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Long time behaviour for a diffusion process associated with a porous medium equation

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0. Introduction

For a given real number $\alpha > 1$, let $\{X(t)\}$ be a d-dimensional diffusion process such that the distribution $P(X(t) \in dx)$ has a density $u(t, x)$ and the generator of $\{X(t)\}$ is $\mathscr{G}_t f(x) = (1/2)u(t, x)^{\alpha-1} \triangle f(x)$, where $\triangle$ is the d-dimensional Laplacian. Then the density function $u = u(t, x)$ has to satisfy

$$(0.1) \qquad (\partial u/\partial t) = (1/2) \triangle (u^{\alpha}), \qquad (t > 0,\ x \in \mathbf{R}^d)$$

in the distribution sense. The equation (0.1) is called a *porous medium equation* ([1]) and the process $\{X(t)\}$ is called a diffusion process associated with (0.1). In the preceding work ([8]), we defined a simple model of many particles flowing through a homogeneous porous medium, and constructed the process $\{X(t)\}$ as a macroscopic limit of the path of each tagged particle and the density u as the same limit of the empirical density of the set of positions of all particles. In this paper, we consider the long time behaviour of the process $\{X(t)\}$ in the following two cases.

Firstly, we consider a random scaling limit. Put

$$K(t) = \int_0^t u(s, X(s))^{\alpha-1} ds,$$

then

$$(0.2) \qquad \lim_{t\to\infty} K(t) = \infty \qquad \text{with probability } 1$$

and

$$(0.3) \qquad \lim_{t\to\infty} E[f(K(t)^{-1/2} X(t))] = \int_{\mathbf{R}^d} f(x)(2\pi)^{-d/2} \exp\{-|x|^2/2\} dx$$

for each $f \in C_b(\mathbf{R}^d \to \mathbf{R})$ (see Theorem 1 in § 1).

Secondly, we consider a non-random scaling limit. Put

$$\bar{K}(t) = E[K(t)] \quad \text{and} \quad \beta = 1/(d(\alpha-1)+2),$$

then

(0.4) $$\lim_{t\to\infty} \bar{K}(t)^{-1} \cdot t^{2\beta} = \text{const.}$$

and

(0.5) $$\lim_{t\to\infty} E[f(\bar{K}(t)^{-1/2} X(t))] = \int_{\mathbf{R}^d} f(x) g_\alpha(x) dx$$

for each $f \in C_b(\mathbf{R}^d \to \mathbf{R})$, where

(0.6) $$g_\alpha(x) = \zeta^{d/2} \gamma \{1 - |x|^2 \zeta\}_+^{1/(\alpha-1)}, \qquad (x \in \mathbf{R}^d),$$

$\zeta = (\alpha - 1)/(2\alpha + d(\alpha - 1))$, $\gamma = \{\Gamma(1/2)\}^d \Gamma(\alpha/(\alpha - 1))/\Gamma((d/2) + (\alpha/(\alpha - 1)))$

and $\{x\}_+ = \max\{x, 0\}$ (see Theorem 2 in §2).

As an application of (0.3) and (0.5), we find a random sequence such that the limit distribution of the standard normalized sum is non-Gaussian but the limit distribution of the self-normalized sum is Gaussian (see §3).

1. Random scaling limit

In this section, we show a random scaling limit (such as (0.3)) for a diffusion process associated with the following non-linear parabolic equation. Let us consider the Cauchy problem

(1.1) $$(\partial/\partial t)u = (1/2)\sum_{j=1}^d (\partial^2/\partial x_j^2)(\varphi(u)u), \qquad (t > 0,\ x \in \mathbf{R}^d),$$

(1.2) $$u(0, x) = u_0(x), \qquad (x \in \mathbf{R}^d),$$

where φ is a given function satisfying the following conditions:

(1.3) $\varphi \in C([0, \infty) \to [0, \infty)) \cap C^1((0, \infty) \to (0, \infty))$ is uniformly Hölder continuous in any finite sub-interval of $[0, \infty)$ such that $\varphi'(x) \geq 0$ for $x > 0$, and $\Phi(x) = \varphi(x) \cdot x$ is convex on $[0, \infty)$.

