

Estimating spatial quantile regression with functional coefficients: A robust semiparametric framework

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This paper considers an estimation of semiparametric functional (varying)-coefficient quantile regression with spatial data. A general robust framework is developed that treats quantile regression for spatial data in a natural semiparametric way. The local M-estimators of the unknown functional-coefficient functions are proposed by using local linear approximation, and their asymptotic distributions are then established under weak spatial mixing conditions allowing the data processes to be either stationary or nonstationary with spatial trends. Application to a soil data set is demonstrated with interesting findings that go beyond traditional analysis.

Keywords: asymptotic distributions; functional (varying) coefficient spatial regression; local M-estimators; quantile regression; robust framework; soil data analysis; spatial data

1. Introduction

Spatial data, which are collected at different sites on the surface of the earth, arise in various areas of research, including econometrics, epidemiology, environmental science, image analysis, oceanography and many others. Numerous applications of spatial models and important developments in the general area of spatial statistics under linear correlation structures can be found in [1,2,4,12], and a more recent comprehensive review by Gelfand *et al.* [10], among others. However, linear correlation structures may not be always reasonable in spatial applications. In the last ten years, efforts have been made in the literature to explore nonlinear relationship in spatial data. See, for example, [9,14,25–28], who explored the nonlinear spatial interdependence from the perspective of conditional mean regressions. Differently from these references, Hallin *et al.* [15] recently proposed to investigate the nonlinear spatial interaction by using conditional quantile regression, showing that spatial quantile regression can provide much more information on spatial data than the conditional mean regression analysis. In this paper, following the above efforts, we aim to develop a structure of spatial quantile regression allowing functional coefficients, *under a robust semiparametric framework*, to reduce the “curse of dimensionality” that spatial quantile regression analysis suffers from when the dimension of the covariates is higher than 3. We will demonstrate in Section 5 that the proposed semiparametric functional-coefficient spatial quantile structure will be useful in the analysis of a soil data set.

To make our results widely applicable, we shall consider the quantile regression for spatial data in a general context. Firstly, we treat data as observed over a space of general dimension N . Denote the set of integer lattice points in N -dimensional Euclidean space by Z^N , where $N \geq 1$ and $Z = \{0, \pm 1, \pm 2, \dots\}$. A point $\mathbf{i} = (i_1, \dots, i_N)$ in Z^N is referred to as a site. Spatial data are modeled as finite realizations of vector stochastic processes indexed by $\mathbf{i} \in Z^N$, that is, random fields. We will consider strictly stationary $(d + k + 1)$ -dimensional random fields of the form

$$\{(Y_{\mathbf{i}}, X_{\mathbf{i}}, U_{\mathbf{i}}) : \mathbf{i} \in Z^N\},$$

where $Y_{\mathbf{i}}$, with values in R , $X_{\mathbf{i}}$, with values in R^d , and $U_{\mathbf{i}}$, with values in R^k , are defined over a probability space (Ω, \mathcal{F}, P) .

Secondly, we treat spatial quantile regression in a general context of robust spatial regression. In a number of applications, a crucial problem consists in describing and analyzing the influence of the covariates $(U_{\mathbf{i}}, X_{\mathbf{i}})$ on the real-valued response $Y_{\mathbf{i}}$. In spatial context, this study is particularly difficult due to the possibly highly complex spatial dependence among the various sites. The traditional approach to this problem consists in assuming that $Y_{\mathbf{i}}$ has finite expectation, so that spatial conditional mean regression function $g : (x, u) \mapsto g(x, u) := E[Y_{\mathbf{i}} | X_{\mathbf{i}} = x, U_{\mathbf{i}} = u]$ may be well defined and clearly carries relevant information on the dependence of Y on X and U (cf., [14,25,26]). Differently, Hallin *et al.* [15] proposed spatial conditional quantile regression, defined by

$$q_{\tau} : (x, u) \mapsto q_{\tau}(x, u) := Q[Y_{\mathbf{i}} | X_{\mathbf{i}} = x, U_{\mathbf{i}} = u], \tag{1.1}$$

which provides more comprehensive information on the dependence of Y on X and U through different $0 < \tau < 1$ (see [23] and [41]), where $q_{\tau}(x, u)$ satisfies $P[Y_{\mathbf{i}} < q_{\tau}(x, u) | X_{\mathbf{i}} = x, U_{\mathbf{i}} = u] = \tau$; see also the robust spatial conditional regression in [24]. As is well known in the nonparametric literature, when $d + k > 3$, both spatial regression functions $g(x, u)$ and $q_{\tau}(x, u)$ can not be well estimated nonparametrically with reasonable accuracy owing to the curse of dimensionality. Because of complex spatial interaction, this issue on how to avoid the curse of dimensionality becomes particularly important, which has been addressed by Gao *et al.* [9] and Lu *et al.* [27] for spatial conditional mean regression $g(x, u)$ under *least squares partially linear and additive approximation structures*, respectively. In this paper, we are particularly concerned with avoiding the curse of dimensionality for spatial quantile regression analysis, and, for generality, consider a general spatial regression that takes conditional quantile regression $Q(Y_{\mathbf{i}} | U_{\mathbf{i}}, X_{\mathbf{i}})$ as a special case, to be approximated by a popular linear structure allowing for functional coefficients in the form

