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On the notion of multiple Markov SαS processes

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0. Introduction

In this paper we introduce a notion of multiple Markov symmetric α-stable (for short, SαS) processes.

The notion of multiple Markov property for stochastic processes with continuous parameter has been studied by many mathematicians. For example, T. Hida [1] proposed a definition of multiple Markov processes for centered Gaussian processes (SαS processes with index $\alpha = 2$). He stated his definition using the notion of conditional expectation (Definition 2.1). V. Mandrekar and B. Thelen [6] used metric projection and extended Hida's definition to SαS processes with index $\alpha > 1$ (Definition 2.2).

However, these definitions of multiple Markov processes cannot be applied directly to SαS processes with index $\alpha \leq 1$, because these processes do not have first moments. In this paper we consider an extension of these definitions, which can be also applied to SαS processes with $\alpha \leq 1$. Here we keep in mind that the notion of multiple Markov processes we will introduce should be a generalization of the notion of Markov processes in the usual sense. When we recall the definition of Markov processes, we see that the definition is stated by conditional probability and therefore it can be also applied to processes without first moments. So we start to characterize multiple Markov processes in Hida's or Mandrekar-Thelen's sense by conditional probability. After some arguments, we obtain a new definition of multiple Markov processes which can be applied to SαS processes with $0 < \alpha \leq 2$ (Definition 3.1). In fact, our definition is an extension of Hida's one (Proposition 3.3) and gives us a generalization of the definition of Markov processes in a certain sense. Moreover, similarly to Hida's arguments in Gaussian case, we obtain a theorem which states that an SαS process which has a canonical representation is multiple Markov in our sense if and only if the representation kernel is a Goursat kernel (Theorem 3.5).

1. Preliminaries

1.1. SαS processes

We take the time domain T for either $[0, \infty)$ or $(-\infty, \infty)$. In this paper, we assume that any stochastic process $X = \{X(t); t \in T\}$ is separable. For every $t \in T$, $\mathfrak{B}_t(X)$ denotes the σ-field generated by $\{X(s); s \leq t\}$.

For fixed α $(0 < \alpha \leq 2)$, a real-valued random variable X is said to be *symmetric α-stable* (for short, *SαS*) if $E[\exp(izX)] = \exp(-c|z|^\alpha)$ with a constant $c \geq 0$. A family of random variables is called an *SαS system* if any finite linear combination of elements of the family is SαS. A process $X = \{X(t); t \in T\}$ is said to be *an SαS process* if X is an SαS system as a family of random variables. An SαS process $X = \{X(t); t \in [0, \infty)\}$ is called an *SαS motion* if X has independent stationary increments and $X(0) = 0$.

Let X be an SαS random variable $(0 < \alpha \leq 2)$ such that $E[\exp(izX)] = \exp(-c|z|^\alpha)$, $c \geq 0$. Then we have $E|X|^p = C(\alpha, p)c^{p/\alpha}$ for any p $(0 < p < \alpha$ if $0 < \alpha < 2$, $0 < p \leq 2$ if $\alpha = 2)$, where $C(\alpha, p)$ is a positive constant which does not depend on the distribution of X. Let $\mathscr{X}$ be an SαS system. We can introduce a metric $d^{[\alpha]}$ into $\mathscr{X}$, defined as $d^{[\alpha]}(X, Y) = c_{Y-X}^{1 \wedge (1/\alpha)}$ where $X, Y \in \mathscr{X}$ and $E[\exp(iz(Y-X))] = \exp(-c_{Y-X}|z|^\alpha)$. The convergence in $d^{[\alpha]}$ on $\mathscr{X}$ is equivalent to the convergence in p-th moment for any $p < \alpha$ and is equivalent to the convergence in probability. Especially in the case $1 \leq p < \alpha < 2$ or $1 \leq p \leq \alpha = 2$, $d^{[\alpha]}$ is a norm which is isometric to L^p-norm up to the constant $C(\alpha, p)$. So we can consider the closure of any SαS system $(0 < \alpha \leq 2)$ in $d^{[\alpha]}$. If $X = \{X(t); t \in T\}$ is an SαS process, $\mathfrak{M}_t^\alpha(X)$ denotes the closed linear hull of $\{X(s); s \leq t\}$ in $d^{[\alpha]}$ for every $t \in T$. (For details, see M. Schilder [10].)

Let $(S, \mathfrak{B}, \mu)$ be a σ-finite measure space. An SαS system $\mathscr{Y} = \{Y(B); B \in \mathfrak{B}, \mu(B) < \infty\}$ which satisfies the conditions i) and ii) below is called an *SαS random measure controlled by* $(S, \mathfrak{B}, \mu)$.

i) $E[\exp(izY(B))] = \exp(-\mu(B)|z|^\alpha)$.

ii) If B_j, $j = 1, 2, \cdots$ are disjoint, then $Y(B_j)$, $j = 1, 2, \cdots$ are independent, and $Y(\bigcup_j B_j) = \sum_j Y(B_j)$ a.s. if $\mu(\bigcup_j B_j) < \infty$.