We assume the following conditions for the initial function $u_0(x)$:

(1.4) u_0 is a probability density function on $\mathbf{R}^d$ such that u_0 is uniformly Hölder continuous and satisfies

$$\int |x|^2 u_0(x) dx + \int \sum_{j=1}^d |(\partial/\partial x_j) u_0(x)| dx < \infty.$$

By the same arguments as stated in [7] and [8], we have the following

LEMMA 1.1. *Under* (1.3) *and* (1.4), *there exists a d-dimensional diffusion process* $\{X(t)\}$ *on* $(\Omega, \mathscr{F}, P)$ *such that the distribution* $P(X(t) \in dx)$ *has a density* $u(t, x)$ *which is the unique weak solution* (*in the distribution sense*) *of the Cauchy* *problem* (1.1)–(1.2). *And the generator of the process is*

$$\mathscr{G}_t f(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\{E[f(X(t+\varepsilon)) \mid X(t) = x] - f(x)\} = (1/2)\varphi(u(t, x)) \triangle f(x)$$

for $f \in C_b^3(\mathbf{R}^d \to \mathbf{R})$.

In this section, we fix the process $X(t)$ and the density u in the above lemma. The purpose of this section is to prove the following

THEOREM 1. *Assume* (1.3) *and* (1.4). *Put*

$$K(T) = \int_0^T \varphi(u(t, X(t)))dt,$$

then

$$\lim_{T\to\infty} K(T) = \infty \qquad \text{with probability } 1 \tag{1.5}$$

and

$$\lim_{T\to\infty} E[f(K(T)^{-1/2} X(T))] = \int_{\mathbf{R}^d} f(x)(2\pi)^{-d/2} \exp\{-|x|^2/2\} dx, \tag{1.6}$$

for each $f \in C_b(\mathbf{R}^d \to \mathbf{R})$.

First we note the martingale property of $\{X(t)\}$ as follows.

LEMMA 1.2 (= *Lemma* 5.2 *in* [8]). *Let* $\mathscr{F}_t$ *be the σ-field generated by* $\{X(s): 0 \le s \le t\}$ *and all P-null sets in* $\mathscr{F}$. *Then, for each* $f \in C_b^3([0, \infty) \times \mathbf{R}^d \to \mathbf{R})$ *and* $g \in C_b([0, \infty) \times \mathbf{R}^d \to \mathbf{R})$, *the process*

$$\left\{f(G(t), X(t)) - \int_0^t \mathscr{L}_{f,g}(s, X(s))ds : t \ge 0\right\}$$

is an $\mathscr{F}_t$*-martingale on* $(\Omega, \mathscr{F}, P)$, *where* $G(t) = \int_0^t g(s, X(s))ds$ *and*

$$\mathscr{L}_{f,g}(s, x) = g(s, x) f_t(G(s), x) + (1/2)\varphi(u(s, x)) \triangle f(G(s), x).$$

Here f_t *is the derivative of* $f(t, x)$ *with respect to the first variable* $t \in [0, \infty)$ *and* $\triangle$ *is the d-dimensional Laplacian with respect to the variable* $x \in \mathbf{R}^d$.

To prove (1.5), we prepare the following

LEMMA 1.3. *For each* $\varepsilon > 0$, *put* $\sigma_0^\varepsilon = 0$,

$$\sigma_{n+1}^\varepsilon = \inf\{t > \sigma_n^\varepsilon : |X(t) - X(\sigma_n^\varepsilon)| > \varepsilon\}$$

and

$$V_\varepsilon(t) = \sum_{n=0}^\infty |X(\sigma_{n+1}^\varepsilon \wedge t) - X(\sigma_n^\varepsilon \wedge t)|^2,$$

then

$$E[V_\varepsilon(T)/(1+K(T))^2] \le d$$

for all $T \ge 0$.