$$\Psi(U_{\mathbf{i}}, X_{\mathbf{i}}) = X_{\mathbf{i}1}\beta_1(U_{\mathbf{i}}) + \dots + X_{\mathbf{i}d}\beta_d(U_{\mathbf{i}}), \tag{1.2}$$

with the functional coefficients $\beta_j(\cdot)$'s defined by minimizing

$$E\rho(Y_{\mathbf{i}} - \Psi(U_{\mathbf{i}}, X_{\mathbf{i}})) = E\rho(Y_{\mathbf{i}} - X_{\mathbf{i}1}\beta_1(U_{\mathbf{i}}) - \dots - X_{\mathbf{i}d}\beta_d(U_{\mathbf{i}})), \tag{1.3}$$

associated with $\rho(y)$ by which we denote hereafter for a general loss function [see Section 2], over a class of functional coefficient linear functions of the form $\Psi(U_{\mathbf{i}}, X_{\mathbf{i}})$ in (1.2). In the subsequent, when considering τ th quantile regression, we will denote by $\rho_{\tau}(z) = |z| + (2\tau - 1)z$ with $0 < \tau < 1$, instead of $\rho(\cdot)$, for the loss function, under which the resulting $\Psi(U_{\mathbf{i}}, X_{\mathbf{i}})$ in (1.2) is

the spatial quantile regression with functional coefficients that we are mainly concerned with in this paper. Let $X_{\mathbf{i}} = (X_{i1}, \dots, X_{id})^T$. As in traditional linear regression when a baseline effect is desired, we set $X_{i1} \equiv 1$. The regime $U_{\mathbf{i}}$ is a vector of explanatory variables, and $\beta_1(u), \dots, \beta_d(u)$ are unknown smooth functions of u to be estimated, with the dimension k of $U_{\mathbf{i}}$ usually small, say $k = 1$ or 2 .

Functional (varying)-coefficient regressions are a useful extension of the classical linear regressions. One of the advantages of such models is that the effects of the regressor vector $X_{\mathbf{i}}$ can be well measured by the functional coefficients through $U_{\mathbf{i}}$ and the dimensionality curse is therefore reduced when k is small. Functional coefficient regression models are popular in traditional regression and time series analysis. A comprehensive theory in the nonspatial case has been well explored, see, for example, [3,7,8,19,20,34,36,38,40]. However, the varying coefficient models with spatial data are still rather rarely investigated in the literature. Some exceptions include an extension of the useful semiparametric model studied by Moyeed and Diggle [30], where the intercept coefficient β_1 is assumed to be time-varying, while β_2, \dots, β_d are constants; see also [28] for a varying-coefficient spatiotemporal model under the least squares mean regression perspective.

In this paper, we will develop in Section 2 a general robust M-type semiparametric framework for approximating a spatial conditional regression, under $\rho(\cdot)$, by the linear structure with functional coefficients, $\Psi(u, x)$ in (1.2), via minimizing (1.3). We apply local linear method to approximate the unknown coefficient functions $\beta_r(u), r = 1, \dots, d$ and obtain their local M-estimators in Section 2.1. The main results on asymptotic distribution for the local M-estimators of $\beta_r(u)$'s at both interior and boundary points with stationary spatial data are established in Section 2.2. Applications of the main results to conditional quantile coefficient functions and robust conditional regression coefficient functions will be presented in Section 3. Section 4 extends the main results to the case of allowing a nonstationary random field with spatial trend, which is of importance in practice. A real data example will be reported in Section 5. The proofs of the main theorems are relegated in Appendix, with details of the proof of necessary lemmas provided in the supplementary material [29].

