Beukers [6, Th. A9] because the non-holomorphy of η-products of the right hand of Proposition obstructs that we apply the theory of the Hecke operators to them. But the following lemma is useful.

LEMMA. *Let p be a prime and*

$$\omega(t) = \sum_{n=1}^{\infty} b_n t^{n-1}\, dt$$

be a differential form with $b_n \in \mathbb{Z}_p$. *Let* $t(u) = \sum_{n=1}^{\infty} c_n u^n$ *with* $c_n \in \mathbb{Z}_p$, c_1 *is a p-adic unit, and suppose*

$$\omega(t(u)) = \sum_{n=1}^{\infty} d_n u^{n-1}\, du.$$

Then $d_p \equiv c_1 b_p$ *mod p*.

Proof. It is clear that

$$\omega(t) - b_p t^{p-1}\, dt = t^p G_1(t)\, dt + dG_2(t), \qquad G_1(t),\ G_2(t) \in \mathbb{Z}_p[[t]].$$

It is straightforward to see that

$$t^{p-1}\, dt = c_1^p u^{p-1}\, du + u^p G_3(u)\, du, \qquad G_3(u) \in \mathbb{Z}_p[[u]].$$

Then we can write

$$\omega(t(u)) - b_p c_1^p u^{p-1}\, du = u^p G_4(u)\, du + dG_5(u), \qquad G_4(u),\, G_5(u) \in \mathbb{Z}_p[[u]]$$

Hence

$$d_p - b_p c_1^p \equiv d_p - b_p c_1 \equiv 0 \ mod\ p.$$

Now, (2) and (3) satisfy the condition of Lemma because the denominators of the coefficients of q-expansion do not divide p. Comparing with the equation

$$\frac{1}{l} F(\lambda^k) d(\lambda^l) = \sum_{n=1}^{\infty} (-1)^n \binom{2n}{n} \lambda^{kn+l-1} \, d\lambda = \sum_{n=0}^{\infty} \gamma_n^{(k,l)} q^{n-1} \, dq,$$

we have proof of our Theorem 1.

EXAMPLES. Let $k = 4$ and $l = 3$. Then

$$\sum_{n=1}^{\infty} \gamma_n^{(4,3)} q^n = \eta(4\tau)^2 \eta(8\tau)^4 \eta(16\tau)^{-6} \eta(32\tau)^4.$$

$$= q^3 - 2q^7 - 5q^{11} + 10q^{15} + 13q^{19} + \cdots.$$

If $p = 11$ then $\binom{2f}{f} = \binom{4}{2} = 6 \equiv -5 = \gamma_{11}^{(4,3)} \quad mod\ 11.$

If $p = 19$ then $\binom{2f}{f} = \binom{8}{4} = 70 \equiv 13 = \gamma_{19}^{(4,3)} \quad mod\ 19.$

This form is the non-holomorphic automorphic form of weight 2 with respect to $\Gamma_0(32)$, but we do not know about the properties of $\gamma_p^{(4,3)}$.

Let $k = 5$ and $l = 2$. Then

$$\sum_{n=1}^{\infty} \gamma_n^{(5,2)} q^n = \eta(5\tau)^2 \eta(10\tau)^{13/5} \eta(20\tau)^{-9/5} \eta(40\tau)^{6/5}.$$

$$= q^2 - 2q^7 - \frac{18}{5} q^{12} + \frac{36}{5} q^{17} + \frac{122}{25} q^{22} - \cdots.$$

If $p = 7$ then $\binom{2f}{f} = \binom{2}{1} = 2 \equiv -(-2) = (-1)\gamma_7^{(5,2)} \quad mod\ 7.$

If $p = 17$ then $\binom{2f}{f} = \binom{6}{3} = 20 \equiv -\left(\frac{36}{5}\right) = (-1)^3 \gamma_{17}^{(5,2)} \quad mod\ 17.$

The following corollary is obtained by applying the consequence of our theorem to the recurrence (1).

COROLLARY 1. *Let k, l and $\gamma_n^{(k,l)}$ be the above. Then, for $p \equiv l$ mod k,*

$$l\gamma_p^{(k,l)} \equiv -2(2l + k)\gamma_p^{(k,k+l)} \ mod\ p.$$

§ 3. Applications

We can try to apply our method to other numbers of which the generating function satisfy the differential equation of the form

$$F(\lambda(\tau)^k)\,d\lambda(\tau) = G(\tau)\frac{dq}{q}.$$

and several examples can be seen in Beukers [2] and Stienstra-Beukers [6].