Proof. Put

$$f(t, x) = |x - X(\sigma_n^\varepsilon \wedge T)|^2/(1+t)^2 \quad \text{and} \quad G(t) = K(t) = \int_0^t \varphi(u(s, X(s)))ds.$$

By Lemma 1.2, we have

$$\begin{aligned} &E[V_\varepsilon(T)/(1+K(T))^2] \\ &\le \sum_{n=0}^\infty E[|X(\sigma_{n+1}^\varepsilon \wedge T) - X(\sigma_n^\varepsilon \wedge T)|^2/(1+K(\sigma_{n+1}^\varepsilon \wedge T))^2] \\ &= \sum_{n=0}^\infty E\left[\int_{\sigma_n^\varepsilon \wedge T}^{\sigma_{n+1}^\varepsilon \wedge T} \varphi(u(s, X(s)))\left\{-\frac{2|X(s)-X(\sigma_n^\varepsilon \wedge T)|^2}{(1+K(s))^3} + \frac{d}{(1+K(s))^2}\right\}ds\right] \\ &\le E\left[\int_0^T d \cdot \frac{\varphi(u(s, X(s)))}{(1+K(s))^2}ds\right] = d \cdot E[1 - 1/(1+K(T))] \le d. \quad \square \end{aligned}$$

The next proof is the main part of the proof of Theorem 1.

Proof of (1.5). If $P(\lim_{T\to\infty} K(T) < \infty) > 0$, then there exists a positive number M such that $P(A_M) > 0$, where $A_M = \{\omega \in \Omega: \lim_{T\to\infty} K(T) \le M\}$. Because

$$P(\{\omega \in A_M: X(t) \in B\}) \le \int_B u(t, x)dx$$

for $B \in \mathscr{B}(\mathbf{R}^d)$, there exists a function $v_M = v_M(t, x)$ such that

$$(1.7) \qquad v_M(t, x) \le u(t, x) \quad \text{and} \quad E[f(X(t))1_{A_M}] = \int f(x) v_M(t, x)dx$$

for any function $f \in L^1(\mathbf{R}^d \to \mathbf{R})$, where 1_A is the indicator function of the set A. Let σ_n^ε and $V_\varepsilon(t)$ be the same as in Lemma 1.3. Let N_ε be a N-valued process defined by

$$N_\varepsilon(t) = n \longleftrightarrow \sigma_n^\varepsilon \le t < \sigma_{n+1}^\varepsilon,$$

then $\varepsilon^2 N_\varepsilon(t) \le V_\varepsilon(t) \le \varepsilon^2\{N_\varepsilon(t) + 1\}$. By Lemma 1.3, we have

$$d \ge E\left[\frac{V_\varepsilon(t)}{(1+K(t))^2} \cdot 1_{A_M}\right] \ge E\left[\frac{\varepsilon^2 N_\varepsilon(t)}{(1+M)^2} \cdot 1_{A_M}\right], \qquad (t \ge 0),$$

which implies

$$P(\omega \in A_M : \lim_{t\to\infty} N_\varepsilon(t) = \infty) = 0.$$

Put $A_M^\varepsilon = \{\omega \in A_M : \lim_{t\to\infty} N_\varepsilon(t) < \infty\}$, then we get

$$P(A_M^\varepsilon) = P(A_M) > 0.$$

Let $B_\varepsilon(x)$ be the ε-neighborhood of a point $x \in \mathbf{R}^d$, then there exists a sequence $\{x_k\}_{k=1}^\infty$ such that $\bigcup_{k=1}^\infty B_\varepsilon(x_k) \supset \mathbf{R}^d$. For each integer m, $k \geq 1$, put $A_M^\varepsilon(m) = \{\omega \in A_M^\varepsilon : |X(t) - X(m)| < 2\varepsilon$ for all $t \geq m\}$ and $B_{m,k}^\varepsilon = \{\omega \in \Omega : X(m) \in B_\varepsilon(x_k)\}$, then we have

$$\begin{aligned} P(A_M^\varepsilon) &\leq \textstyle\sum_{m=1}^\infty P(\omega \in A_M : N_\varepsilon(t) = N_\varepsilon(m) \text{ for all } t \geq m) \\ &\leq \textstyle\sum_{m=1}^\infty P(A_M^\varepsilon(m)) \leq \sum_{m=1}^\infty \sum_{k=1}^\infty P(A_M^\varepsilon(m) \cap B_{m,k}^\varepsilon). \end{aligned}$$