Let $L^{(\alpha)}(S, \mathfrak{B}, \mu)$ be the metric space $\left\{f; \int_S |f|^\alpha d\mu < \infty\right\}$ equipped with metric $d^{(\alpha)}(f, g) = \left(\int_S |f-g|^\alpha d\mu\right)^{1 \wedge (1/\alpha)}$. We define a Wiener-type stochastic integral $\int_S f dY$ of $f \in L^{(\alpha)}(S, \mathfrak{B}, \mu)$ with respect to $\mathscr{Y} = (Y(B); B \in \mathfrak{B}, \mu(B) < \infty\}$ as follows: If f is a simple function $\sum_j a_j I_{B_j}$ where $\{B_j; \mu(B_j) < \infty\}$ is a family of disjoint sets and I_B denotes the indicator function of B, then $\int_S f dY$ is defined as $\sum_j a_j Y(B_j)$. For a general f, we take a sequence of simple functions $\{f_j\}_{j=1,2,\dots}$ which converges to f in $d^{(\alpha)}$ and we define $\int_S f dY$ as the limit of the sequence $\left\{\int_S f_j dY\right\}_{j=1,2,\dots}$ in $d^{[\alpha]}$. (Schilder [10])

1.2. Canonical representations of SαS processes

Assume that an SαS process $X=\{X(t);\, t\in T\}$ has an expression as follows:

$$X(t)=\int^t F(t,u)dZ(u), \qquad \text{for every } t\in T, \tag{1.1}$$

where

i) $Z=\{Z(t);\, t\in T\}$ is an SαS process with independent increments (the control measure of Z is denoted by μ);

ii) $F(t,u)$ is a function on $T\times T$ which vanishes on $\{(t,u);\, u>t\}$ and $F(t,\cdot)\in L^{(\alpha)}(T,\mu)$ for every $t\in T$.

Here $\int^t$ means $\int_{(-\infty,t]\cap T}$. Then the expression (1.1) is said to be a *causal representation* of X. If $\mathfrak{B}_t(X)=\mathfrak{B}_t(Z)$ for every $t\in T$, the representation (1.1) is said to be *canonical*. If $\mathfrak{M}_t^\alpha(X)=\mathfrak{M}_t^\alpha(Z)$ for every $t\in T$, (1.1) is said to be *proper*. In Gaussian case ($\alpha=2$), a causal representation is canonical if and only if it is proper. On the contrary, in non-Gaussian case ($0<\alpha<2$), a proper representation is canonical, but the converse is not true. (K. Kojo [3])

2. Arguments for our new definition

2.1. Hida's and Mandrekar-Thelen's definitions

Firstly let us recall T. Hida's definition of multiple Markov Gaussian processes.

Definition 2.1 (Hida [1]). A centered Gaussian process $X=\{X(t);\, t\in T\}$ is said to be an *n-ple Markov process* if X satisfies the following two conditions:

(H1) $E(X(t_j)\,|\,\mathfrak{B}_{t_0}(X))$, $1\le j\le n$ are linearly independent for any $t_0, t_1,\cdots, t_n\in T^\circ$ ($t_0\le t_1<\cdots<t_n$).

(H2) $E(X(t_j)\,|\,\mathfrak{B}_{t_0}(X))$, $1\le j\le n+1$ are linearly dependent for any $t_0, t_1,\cdots,t_{n+1}\in T^\circ$ ($t_0\le t_1<\cdots<t_{n+1}$).

In Gaussian case, for any fixed s, t ($s\le t$), let $Y(t;s)$ be the nearest element to $X(t)$ in $\mathfrak{M}_s^2(X)$, then $Y(t;s)$ coincides with the conditional expectation $E(X(t)\,|\,\mathfrak{B}_s(X))$. Similarly, in non-Gaussian SαS case ($0<\alpha<2$), let

$Y(t;s)$ $(s \leq t)$ be (one of) the nearest element(s) to $X(t)$ in $\mathfrak{M}_s^\alpha(X)$. According to I. Singer ([11], p. 111 Chapter I. Corollary 3.5), $Y(t;s)$ always exists uniquely if the index $\alpha > 1$. V. Mandrekar and B. Thelen used this fact and extended the above definition of multiple Markov processes to SαS processes with $\alpha > 1$.

Definition 2.2 (Mandrekar and Thelen [6]). An SαS process $X = \{X(t);\ t \in T\}$ with index $\alpha > 1$ is said to be an *n-ple Markov process* if X satisfies the following two conditions:

(MT1) $Y(t_j;\ t_0)$, $1 \leq j \leq n$ are linearly independent for any $t_0, t_1, \cdots, t_n \in T^\circ$ $(t_0 \leq t_1 < \cdots < t_n)$.

(MT2) $Y(t_j;\ t_0)$, $1 \leq j \leq n+1$ are linearly dependent for any $t_0, t_1, \cdots, t_{n+1} \in T^\circ$ $(t_0 \leq t_1 < \cdots < t_{n+1})$.

This definition cannot be applied directly to SαS processes with $\alpha \leq 1$. Because $Y(t;s)$ $(s \leq t)$ does not always exist and is not always unique even if it exists.

2.2. Arguments

Our purpose is to obtain the notion of multiple Markov processes, which can be also applied to SαS processes with $\alpha \leq 1$. We accomplish our purpose with a view that the notion of multiple Markov processes we require should be a generalization of the notion of Markov processes. Let us recall the definition of Markov processes.

Definition 2.3. A stochastic process $X = \{X(t);\ t \in T\}$ is said to be *Markov* if, for any fixed s, $t\,(s < t)$, X satisfies the equation

$$P(X(t) \in B \mid \mathfrak{B}_s(X)) = P(X(t) \in B \mid X(s)) \quad \text{for any Borel set } B \text{ of } \boldsymbol{R}. \tag{2.1}$$

We emphasize that a Markov process is defined by the notion of conditional probability and is not defined by the notion of conditional expectation. Therefore the definition can be also applied to processes without first moments like SαS processes with $\alpha \leq 1$.