Let

$$a(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad n > 0$$

be Apéry numbers with the proof of irrationality of $\zeta(3)$. Beukers [2, Prop. 1] proved that the generating function

$$A(t) = \sum_{n=0}^{\infty} a(n)t^n$$

satisfies

$$A(\lambda^2)d\lambda = \{\eta(2\tau)^4\eta(4\tau)^4 - 9\eta(6\tau)^4\eta(12\tau)^4\}\frac{dq}{q},$$

where $\lambda(\tau) = \eta(2\tau)^6\eta(4\tau)^{-6}\eta(6\tau)^{-6}\eta(12\tau)^6$. Extending of this in the same method of Proposition, we have

$$\begin{aligned} A(\lambda^k)\,d(\lambda^l) = l\{&\eta(k\tau)^{m-2}\eta(2k\tau)^{10-m}\eta(3k\tau)^{6-m}\eta(6k\tau)^{m-6} \\ &- 9\eta(k\tau)^{m-6}\eta(2k\tau)^{6-m}\eta(3k\tau)^{10-m}\eta(6k\tau)^{m-2}\}\frac{dq}{q}, \end{aligned}$$

where $\lambda(\tau) = \{\eta(k\tau)\eta(2k\tau)\eta(3k\tau)\eta(6k\tau)\}^{12/k}$ and $m = 12l/k$. Consequently, by Lemma, we have

THEOREM 2. *Let k, l be positive integers with $(k, l) = 1$ and write for $m = 12l/k$,*

$$\begin{aligned} \sum_{n=1}^{\infty} \xi_n^{(k,l)} q^n = {}&\eta(k\tau)^{m-2}\eta(2k\tau)^{10-m}\eta(3k\tau)^{6-m}\eta(6k\tau)^{m-6} \\ &- 9\eta(k\tau)^{m-6}\eta(2k\tau)^{6-m}\eta(3k\tau)^{10-m}\eta(6k\tau)^{m-2}. \end{aligned}$$

Then, for any prime $p \equiv l$ mod k,

$$a\left(\frac{p-l}{k}\right) \equiv \xi_p^{(k,l)} \qquad \text{mod } p.$$

Since the Apéry numbers $a(n)$ satisfy the recurrence

$$(n+1)^3 a(n+1) - (34n^3 + 51n^2 + 27n + 5)a(n) + n^3 a(n-1) = 0, \qquad n > 1,$$

the following corollary is an easy consequence.

COROLLARY 2. *Let k, l and $\xi_n^{(k,l)}$ be the above. Then for any prime $p \equiv l$ mod k,*

$$l^3 \xi_p^{(k,l)} + (k+l)^3 \xi_p^{(k,l+2k)} \equiv (34l^3 + 52l^2 k + 27lk^2 + 5k^3) \xi_p^{(k,l+k)} \qquad \text{mod } p.$$

We cite an another example. For the numbers $\binom{2n}{n}^2$, $n \geq 0$, Steinstra and Beukers [6] proved that the generating function

$$F_1(t) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 t^n$$

satisfies

$$F_1(\lambda^4) d\lambda = \eta(4\tau)^6 \frac{dq}{q},$$

where $\lambda(\tau) = \eta(4\tau)^2 \eta(8\tau)^{-6} \eta(16\tau)^4$. Extending of this in same method, we have

$$F_1(\lambda^k) d(\lambda^l) = l\eta(k\tau)^{m+2} \eta(2k\tau)^{6-3m} \eta(4k\tau)^{2m-8} \frac{dq}{q},$$

where $\lambda(\tau) = \{\eta(k\tau)\eta(2k\tau)^{-3}\eta(4k\tau)^2\}^{8/k}$ and $m = 8l/k$. Consequently,

THEOREM 3. *Let k, l be positive integers with $(k, l) = 1$ and write for $m = 8l/k$,*

$$\sum_{n=1}^{\infty} \alpha_n^{(k,l)} q^n = \eta(k\tau)^{m+2} \eta(2k\tau)^{6-2m} \eta(4k\tau)^{2m-8}$$

Then, for any prime $p \equiv l$ mod k and $p = kf + l$,

$$\binom{2f}{f}^2 \equiv \alpha_p^{(k,l)} \qquad \text{mod } p.$$

Combining this with Theorem 1, we can obtain the congruences of Fourier coefficients of the automorphic forms of the different weights.

COROLARY 3. *Let k, l, $\gamma_n^{(k,l)}$ and $\alpha_n^{(k,l)}$ be the above. Then, for $p \equiv l$ mod k,*

$$\alpha_p^{(k,l)} \equiv \{\gamma_p^{(k,l)}\}^2 \qquad \text{mod } p.$$

References

[1] A. O. L. Atkin and H. P. F. Swinnerton-Dyer, Modular forms on noncongruence subgroups, "Combinatorics", 1–25, Providence, Amer. Math. Soc. 1979.

[2] F. Beukers, Another congruences of the Apéry numbers, J. Number Theory **25** (1987), 201–210.

[3] T. Honda, Invariant differentials and L-function. Reciprocity law for quadratic fields and elliptic curves over $\mathbb{Q}$, Rend. Sem. Math. Univ. Padova **49** (1973), 322–335.

[4] R. H. Hudson and K. S. Williams, Binomial coefficients and Jacobi sums, Trans. Amer. Math. Soc. **281** (1984), 431–505.

[5] S. Lang, Introduction to modular forms, G. M. W. 222, Springer, 1976.

[6] J. Stienstra and F. Beukers, On the Picard-Fuchs equation and the formal Brauer group of certain elliptic $K3$ surfaces, Math. Ann. **271** (1985), 269–304.

Department of Mathematics,
Faculty of Science,
Kobe University