Therefore there exist integers m, $k \geq 1$ such that

$$\begin{aligned} 0 &< P(A_M^\varepsilon(m) \cap B_{m,k}^\varepsilon) \leq P(\omega \in A_M : |X(t) - x_k| < 3\varepsilon \text{ for all } t \geq m) \\ &\leq E[1_{B_{3\varepsilon}(x_k)}(X(t)) \cdot 1_{A_M}] = \int_{|x-x_k|<3\varepsilon} v_M(t, x)dx \end{aligned}$$

for all $t \geq m$. By the convexity of the function $\Phi(x) = \varphi(x)x$, we have

$$\int h(x)\Phi(v_M(x))dx \Big/ \int h(x)dx \geq \Phi\left(\int h(x)v_M(x)dx \Big/ \int h(x)dx\right),$$

where $h(x) \doteq 1_{\{|x-x_k|<3\varepsilon\}}$. It follows that there exists a positive constant $\delta = \delta(m, k, \varepsilon)$ such that

$$\int_{|x-x_k|<3\varepsilon} \Phi(v_M(t, x))dx \geq \delta$$

for all $t \geq m$. By (1.7), we get

$$\begin{aligned} &\limsup_{T\to\infty} E[K(T) \cdot 1_{A_M}] = \limsup_{T\to\infty} E\left[\int_0^T \varphi(u(t, X(t)))dt \cdot 1_{A_M}\right] \\ &\geq \limsup_{T\to\infty} \int_m^T E[\varphi(v_M(t, X(t))) \cdot 1_{A_M}]dt \geq \limsup_{T\to\infty} (T-m)\delta = \infty. \end{aligned}$$

But, by the definition of A_M, we see

$$M \geq \limsup_{T\to\infty} E[K(T) \cdot 1_{A_M}].$$

It is a contradiction. □

LEMMA 1.4. *Put $\lambda(t) = K^{-1}(t)$ be the inverse of $K(t)$, then there exists a Brownian motion $\{B_\infty(t)\}$ starting at $0 \in \mathbf{R}^d$ such that $a^{-1/2}X(\lambda(at))$ converges to $B_\infty(t)$ in law as $a \to \infty$.*

PROOF. Put $B(t) = X(\lambda(t)) - X(0)$, then the process

$$\left\{ f(B(t)) - \int_0^t (1/2)\Delta f(B(s))ds : t \geq 0 \right\}$$

is an $\mathscr{F}_{\lambda(t)}$-martingale for each $f \in C_b^3(\mathbf{R}^d \to \mathbf{R})$. That is, $\{B(t)\}$ is an $\mathscr{F}_{\lambda(t)}$-Brownian motion and $X(t) = X(0) + B(K(t))$. For each $a > 0$, put $X_a(t) = a^{-1/2} X(\lambda(at))$, then

$$X_a(t) = a^{-1/2}\{X(0) + B(at)\}.$$

By checking the conditions of the weak convergence in the space C (see [3]), we see that $\{a^{-1/2}B(at)\}$ converges to a certain Brownian motion $\{B_\infty(t)\}$ (starting at $0 \in \mathbf{R}^d$) in law as $a \to \infty$. □

PROOF OF THEOREM 1. By Lemma 1.4 and Skorohod's theorem ([9]), there exist a diffusion process $\{\hat{X}(t)\}$ on a probability space $(\hat{\Omega}, \hat{P})$ and a Brownian motion $\{\hat{B}_\infty(t)\}$ on $(\hat{\Omega}, \hat{P})$ such that $\{X(t)\} \overset{\mathscr{L}}{\sim} \{\hat{X}(t)\}$, $\{B_\infty(t)\} \overset{\mathscr{L}}{\sim} \{\hat{B}_\infty(t)\}$ and