Example 2.4. For any fixed α $(0 < \alpha \leq 2)$, the following SαS processes are Markov.

i) $Z = \{Z(t);\ t \in T\}$ which is an SαS process with independent increments.

ii) $X = \{X(t) \equiv f(t)Z(t);\ t \in T\}$ where the function f never vanishes on T°. In fact, for any $s < t$, $X(t)$ has the following decomposition.

$$\begin{aligned} X(t) &= f(t)Z(t) = f(t)\{Z(s) + (Z(t) - Z(s))\} \\ &= f(s)^{-1}f(t)X(s) + f(t)(Z(t) - Z(s)). \end{aligned}$$

The first term is a multiple of $X(s)$ while the second term is independent of $\mathfrak{B}_s(X)$. Thus X satisfies equation (2.1).

In Gaussian case ($\alpha = 2$), these examples are simple Markov in Hida's sense if $Z(\cdot)$ never vanishes on T°. Therefore it is natural that we hope a multiple Markov Gaussian process has a property which is stated by conditional probability and which characterizes the multiple Markov property. Now let us investigate it.

EXAMPLE 2.5. For any fixed $k \in \boldsymbol{R}$, let us consider a Gaussian process $X = \{X(t); t \in [0, \infty)\}$ defined as

$$X(t) = \int_0^t (kt - u) dB_0(u) \tag{2.2}$$

where $B_0 = \{B_0(t); t \in [0, \infty)\}$ is a Brownian motion. The expression (2.2) is proper canonical as a causal representation of X if $k \leq 1/3$ or $k \geq 1$, and is not proper canonical otherwise. In the latter case, X has a proper canonical representation,

$$X(t) = \int_0^t ((2k-1)t - (3k-2)u) d\tilde{B}_0(u),$$

where $\tilde{B}_0 = \{\tilde{B}_0(t); t \in [0, \infty)\}$ is a Brownian motion. We can easily see that X is double Markov in Hida's sense for any k except $k = 0, 1/2, 2/3$, and is simple Markov otherwise.

Here, set $a_1 = t_2 - t_0$, $a_2 = -(t_1 - t_0)$ for any $t_0 < t_1 < t_2$, then the linear combination $a_1 X(t_1) + a_2 X(t_2)$ has the following decomposition:

$$\begin{aligned}
&a_1 X(t_1) + a_2 X(t_2) \\
&= (t_2 - t_0)\left(\int_0^{t_0} + \int_{t_0}^{t_1}\right)(kt_1 - u)dB_0(u) - (t_1 - t_0)\left(\int_0^{t_0} + \int_{t_0}^{t_2}\right)(kt_2 - u)dB_0(u) \\
&= (t_2 - t_1)X(t_0) + \left\{(t_2 - t_0)\int_{t_0}^{t_1}(kt_1 - u)dB_0(u) - (t_1 - t_0)\int_{t_0}^{t_2}(kt_2 - u)dB_0(u)\right\},
\end{aligned}$$

The second term is independent of $\mathfrak{B}_{t_0}(X)$ and therefore the following equation holds:

$$P(a_1 X(t_1) + a_2 X(t_2) \in B \mid \mathfrak{B}_{t_0}(X)) = P(a_1 X(t_1) + a_2 X(t_2) \in B \mid X(t_0)) \tag{2.3}$$
$$\text{for any Borel set } B \text{ of } \boldsymbol{R}.$$

We note that the number of terms in the above linear combination is two and that the number is equal to the multiplicity of multiple Markov property of X except in the case $k = 0, 1/2, 2/3$. This fact gives us an idea for a new definition of multiple Markov processes. For example, if a given stochastic process $X = \{X(t); t \in T\}$ satisfies the condition that, for any fixed $t_0 < t_1 < t_2$, there exists a pair of coefficients $(a_1, a_2) \in \boldsymbol{R}^2 \setminus \{(0, 0)\}$ which satisfies equation (2.3), we may say that X is double Markov. Moreover, we require our definition to satisfy the additional conditions i) and ii) below, following to Hida's and Mandrekar-Thelen's definitions:

i) Any multiple Markov process has a unique multiplicity. In other words, an n-ple Markov process is not $(n-1)$-ple Markov.

ii) Any multiple Markov process satisfies a uniform condition in multiplicity. In other words, the "local multiplicity" of a multiple Markov process does not vary with the choice of distinct times.

3. Multiple Markov processes

3.1. Definition

Definition 3.1. A stochastic process $X = \{X(t); t \in T\}$ is said to be an *n-ple Markov process of linear combination type* (for short, an *n-ple Markov process of LC type*) if X satisfies the following three conditions:

(LC1) For any fixed $t_0, t_1, \cdots, t_n \in T^\circ$ $(t_0 < t_1 < \cdots < t_n)$, there exists an n-tuple of coefficients $(a_1, \cdots, a_n) \in \boldsymbol{R}^n \setminus \{(0, \cdots, 0)\}$ which satisfies

$$P\Big(\sum_{j=1}^{n} a_j X(t_j) \in B \,|\, \mathfrak{B}_{t_0}(X)\Big) = P\Big(\sum_{j=1}^{n} a_j X(t_j) \in B \,|\, X(t_0)\Big) \tag{3.1}$$

for any Borel set B of $\boldsymbol{R}$.

(LC2) There exist no time points $t_0, t_1, \cdots, t_{n-1} \in T^\circ$ $(t_0 < t_1 < \cdots < t_{n-1})$ and no $(n-1)$-tuples of coefficients $(a_1, \cdots, a_{n-1}) \in \boldsymbol{R}^{n-1} \setminus \{(0, \cdots, 0)\}$ which satisfy

$$P\Big(\sum_{j=1}^{n-1} a_j X(t_j) \in B \,|\, \mathfrak{B}_{t_0}(X)\Big) = P\Big(\sum_{j=1}^{n-1} a_j X(t_j) \in B \,|\, X(t_0)\Big) \tag{3.2}$$

for any Borel set B of $\boldsymbol{R}$.