2. Spatial quantile regression under general M-estimation framework: Asymptotic results

Consider a rectangular sampling region by

$$G_N = \{\mathbf{i} = (i_1, \dots, i_N) \in Z^N : 1 \leq i_l \leq n_l, l = 1, \dots, N\},$$

with $\mathbf{n} = (n_1, \dots, n_N)$. In this paper, we write $\mathbf{n} \rightarrow \infty$ if $n_l \rightarrow \infty$ for some $1 \leq l \leq N$. Assume that we observe $(Y_{\mathbf{i}}, X_{\mathbf{i}}, U_{\mathbf{i}})$ on G_N . The total sample size is thus $\tilde{\mathbf{n}} = \prod_{l=1}^N n_l$. We will assume that $\{(Y_{\mathbf{i}}, X_{\mathbf{i}}, U_{\mathbf{i}})\}$ satisfies the following mixing condition as defined in the literature (cf., [14,15,37]). Let S and S' be two sets of sites. The Borel fields $\mathcal{B}(S) = \mathcal{B}((Y_{\mathbf{i}}, X_{\mathbf{i}}, U_{\mathbf{i}}) : \mathbf{i} \in S)$ and $\mathcal{B}(S') = \mathcal{B}((Y_{\mathbf{i}}, X_{\mathbf{i}}, U_{\mathbf{i}}) : \mathbf{i} \in S')$ are the σ -fields generated by $(Y_{\mathbf{i}}, X_{\mathbf{i}}, U_{\mathbf{i}})$ with \mathbf{i} being the elements of S and S' , respectively. Let $d(S, S')$ be the Euclidean distance between S and S' . Then the spatial

mixing defines that there exists a function $\varphi(t) \downarrow 0$ as $t \rightarrow \infty$, such that whenever $S, S' \subset Z^N$,

$$\begin{aligned} \alpha(\mathcal{B}(S), \mathcal{B}(S')) &= \sup\{|P(AB) - P(A)P(B)|, A \in \mathcal{B}(S), B \in \mathcal{B}(S')\} \\ &\leq \chi(\text{Card}(S), \text{Card}(S'))\varphi(d(S, S')), \end{aligned} \tag{2.1}$$

where $\text{Card}(S)$ denotes the cardinality of S , and χ is a symmetric positive function nondecreasing in each variable. If $\chi \equiv 1$, then $\{(Y_i, X_i, U_i)\}$ is called strongly mixing.

2.1. A general M-type semiparametric framework

Consider $u \in E^0 = \{u = (u_1, \dots, u_k) \mid u_{*j} \leq u_j \leq u_j^*, 1 \leq j \leq k\}$, where u_{*j} and u_j^* are constants of lower and upper limits of u_j , respectively. Let $\beta_r(u)$, $r = 1, \dots, d$, in (1.2) be defined by minimizing (1.3) with $\rho(\cdot)$. Then, given $u_0 \in E^0$, for u in the neighborhood of u_0 , we can use $a_r + b_r^T(u - u_0)$ to approximate the unknown coefficient function $\beta_r(u)$ ($r = 1, \dots, d$), where $b_r = (b_{r1}, \dots, b_{rk})^T$. Based on spatial observations $\{(Y_i, X_i, U_i) : i \in G_N\}$, by using the idea of local linear fitting (see, e.g., [5] and [33]), we solve the following minimization problem

$$\min_{a_r, b_r, r=1, \dots, d} \sum_{i \in G_N} \rho \left(Y_i - \sum_{r=1}^d [a_r + (U_i - u_0)^T b_r] X_{ir} \right) K \left(\frac{U_i - u_0}{h_n} \right), \tag{2.2}$$

where $K(\cdot)$ is a given kernel function, h_n is a chosen bandwidth. Let \hat{a}_r, \hat{b}_r , $r = 1, 2, \dots, d$, be the minimizer of (2.2). Then the M-estimator $\hat{\beta}(u_0)$ of $\beta(u_0) = (\beta_1(u_0), \dots, \beta_d(u_0))^T$, which minimizes (1.3) for $\rho(\cdot)$, is defined by

$$\hat{\beta}(u_0) = (\hat{\beta}_1(u_0), \dots, \hat{\beta}_d(u_0))^T = \hat{a} = (\hat{a}_1, \dots, \hat{a}_d)^T. \tag{2.3}$$

Typical choices for ρ are convex and symmetric about 0. Here, we only require ρ a convex function so that the optimisations (2.2) are well defined and the problem of local minima is avoided. It can be asymmetric. For example, an estimator with $\rho_\tau(z)$ for $0 < \tau < 1$ gives the τ th conditional quantile of Y , defined in (1.1). For robustness consideration, we may take ρ having a bounded derivative $\rho'(z) = \max\{-1, \min\{z/c, 1\}\}$, $c > 0$; see [21] and [16] for more details about the robustness of M-estimators.

2.2. Asymptotic results

In this subsection, we state asymptotic properties of the estimates $\hat{\beta}(u_0)$. Let $\psi(z)$ be the derivative function of $\rho(z)$ with respect to z almost everywhere. The following assumptions are required for our asymptotic results.