$$\lim_{a\to\infty} a^{-1/2} \hat{X}(\hat{\lambda}(at)) = \hat{B}_\infty(t)$$

with probability 1, where $\hat{\lambda}(t) = \hat{K}^{-1}(t)$ and $\hat{K}(t) = \int_0^t \varphi(u(s, \hat{X}(s)))ds$. Put $t = 1$ and $a = \hat{K}(T)$, then we get

$$\begin{aligned}
&\lim_{T\to\infty} E[f(K(T)^{-1/2}X(T))] = \lim_{T\to\infty} E[f(\hat{K}(T)^{-1/2}\hat{X}(T))] \\
&= E[\lim_{T\to\infty} f(\hat{K}(T)^{-1/2}\hat{X}(T))] = E[\lim_{a\to\infty} f(a^{-1/2}\hat{X}(\hat{\lambda}(a)))] \\
&= E[f(\hat{B}_\infty(1))] = \int_{\mathbf{R}^d} f(x)(2\pi)^{-d/2} \exp\{-|x|^2/2\}dx,
\end{aligned}$$

which implies (1.6). Thus we complete the proof of Theorem 1. □

2. Non-random scaling limit

In this section, we show the non-random scaling limit (0.5) for the diffusion process associated with the porous medium equation (0.1).

THEOREM 2. *For given $\alpha > 1$, let $\{X(t)\}$ be the d-dimensional diffusion process in Lemma* 1.1 *with $\varphi(u) = u^{\alpha-1}$ and $u = u(t, x)$ be the density of the distribution $P(X(t) \in dx)$. Put*

$$\bar{K}(t) = E\left[\int_0^t u(s, X(s))^{\alpha-1}ds\right] \quad \textit{and} \quad \beta = 1/(d(\alpha-1)+2),$$

then

(2.1) $$\lim_{t\to\infty} \bar{K}(t)^{-1}\cdot t^{2\beta} = const.$$

and

(2.2) $$\lim_{t\to\infty} E[f(\bar{K}(t)^{-1/2}X(t))] = \int f(x)g_\alpha(x)dx$$

where g_α is the function given by (0.6).

To prove this, we will use the analytic results for the porous medium equation as follows. For each A, t_0, $x_0 \in \mathbf{R}^d$, put

$$w(t, x; A, t_0, x_0) = (t+t_0)^{-d\beta}\{A - c|x - x_0|^2/(t+t_0)^{2\beta}\}_+^{1/(\alpha-1)}$$

where $c = (\alpha-1)\beta/2\alpha$, $\beta = 1/((\alpha-1)d+2)$ and $\{x\}_+ = \max\{x, 0\}$, then the function $w(t, x) = w(t, x; A, t_0, x_0)$ is a solution of the porous medium equation $w_t = \triangle(w^\alpha)$. The explicit solution $w = w(t, x)$ was discovered by Barenblatt [2]. For the general solution of the Cauchy problem of the porous medium equation, Friedman and Kamin [5] proved the following

LEMMA 2.1. (*Friedman-Kamin* [5]) *Let v_0 be a bounded continuous function on $\mathbf{R}^d$ such that $v_0 \geq 0$ and $v_0 \in L^1(\mathbf{R}^d)\cap L^2(\mathbf{R}^d)$. Let $v = v(t, x)$ be the unique weak solution of the Cauchy problem*

(2.3) $$v_t = \triangle(v^\alpha), \quad (t>0,\ x\in\mathbf{R}^d), \qquad v(0, x) = v_0(x), \quad (x\in\mathbf{R}^d),$$

then

$$\lim_{t\to\infty} t^{d\beta}|v(t, x) - w(t, x; A, 0, 0)| = 0$$

uniformly with respect to x in any set $|x| \leq Mt^\beta (M>0)$, where the positive constant A is determined by

$$\int_{\mathbf{R}^d} v_0(x)dx = \int_{\mathbf{R}^d} w(t, x; A, 0, 0)dx \qquad (t>0).$$

By using Lemma 2.1, we have the following

LEMMA 2.2.