(LC3) There exist no time points $t_0, t_1, \cdots, t_n \in T^\circ$ $(t_0 < t_1 < \cdots < t_n)$ and no n-tuples of coefficients $(a_1, \cdots, a_n) \in \boldsymbol{R}^n \setminus \{(0, \cdots, 0)\}$ such that $\sum_{j=1}^{n} a_j X(t_j)$ is independent of $\mathfrak{B}_{t_0}(X)$.

3.2. In the case of centered Gaussian processes

For centered Gaussian processes, we have the following proposition.

PROPOSITION 3.2. *If a Gaussian process $X = \{X(t); t \in T\}$ is n-ple Markov of LC type, then n-tuple of coefficients $(a_1, \cdots, a_n) \in \boldsymbol{R}^n$ which satisfies equation* (3.1) *is unique up to a constant for each $t_0 < t_1 < \cdots < t_n$.*

PROOF. Suppose that, for a fixed $t_0 < t_1 < \cdots < t_n$, there exist two linearly independent n-tuples $(a_1^{(i)}, \cdots, a_n^{(i)}) \in \boldsymbol{R}^n \setminus \{(0, \cdots, 0)\}$ $(i = 1, 2)$ which satisfy equation (3.1). Since X is Gaussian, the equation

$$E(\sum_{j=1}^{n} a_j^{(i)} X(t_j) \mid \mathfrak{B}_{t_0}(X)) = E(\sum_{j=1}^{n} a_j^{(i)} X(t_j) \mid X(t_0))$$

holds and we can set the right hand side of the equation as $-a_0^{(i)} X(t_0)$ where $a_0^{(i)}$ is a constant for each $i = 1, 2$. Thus the random variable $\sum_{j=0}^{n} a_j^{(i)} X(t_j)$ belongs to $(\mathfrak{M}_{t_0}^2(X))^{\perp}$, the orthogonal complement of $\mathfrak{M}_{t_0}^2(X)$. We note that condition (LC2) implies $a_n^{(2)} \neq 0$. Set $\tilde{a}_j = a_j^{(1)} - a_n^{(1)} a_n^{(2)^{-1}} a_j^{(2)}$ for $0 \leq j \leq n$. Then we have $\tilde{a}_n = 0$, $(\tilde{a}_1, \cdots, \tilde{a}_{n-1}) \neq (0, \cdots, 0)$ by the assumption and

$$\sum_{j=1}^{n-1} \tilde{a}_j X(t_j) = -\tilde{a}_0 X(t_0) + (\sum_{j=0}^{n} a_j^{(1)} X(t_j) - a_n^{(1)} a_n^{(2)^{-1}} \sum_{j=0}^{n} a_j^{(2)} X(t_j)).$$

The second term belongs to $(\mathfrak{M}_{t_0}^2(X))^{\perp}$ and thus is independent of $\mathfrak{B}_{t_0}(X)$. This is contradictory to (LC2). $\square$

The following proposition assures us that our definition is an extension of Hida's one.

PROPOSITION 3.3. *A Gaussian process $X = \{X(t); t \in T\}$ is n-ple Markov of LC type if and only if X is n-ple Markov in Hida's sense* (*Definition* 2.1).

PROOF. '*Only if*' *part*: Firstly we prove '(LC2) and (LC3)$\rightarrow$(H1)'. Suppose that (H1) is not satisfied, that is, there exist $t_0 \leq t_1 < \cdots < t_n$ and $(a_1, \cdots, a_n) \in \boldsymbol{R}^n \setminus \{(0, \cdots, 0)\}$ such that $\sum_{j=1}^{n} a_j E(X(t_j) \mid \mathfrak{B}_{t_0}(X)) = 0$. This means that $\sum_{j=1}^{n} a_j X(t_j)$ is independent of $\mathfrak{B}_{t_0}(X)$. In the case $t_0 < t_1$, this is contradictory to (LC3). In the case $t_0 = t_1$, the second term of the equation

$$\sum_{j=2}^{n} a_j X(t_j) = -a_1 X(t_1) + \sum_{j=1}^{n} a_j X(t_j)$$

is independent of $\mathfrak{B}_{t_1}(X)$ by the assumption. If $(a_2, \cdots, a_n) \neq (0, \cdots, 0)$, this is contradictory to (LC2). If $(a_2, \cdots, a_n) = (0, \cdots, 0)$, we have $a_1 \neq 0$ and $a_1 X(t_1)$ is independent of $\mathfrak{B}_{t_1}(X)$, so that $X(t_1) = 0$. This is contradictory to (LC2).

Secondly we prove '(LC1)$\rightarrow$(H2)'. From (LC1), for any $t_1 < \cdots < t_{n+1}$, there exists $(a_2, \cdots, a_{n+1}) \in \boldsymbol{R}^n \setminus \{(0, \cdots, 0)\}$ which satisfies

$$P(\sum_{j=2}^{n+1} a_j X(t_j) \in B \mid \mathfrak{B}_{t_1}(X)) = P(\sum_{j=2}^{n+1} a_j X(t_j) \in B \mid X(t_1))$$

for any Borel set B of $\boldsymbol{R}$.

Since X is Gaussian, the equation

$$E(\sum_{j=2}^{n+1} a_j X(t_j) \mid \mathfrak{B}_{t_1}(X)) = E(\sum_{j=2}^{n+1} a_j X(t_j) \mid X(t_1)) = -a_1 X(t_1),$$

holds where a_1 is a constant. Therefore, for any $t_0 \leq t_1$, we have

$$\sum_{j=1}^{n+1} a_j E\,(X(t_j) \mid \mathfrak{B}_{t_0}(X)) = E(E(a_1 X(t_1) + \sum_{j=2}^{n+1} a_j X(t_j) \mid \mathfrak{B}_{t_1}(X)) \mid \mathfrak{B}_{t_0}(X)) = 0.$$

Proof of 'if' part: Firstly we prove '(H1) $\rightarrow$ (LC2) and (LC3)'. Suppose that (LC2) is not satisfied, that is, there exist $t_0 < t_1 < \cdots < t_{n-1}$ and $(a_1, \cdots, a_{n-1}) \in \boldsymbol{R}^{n-1} \setminus \{(0, \cdots, 0)\}$ such that the equation (3.2) holds. Then the equation

$$E(\sum_{j=1}^{n-1} a_j X(t_j) \mid \mathfrak{B}_{t_0}(X)) = E(\sum_{j=1}^{n-1} a_j X(t_j) \mid X(t_0)) = -a_0 X(t_0),$$

holds where a_0 is a constant. This implies that $\sum_{j=0}^{n-1} a_j E(X(t_j) \mid \mathfrak{B}_{t_0}(X)) = 0$, which is a contradiction.

Suppose that (LC3) is not satisfied, that is, there exist $t_0 < t_1 < \cdots < t_n$ and $(a_1, \cdots, a_n) \in \boldsymbol{R}^n \setminus \{(0, \cdots, 0)\}$ such that $\sum_{j=1}^{n} a_j X(t_j)$ is independent of $\mathfrak{B}_{t_0}(X)$. Then we have $\sum_{j=1}^{n} a_j E(X(t_j) \mid \mathfrak{B}_{t_0}(X)) = 0$. This is a contradiction.

Secondly we prove '(H1) and (H2) $\rightarrow$ (LC1)'. From (H2), for any $t_0 < t_1 < \cdots < t_n$, there exists $(a_0, \cdots, a_n) \in \boldsymbol{R}^{n+1} \setminus \{(0, \cdots, 0)\}$ such that $\sum_{j=0}^{n} a_j E(X(t_j) \mid \mathfrak{B}_{t_0}(X)) = 0$. Then the second term of the equation

$$\sum_{j=1}^{n} a_j X(t_j) = -a_0 X(t_0) + \sum_{j=0}^{n} a_j X(t_j)$$

is independent of $\mathfrak{B}_{t_0}(X)$ and therefore the equation (3.1) holds. Let us verify $(a_1, \cdots, a_n) \neq (0, \cdots, 0)$. If $(a_1, \cdots, a_n) = (0, \cdots, 0)$, we have $a_0 \neq 0$ and $a_0 X(t_0)$ is independent of $\mathfrak{B}_{t_0}(X)$, so that $X(t_0) = 0$, which is contradictory to (H1). $\square$

3.3. In the case of SαS processes which have canonical representations

For SαS processes which have canonical representations, we obtain an analogue of Proposition 3.2.

PROPOSITION 3.4. *If an SαS process* $X = \{X(t); t \in T\}$ $(0 < \alpha \leq 2)$ *which has a canonical representation is n-ple Markov of LC type, then n-tuple of coefficients* $(a_1, \cdots, a_n) \in \boldsymbol{R}^n$ *which satisfies equation* (3.1) *is unique up to a constant for each* $t_0 < t_1 < \cdots < t_n$.

We will prove this proposition later. The main theorem in this paper is as follows.

THEOREM 3.5. *Assume that* $X = \{X(t); t \in T\}$ *is an SαS process* $(0 < \alpha \leq 2)$ *which has a canonical representation*

$$X(t) = \int^{t} F(t, u) dZ(u) \tag{3.3}$$

and which is continuous in $d^{[\alpha]}$. *Then X is n-ple Markov of LC type if and only if* $F(t, u)$ *has the following expression*:

$$F(t, u) = \sum_{j=1}^{n} f_j(t) g_j(u) \tag{3.4}$$

where

i) $g_j \in L^{(\alpha)}(T, \mu)$, $1 \leq j \leq n$ *are linearly independent on* $(-\infty, t] \cap T$ *for any* $t \in T^{\circ}$;

ii) *the determinant of* $n \times n$ *matrix* $(f_i(t_j))_{i,j}$ *never vanishes for any* $t_1, \cdots, t_n \in T^{\circ}$ $(t_1 < \cdots < t_n)$.

In Gaussian case $(\alpha = 2)$, this theorem is proved by Hida ([1], Theorem II. 2). In his paper, $F(t, u)$ is called a Goursat kernel of order n if $F(t, u)$ has expression (3.4) satisfying conditions i) and ii). Similarly, in non-Gaussian case $(0 < \alpha < 2)$, we also call $F(t, u)$ a *Goursat kernel of order n* if $F(t, u)$ has expression (3.4) satisfying i), ii).

The following property of SαS random variables is essential to prove Proposition 3.4 and Theorem 3.5.

LEMMA 3.6 (a specific case of Theorem in K. Sato [8]). *Let random variables X and Y be elements of a certain SαS system* $(0 < \alpha \leq 2)$. *If X and Y are linearly independent, the joint distribution of X and Y is absolutely continuous* (*with respect to the* 2-*dimensional Lebesgue measure*).

PROPOSITION 3.7. *Let X and Y be elements of a certain SαS system* $(0 < \alpha \leq 2)$. *If Y is* $\sigma(X)$-*measurable* (*here* $\sigma(X)$ *denotes the* σ-*field generated by* X), *then Y is a multiple of X*.

PROOF. If $X = 0$, it is trivial. Assume $X \neq 0$. Suppose that Y is not a multiple of X while Y is $\sigma(X)$-measurable. Since Y is not a multiple of X, the joint distribution of X and Y is absolutely continuous by Lemma 3.6. On the other hand, since Y is $\sigma(X)$-measurable, there exists a function $\varphi: \boldsymbol{R} \to \boldsymbol{R}$ such that $Y(\omega) = \varphi(X(\omega))$ a.s. Thus the joint distribution of X and Y concentrates on $\{(x, \varphi(x)); x \in \boldsymbol{R}\}$ and is not absolutely continuous. This is a contradiction. $\square$

The following corollary is convenient for the proofs of Proposition 3.4 and Theorem 3.5.