(2.4) $$\lim_{a\to\infty} \{\bar{K}(a)\}^{-1/2\beta}\cdot a = D$$

and

(2.5) $$\lim_{a\to\infty} u(at, \bar{K}(a)^{1/2}x)\bar{K}(a)^{d/2} = u_\infty(t, x)$$

uniformly in each compact subset of $(0, \infty)\times\mathbf{R}^d$, *where*

(2.6) $$u_\infty(t, x) = \zeta^{d/2}\gamma t^{-d\beta}\{1 - |x|^2\zeta t^{-2\beta}\}_+^{1/(\alpha-1)},$$
$$D = ((\alpha-1)/\alpha)\beta\zeta^{-1/2\beta}\gamma^{\alpha-1}$$

and γ, ζ, *are the same as in* (0.6).

PROOF. Note that the density function u is the unique weak solution of the initial value problem (0.1)–(1.2). By Veron [10], there exists a positive constant C_0 such that

$$\sup_{x\in\mathbf{R}^d} u(t, x) \le C_0 t^{-d\beta} \tag{2.7}$$

for all $t \ge 0$. Note

$$\bar{K}(t) = \int_0^t ds \int_{\mathbf{R}^d} u(s, x)^\alpha dx,$$

then

$$\bar{K}(t) \le C_0^{\alpha-1}(2\beta)^{-1} t^{2\beta} \tag{2.8}$$

for all $t \ge 0$. By the comparison theorem for the porous medium equation, there exist constants A, $t_0 > 0$ and $x_0 \in \mathbf{R}^d$ such that

$$u(s, x) \ge w(s/2, x; A, t_0, x_0)$$

for a.e. $(s, x) \in (0, \infty) \times \mathbf{R}^d$, where w is the Barenblatt's explicit solution. Therefore there exists a constant $C_1 > 0$ such that

$$\bar{K}(t) \ge C_1((t + t_0)^{2\beta} - t_0^{2\beta}) \tag{2.9}$$

for all $t \ge 0$. By (2.8) and (2.9), we have

$$0 < \liminf_{t\to\infty} \{\bar{K}(t)\}^{-1/2\beta} \cdot t \le \limsup_{t\to\infty} \{\bar{K}(t)\}^{-1/2\beta} \cdot t < \infty.$$

Let $\mathscr{A}$ be the set of all increasing sequences $\{a_n\}$ satisfying $a_n \uparrow \infty$ as $n \to \infty$ and D be one of accumulating points of the set $\{\bar{K}(a_n)^{-1/2\beta} \cdot a_n : n \ge 1, \{a_n\} \in \mathscr{A}\}$: i.e.

$$\lim_{n\to\infty} \{\bar{K}(a_n)\}^{-1/2\beta} \cdot a_n = D. \tag{2.10}$$

We will show that the positive constant D is independent of the choice of the sequence $\{a_n\} \in \mathscr{A}$. Put $v(t, x) = u(2t, x)$, then v is the weak solution of (2.3) with the initial value $v(0, x) = u_0(x)$. Let B be the positive constant satisfying $\int w(t, x; B, 0, 0)dx = 1$ $(t > 0)$. Put $w(t, x) = w(t, x; B, 0, 0)$, then $w(at, a^\beta x)a^{d\beta} = w(t, x)$ for any $a > 0$. By Lemma 2.1,

$$|v(at, a^\beta x)a^{d\beta} - w(t, x)| \longrightarrow 0 \qquad (\text{as } a \longrightarrow \infty)$$

uniformly in $\{(t, x): |x| \le Mt^\beta\}$ for each $M > 0$. It follows that

$$\lim_{a\to\infty} u(at, a^\beta x)a^{d\beta} = w(t/2, x)$$

uniformly in each compact subset of $(0, \infty) \times \mathbf{R}^d$. Put $u_\infty(t, x)$

$= w(t/2, D^{-\beta}x)D^{-d\beta}$, then

(2.11) $$\lim_{n\to\infty} u(a_n t, \bar{K}(a_n)^{1/2}x)\bar{K}(a_n)^{d/2} = u_\infty(t, x),$$

(2.12) $(\partial/\partial t)u_\infty = (D/2)\triangle(u_\infty)^\alpha$ in $\mathscr{D}'((0, \infty)\times \mathbf{R}^d)$, $\lim_{t\downarrow 0} u_\infty(t, x) = \delta_0(x)$.

Because

$$\int_{\mathbf{R}^d} |x|^2 u(a, \bar{K}(a)^{1/2}x)\bar{K}(a)^{d/2}dx = \{\bar{K}(a)\}^{-1}E[|X(a)|^2] \longrightarrow d \qquad (\text{as } a \longrightarrow \infty),$$

we have

(2.13) $$\int_{\mathbf{R}^d} |x|^2 u_\infty(1, x)dx = d.$$

By the condition (2.13) and the uniqueness of the solution of (2.12) ([4]), u_∞ and D coincide the explicit form (2.6). By the uniqueness of the limits u_∞ and D, (2.10) (resp. (2.11)) yields (2.4) (resp. (2.5)). □

PROOF OF THEOREM 2. Note

(2.14) $$E[f(\bar{K}(a)^{-1/2}X(at))] = \int f(x)u(at, \bar{K}(a)^{1/2}x)\bar{K}(a)^{d/2}dx,$$

and $u_\infty(1, x) = g_\alpha(x)$, then Lemma 2.2 implies Theorem 2. □

Moreover, we can see the following

PROPOSITION 2.3. *Put $X_a(t) = \bar{K}(a)^{-1/2}X(at)$, then the process $X_a = \{X_a(t)\}$ converges in law to a diffusion process $X_\infty = \{X_\infty(t)\}$ as $a \to \infty$ such that the distribution $P(X_\infty(t)\in dx)$ has a density $u_\infty(t, x)$ and the generator of $\{X_\infty(t)\}$ is*

(2.15) $$\mathscr{G}_t^\infty f(x) = (D/2)u_\infty(t, x)^{\alpha-1}\triangle f(x)$$

for $f \in C_b^3(\mathbf{R}^d \to \mathbf{R})$. Here the density u_∞ is the function given by (2.6).

PROOF. By the martingale property, we have

(2.16) $$E[|X_a(t) - X_a(s)|^{2m}] \le (\textstyle\prod_{i=1}^m c_i)\{\bar{K}(a)\}^{-m}\left\{\displaystyle\int_{as}^{at} (\sup_x u(\theta, x)^{\alpha-1})d\theta\right\}^m$$

for each integer $m \ge 1$, where $c_i = i\{2(i-1)+d\}$. By the estimates (2.7), (2.9), (2.16) and $\int |x|^2 u_0(x)dx < \infty$, we see that the family of probability measures $\{P_{X_a}\}$ is tight. Then there exist an increase sequence $\{a_n\}$ ($a_n \uparrow \infty$ as $n \to \infty$) and a process $X_\infty = \{X_\infty(t)\}$ such that X_{a_n} converges in law to X_∞. By (2.5) and (2.14), we have

$$P(X_\infty(t)\in dx) = u_\infty(t, x)dx.$$

By (2.4), (2.5), (2.14) and the martingale property, the process

$$\left\{f(t, X_\infty(t)) - \int_0^t \{f_t(s, X_\infty(s)) + (D/2)u_\infty(s, X_\infty(s))^{\alpha-1} \triangle f(s, X_\infty(s))\} ds\right\}$$

is a martingale for each $f \in C_0^\infty((0, \infty) \times \mathbf{R}^d \to \mathbf{R})$. By the same method as the proof of Lemma 5.3 in [8], we see that $\{X_\infty(t)\}$ is a Markov process with the generator (2.15). The uniqueness of the solution of (2.12) implies the uniqueness of the distribution of the Markov process $\{X_\infty(t)\}$. Therefore any finite distribution of X_a converges to that of X_∞ as $a \to \infty$. □

3. Note for the martingale central limit theorem

Let us consider the results (0.3) and (0.5), then we can find an example of random sequence which does not satisfy the central limit theorem (cf. [6]) as follows. For fixed $\alpha > 1$, let $\{X(t)\}$ be the 1-dimensional diffusion process associated with (0.1) (= the same process as in Lemma 1.1 with $\varphi(u) = u^{\alpha-1}$ and $d = 1$). Let $\{a_n : n \geq 0\}$ be a non-random sequence satisfying