COROLLARY 3.8. *Let X, Y_1 and Y_2 be elements of a certain $S\alpha S$ system $(0 < \alpha \leq 2)$ and let $\mathfrak{B}_0$ be a sub-σ-field such that X and Y_1 are $\mathfrak{B}_0$-measurable while Y_2 is independent of $\mathfrak{B}_0$. If $Y \equiv Y_1 + Y_2$ satisfies the equation*

$$P(Y \in B \mid \mathfrak{B}_0) = P(Y \in B \mid X) \qquad \textit{for any Borel set } B \textit{ of } \boldsymbol{R},$$

then Y_1 is a multiple of X.

PROOF. In the case $Y_2 = 0$ (namely, $Y = Y_1$), we have $P(Y_1 \in B \mid X) = P(Y_1 \in B \mid \mathfrak{B}_0) = 1_{\{Y_1 \in B\}}$. This means that Y_1 is $\sigma(X)$-measurable and hence Y_1 is a multiple of X by Proposition 3.7.

Assume $Y_1 \neq 0$ and $Y_2 \neq 0$. Since Y_1 is $\mathfrak{B}_0$-measurable and Y_2 is independent of $\mathfrak{B}_0$, for any fixed $y \in \boldsymbol{R}$, $P(Y \leq y \mid \mathfrak{B}_0)$ is a random variable with value $\int_{-\infty}^{y-y_1} f_2(y_2)dy_2$ on $Y_1 = y_1$ $(y_1 \in \boldsymbol{R})$, where f_2 is the density function of Y_2. Since f_2 never vanishes on $\boldsymbol{R}$, the mapping $y_1 (\in \boldsymbol{R}) \to \int_{-\infty}^{y-y_1} f_2(y_2)dy_2$ is one-to-one. Since $P(Y \leq y \mid \mathfrak{B}_0)$ is $\sigma(X)$-measurable, this implies that Y_1 is $\sigma(X)$-measurable. Hence Y_1 is a multiple of X by Proposition 3.7. $\square$

PROOF OF PROPOSITION 3.4. Let $X(t) = \int^t F(t, u)dZ(u)$ be a canonical representation of $X = \{X(t); t \in T\}$. Suppose that, for a fixed $t_0 < t_1 < \cdots < t_n$, there exist two linearly independent n-tuples $(a_1^{(i)}, \cdots, a_n^{(i)}) \in \boldsymbol{R}^n \setminus \{(0, \cdots, 0)\}$ $(i = 1, 2)$ which satisfy equation (3.1). We have the following decomposition formula:

$$\sum_{j=1}^n a_j^{(i)} X(t_j) = \int^{t_0} \sum_{j=1}^n a_j^{(i)} F(t_j, u)dZ(u) + \sum_{j=1}^n a_j^{(i)} \int_{t_0}^{t_j} F(t_j, u)dZ(u)$$

for each $i = 1, 2$. Since X is n-ple Markov of LC type, by Corollary 3.8, the first term is equal to $a_0^{(i)} X(t_0)$ where $a_0^{(i)}$ is a constant for $i = 1, 2$. Now, similarly to Proposition 3.2, we can finish the proof. $\square$

PROOF OF THEOREM 3.5. *Proof of 'only if' part*: Since X is n-ple Markov of LC type and the representation (3.3) is canonical, for any $t_1, \cdots, t_n$, $\tau \in T^{\circ}$

$(t_1 < \cdots < t_n < \tau)$, there exists a unique n-tuple of coefficients $(a_1(\tau, \boldsymbol{t}), \cdots, a_n(\tau, \boldsymbol{t}))$ $\in (\boldsymbol{R} \setminus \{0\})^n$ $(\boldsymbol{t} = (t_1, \cdots, t_n))$ such that

$$F(\tau, u) = \sum_{j=1}^{n} a_j(\tau, \boldsymbol{t}) F(t_j, u) \quad \mu\text{-a.e.} \quad \text{on } \{u; u < t_1\},$$

by Corollary 3.8. Similarly, for any $s_1 < \cdots < s_n < t_1$, we have

$$F(t_j, u) = \sum_{k=1}^{n} a_k(t_j, \boldsymbol{s}) F(s_k, u) \quad \mu\text{-a.e.} \quad \text{on } \{u; u < s_1\} \tag{3.5}$$

$(\boldsymbol{s} = (s_1, \cdots, s_n))$ for each j $(1 \leq j \leq n)$ and

$$F(\tau, u) = \sum_{k=1}^{n} a_k(\tau, \boldsymbol{s}) F(s_k, u) \quad \mu\text{-a.e.} \quad \text{on } \{u; u < s_1\}.$$

Therefore we have

$$\sum_{k=1}^{n} a_k(\tau, \boldsymbol{s}) F(s_k, u) = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j(\tau, \boldsymbol{t}) a_k(t_j, \boldsymbol{s}) F(s_k, u) \qquad \text{on } \{u; u < s_1\}.$$

Let us prove that $F(s_k, u)$, $1 \leq k \leq n$ are linearly independent on $\{u; u < s\}$ for any $s < s_1$. Suppose that $F(s_k, u)$, $1 \leq k \leq n$ are linearly dependent, that is, there exist $s(< s_1)$ and $(b_1, \cdots, b_n) \in \boldsymbol{R}^n \setminus \{(0, \cdots, 0)\}$ such that $\sum_{k=1}^{n} b_k F(s_k, u) = 0$ on $\{u; u < s\}$. Then $\sum_{k=1}^{n} b_k X(s_k)$ is independent of $\mathfrak{B}_s(X)$, which is contradictory to (LC3). By this linearly independent property, we obtain

$$a_k(\tau, \boldsymbol{s}) = \sum_{j=1}^{n} a_j(\tau, \boldsymbol{t}) a_k(t_j, \boldsymbol{s})$$

for each k $(1 \leq k \leq n)$. Furthermore we obtain $\det(a_k(t_j, \boldsymbol{s})) \neq 0$ from (3.5) since $F(t_j, u)$, $1 \leq j \leq n$ are linearly independent.