(3.1) $$a_0 = 0,\ a_n < a_{n+1} \quad \text{and} \quad \lim_{n\to\infty} a_n = \infty.$$

Put

(3.2) $$\xi_n = X(a_n) - X(a_{n-1}),$$

then $E[\xi_n] = 0$ and $E[\xi_n \cdot \xi_k] = 0$ for $n \neq k$. For the martingale-difference sequence $\{\xi_n\}$, we have the following

THEOREM 3. *Let ξ_n be the random sequence defined by* (3.2) *with the condition* (3.1). *Put*

$$\bar{S}_n = \textstyle\sum_{k=1}^n \xi_k / (\sum_{k=1}^n E[|\xi_k|^2])^{1/2} \quad \text{and} \quad S_n = \sum_{k=1}^n \xi_k / (\sum_{k=1}^n |\xi_k|^2)^{1/2},$$

then

(3.3) $$\lim_{n\to\infty} E[f(\bar{S}_n)] = \int_{\mathbf{R}} f(x) g_\alpha(x) dx$$

for each $f \in C_b(\mathbf{R} \to \mathbf{R})$, where g_α is the function given by (0.6) *with $d = 1$. If*

(3.4) $$a_n - a_{n-1} \leq n^{-r}$$

for some constant $r \in [0, 1] \cap ((3-\alpha)/4, 1]$, then

(3.5) $$\lim_{n\to\infty} E[f(S_n)] = \int_{\mathbf{R}} f(x)(2\pi)^{-1/2} \exp\{-x^2/2\} dx,$$

for each $f \in C_b(\mathbf{R} \to \mathbf{R})$.

PROOF. Put

$$B_n = \sum_{k=1}^n E[|\xi_k|^2] \quad \text{and} \quad C_n = \sum_{k=1}^n |\xi_k|^2.$$

By the martingale property, we have

$$B_n = \sum_{k=1}^n E[|X(a_k) - X(a_{k-1})|^2] = \sum_{k=1}^n E\left[\int_{a_{k-1}}^{a_k} u(s, X(s))^{\alpha-1} ds\right] = \bar{K}(a_n),$$

which implies

$$\bar{S}_n = \{X(a_n) - X(0)\}/(\bar{K}(a_n))^{1/2}.$$

By Theorem 2, we get (3.3). Next we show (3.5). By the martingale property, we have

$$\begin{aligned} E[|C_n - K(a_n)|^2] &= \sum_{k=1}^n E\left[\left||\xi_k|^2 - \int_{a_{k-1}}^{a_k} u(s, X(s))^{\alpha-1} ds\right|^2\right] \\ &= \sum_{k=1}^n E\left[\int_{a_{k-1}}^{a_k} 4u(s, X(s))^{\alpha-1} |X(s) - X(a_{k-1})|^2 ds\right] \\ &\leq \sum_{k=1}^n 2\left\{\int_{a_{k-1}}^{a_k} (\sup_{x\in\mathbf{R}} u(s, x)^{\alpha-1}) ds\right\}^2 \end{aligned}$$

By the estimate (2.7) and the assumption (3.4), there exists a positive constant M satisfying

$$E[|C_n - K(a_n)|^2] \leq M$$

for all $n \geq 1$. By (1.5), we have

(3.7) $$\lim_{n\to\infty} |(C_n/K(a_n)) - 1| = 0 \quad \text{with probability 1.}$$

By Theorem 1, (3.7) implies (3.5). □

REMARK. Let $\{a_n\}$ be the sequence defined by $a_0 = 0$ and

$$a_n = \inf\{t > a_{n-1} : \bar{K}(t) - \bar{K}(a_{n-1}) > 1\},$$

then $E[|\xi_n|^2] = 1$: i.e.

$$\bar{S}_n = \sum_{k=1}^n \xi_k/\sqrt{n}.$$

It is the standard normalized sum. But the limit distribution is $\int_{-\infty}^x g_\alpha(y)dy$, and is non-Gaussian.

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