Now we can prove that $F(t, u)$ is a Goursat kernel of order n by the same arguments of Hida ([1], Theorem II. 2) in Gaussian case ($\alpha = 2$), and so we omit it.

Proof of 'if' part: Firstly let us prove that X satisfies (LC1). For any $t_0 < t_1 < \cdots < t_n$, set

$$(a_1, \cdots, a_n) = (f_1(t_0), \cdots, f_n(t_0))((f_j(t_i))_{i,j})^{-1},$$

then we have $(a_1, \cdots, a_n) \neq (0, \cdots, 0)$ and

$$\begin{aligned}&\sum_{j=1}^{n} a_j X(t_j) \\ &= (f_1(t_0), \cdots, f_n(t_0))((f_j(t_i))_{i,j})^{-1} {}^t\!\left(\int^{t_1} F(t_1, u) dZ(u), \cdots, \int^{t_n} F(t_n, u) dZ(u)\right)\end{aligned}$$

$$= (f_1(t_0),\cdots,f_n(t_0))((f_j(t_i))_{i,j})^{-1} \times$$
$$\quad {}^t\Big(\sum_{j=1}^n f_j(t_1)\Big(\int^{t_0} + \int_{t_0}^{t_1}\Big)g_j(u)dZ(u),\cdots,\sum_{j=1}^n f_j(t_n)\Big(\int^{t_0} + \int_{t_0}^{t_n}\Big)g_j(u)dZ(u)\Big)$$
$$= (f_1(t_0),\cdots,f_n(t_0))((f_j(t_i))_{i,j})^{-1} \times$$
$$\quad {}^t\Big(\Big(\int^{t_0} g_1(u)dZ(u),\cdots,\int^{t_0} g_n(u)dZ(u)\Big)(f_i(t_j))_{i,j}$$
$$\quad + \Big(\sum_{j=1}^n f_j(t_1)\int_{t_0}^{t_1} g_j(u)dZ(u),\cdots,\sum_{j=1}^n f_j(t_n)\int_{t_0}^{t_n} g_j(u)dZ(u)\Big)\Big)$$
$$= X(t_0) + (f_1(t_0),\cdots,f_n(t_0))((f_j(t_i))_{i,j})^{-1} \times$$
$$\quad {}^t\Big(\sum_{j=1}^n f_j(t_1)\int_{t_0}^{t_1} g_j(u)dZ(u),\cdots,\sum_{j=1}^n f_j(t_n)\int_{t_0}^{t_n} g_j(u)dZ(u)\Big)$$

where ${}^t\boldsymbol{v}$ means the transposed column-vector of row-vector $\boldsymbol{v}$. The second term is independent of $\mathfrak{B}_{t_0}(X)$, so that we find that X satisfies (LC1).

Secondly let us prove that X satisfies (LC2). Suppose that there exist $t_0 < t_1 < \cdots < t_{n-1}$ and $(a_1,\cdots,a_{n-1}) \in \boldsymbol{R}^{n-1}$ which satisfy equation (3.2). By Corollary 3.8, we have $\sum_{j=1}^{n-1} a_j \int^{t_0} F(t_j, u)\,dZ(u) = -a_0 X(t_0)$ where a_0 is a constant. Thus we have

$$0 = \sum_{j=0}^{n-1} a_j \int^{t_0} F(t_j, u)\,dZ(u) = \int^{t_0} \sum_{k=1}^{n} (\sum_{j=0}^{n-1} a_j f_k(t_j)) g_k(u) dZ(u).$$

Therefore $\sum_{k=1}^{n} (\sum_{j=0}^{n-1} a_j f_k(t_j)) g_k(u) = 0$ on $\{u;\, u \le t_0\}$. Since $g_k(u)$, $1 \le k \le n$ are linearly independent, this implies that $\sum_{j=0}^{n-1} a_j f_k(t_j) = 0$ for each k. Since $\det(f_i(t_j)) \ne 0$, we find $(a_0,\cdots,a_{n-1}) = (0,\cdots,0)$.

Finally let us prove that X satisfies (LC3). Suppose that there exist $t_0 < t_1 < \cdots < t_n$ and $(a_1,\cdots,a_n) \in \boldsymbol{R}^n$ such that $\sum_{j=1}^{n} a_j X(t_j)$ is independent of $\mathfrak{B}_{t_0}(X)$. We have

$$\sum_{j=1}^{n} a_j X(t_j) = \sum_{j=1}^{n} a_j \sum_{k=1}^{n} f_k(t_j)\Big(\int^{t_0} + \int_{t_0}^{t_j}\Big) g_k(u) dZ(u)$$
$$\qquad = \int^{t_0} \sum_{k=1}^{n} (\sum_{j=1}^{n} a_j f_k(t_j))\, g_k(u) dZ(u) + \sum_{j=1}^{n} a_j \sum_{k=1}^{n} f_k(t_j) \int_{t_0}^{t_j} g_k(u) dZ(u).$$

Since the representation (3.3) is canonical, the first term vanishes, that is,

$\sum_{k=1}^{n}(\sum_{j=1}^{n} a_j f_k(t_j))g_k(u) = 0$ on $\{u; u \leq t_0\}$. Similarly to the previous paragraph, we find $(a_1, \cdots, a_n) = (0, \cdots, 0)$. We finish the proof that X is n-ple Markov of LC type. □

It is unknown whether our definition is an extension of Mandrekar-Thelen's one or not. However, for SαS processes with $\alpha > 1$ which have proper canonical representations, these definitions are equivalent.

PROPOSITION 3.9. *Let $X = \{X(t); t \in T\}$ be an SαS process with index $\alpha > 1$ which has a proper canonical representation. Then X is n-ple Markov of LC type if and only if X is n-ple Markov in Mandrekar-Thelen's sense (Definition 2.2).*

PROOF. Let $X(t) = \int^t F(t, u)dZ(u)$ be a proper canonical representation of X, then we have $Y(t; s) = \int^s F(t, u)dZ(u)$ for any fixed $s \leq t$. Using this fact and Corollary 3.8, we can prove this proposition similarly to Proposition 3.3. □

3.4. Remark

Here we note that a Markov process in the usual sense without the uniform condition (LC3) for $n = 1$ is not simple Markov in our sense.

REMARK 3.10. Obviously, a simple Markov process of LC type is Markov. However, the converse is not true. For example, let us consider a Gaussian process $X = \{X(t); t \in [0, \infty)\}$ defined as

$$X(t) = B_0(t) \quad \text{for} \quad t \in [0, 1] \quad \text{and} \quad = B_0(t) - B_0(1) \quad \text{for} \quad t \in (1, \infty),$$

where $B_0 = \{B_0(t); t \in [0, \infty)\}$ is a Brownian motion. It is easy to see that X satisfies (LC1) for $n = 1$ and thus X is Markov. However, X does not satisfy (LC3) for $n = 1$ in the case $t_0 \leq 1 < t_1$ and thus X is not simple Markov.

Suppose that we introduce a new definition below: If a stochastic process $X = \{X(t); t \in T\}$ satisfies (LC1) for a certain n, X is said to be *at most n-ple Markov of LC type.* Then X is Markov in the usual sense if and only if X is at most simple Markov of LC type.

4. Examples

The author constructed SαS $M(t)$-processes in his paper [3]. In this section we show that these processes are natural examples of multiple Markov processes of LC type.

Let $X_n^\alpha = \{X_n^\alpha(t);\, t \in \boldsymbol{R}^n\}$ be an SαS Lévy motion $(0 < \alpha \le 2)$ with parameter $\boldsymbol{R}^n$, that is, an SαS random field satisfying $X_n^\alpha(\boldsymbol{0}) = 0$ and $E \exp [iz(X_n^\alpha(\boldsymbol{t}) - X_n^\alpha(\boldsymbol{s}))] = \exp(-cd(\boldsymbol{t}, \boldsymbol{s})|z|^\alpha)$ where $d(\boldsymbol{t}, \boldsymbol{s})$ denotes the Euclidean distance on $\boldsymbol{R}^n$ and c is a positive constant which does not depend on $\boldsymbol{t}, \boldsymbol{s}$. (This random field is unique in distributions and can be constructed as Chentsov type.) We can consider a spherical mean process of X_n^α with a spherical harmonic $v_{l,m}^n$ as its weight (where l is the degree of $v_{l,m}^n$ and m is the associated multi-suffix):

$$M_{n,l,m}^\alpha(t) \equiv \int_{\xi \in S^{n-1}} X_n^\alpha(t\xi) v_{l,m}^n(\xi) d\xi, \qquad t \ge 0.$$

Here the right hand side is well-defined as the limit of Riemannian sums in $d^{[\alpha]}$. The SαS process $M_{n,l,m}^\alpha = \{M_{n,l,m}^\alpha(t);\, t \ge 0\}$ is called *SαS $M(t)$-process.* According to [3], $M_{n,l,m}^\alpha$ has a causal representation as follows:

$$M_{n,l,m}^\alpha(t) = \int_0^t F_{n,l}(t, u) dZ_{n,l,m}^\alpha(u), \tag{4.1}$$

where $Z_{n,l,m}^\alpha = \{Z_{n,l,m}^\alpha(t);\, t \ge 0\}$ is an SαS motion and $F_{n,l}(t, u) = c(n, l)\left[\int_x^1 \frac{d^l}{dx^l} (1 - x^2)^{l+(n-3)/2} dx\right]\bigg|_{x=u/t}$ ($c(n, l)$ is a constant).

In Gaussian case $(\alpha = 2)$, H. P. McKean Jr. [7] showed that the representation (4.1) is proper canonical if $l = 0, 1, 2$, while (4.1) is not canonical if $l \ge 3$. Furthermore, he obtained the proper canonical representation of $M_{n,l,m}^2$ in the latter case. If n is odd, the kernel of the proper canonical representation is a Goursat kernel of order $(n+1)/2$, so that $M_{n,l,m}^2$ is $(n+1)/2$-ple Markov by Theorem 3.5.

On the other hand in non-Gaussian case $(0 < \alpha < 2)$, the representation (4.1) is canonical for any n, l (Kojo [3]). If n is odd, the canonical kernel $F_{n,l}(t, u)$ is a Goursat kernel of order

$$(n+1)/2 \quad \text{for} \quad l = 0, 1, 2, \qquad (n+1)/2 + [(l-1)/2] \quad \text{for} \quad l \ge 3,$$

where [ ] denotes the integer part. By Theorem 3.5, $M_{n,l,m}^\alpha$ $(0 < \alpha < 2)$ is multiple Markov of LC type. We note that the multiplicity of multiple Markov property in non-Gaussian case is different from in Gaussian case if $l \ge 3$.

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