Assumption 1. *The random field $\{(Y_i, X_i, U_i) : i \in Z^N\}$ is strictly stationary. For all distinct i and j in Z^N , the random variables U_i and U_j admit a joint density $f_{i,j}(u, v) \leq C_0$ uniformly with respect to $i, j \in Z^N$ and $u, v \in E^0$, where C_0 is some positive constant. The marginal density $f(u)$ of U_i is continuous and bounded away from 0 uniformly over E^0 .*

Assumption 2. All functions, $\beta_r(u)$'s, are twice continuously differentiable in a neighborhood of u_0 , for $r = 1, \dots, d$.

Assumption 3. The convex loss function $\rho(\cdot)$ satisfies, for some $\delta > 0$,

$$E\psi(\varepsilon_i|X_i, U_i) = 0, \quad E(|\psi(\varepsilon_i)|^{2+\delta}|X_i, U_i) \leq C_1,$$

where $\varepsilon_i = Y_i - X_i^T \beta(U_i)$ and $C_1 > 0$ is a constant. Furthermore, there exist some function $\phi(\cdot)$ and constant $\bar{c}_1 > 0$ such that $|E(\psi(\varepsilon_i + z)|X_i, U_i) - \phi(X_i, U_i)z| \leq C_1 z^2$ for any $|z| \leq \bar{c}_1$.

Assumption 4. There exist constants $0 < \bar{c}_2, C_2 < \infty$ such that

$$|\psi(v+z) - \psi(v)| \leq C_2, \quad E([\psi(\varepsilon_i + z) - \psi(\varepsilon_i)]^2|X_i, U_i) \leq C_2|z|$$

for any $|z| \leq \bar{c}_2$ and $v \in R^1$.

Assumption 5. The bandwidth h_n satisfies that $h_n \leq C_3 \tilde{n}^{-1/(k+4)}$ for some positive constant C_3 and $\tilde{n}h_n^k \rightarrow +\infty$ as $n \rightarrow \infty$.

Assumption 6. $\max_{i \in G_N} \|X_i\| = o_p((\tilde{n}h_n^k)^{1/2})$, $\max_{i \in G_N} \|X_i\| = o_p(h_n^{-2})$ and $E\|X_i\|^{4+2\delta} < \infty$.

Assumption 7. The kernel function $K(\cdot) \geq 0$ is a bounded symmetric function with a compact support $\tilde{M} = [-M_1, M_1] \times \dots \times [-M_k, M_k]$ and $\int_{\tilde{M}} uu^T K(u) du$ is positive definite.

Assumption 8. The function $\chi(\cdot, \cdot)$ and φ satisfy that $\chi(n', n'') \leq \min(n', n'')$ and

$$\lim_{k \rightarrow \infty} k^a \sum_{z=k}^{\infty} z^{N-1} \{\varphi(z)\}^{\delta/(2+\delta)} = 0 \quad (2.4)$$

for some constant $a > (4 + \delta)N/(2 + \delta)$.

Assumption 9. $\min_{1 \leq l \leq N} n_l \rightarrow \infty$ and there exist two sequences of positive integer vectors, $\mathbf{p} = \mathbf{p}_n = (p_1, \dots, p_N) \in Z^N$ and $\mathbf{q} = \mathbf{q}_n = (q, \dots, q) \in Z^N$, with $q \rightarrow \infty$ such that $q/p_l \rightarrow 0$ and $n_l/p_l \rightarrow \infty$ for all $l = 1, \dots, N$, and $\tilde{\mathbf{p}} = \prod_{l=1}^N p_l = o((\tilde{n}h_n^k)^{1/2})$, $\tilde{n}\varphi(q) \rightarrow 0$. Furthermore, $qh_n^{\delta k/[a(2+\delta)]} > 1$.

The above assumptions are standard in the setting of local smoothers needed for asymptotics. See [14], for example, for Assumptions 1, 2, 7, 8 and 9 in the spatial context. Assumptions 3 and 4 are easily checked if the score function ψ is differentiable, but they cover nondifferentiable case including the least absolute deviation estimator with $\psi(z) = \text{sgn}(z)$. Assumptions 5 and 6 can be found in [7], where the moment condition on X_i is technical for the establishment of asymptotic properties in varying coefficient setting. The bounded support restriction on $K(\cdot)$ is technical and can be relaxed by using such kernels with light tails as Gaussian kernel.

To state our main results, we let

$$\begin{aligned}\Phi(u) &= E(\phi(X_{\mathbf{i}}, U_{\mathbf{i}})X_{\mathbf{i}}X_{\mathbf{i}}^T | U_{\mathbf{i}} = u), & \Sigma(u) &= E(E(\psi^2(\varepsilon_{\mathbf{i}}) | X_{\mathbf{i}}, U_{\mathbf{i}})X_{\mathbf{i}}X_{\mathbf{i}}^T | U_{\mathbf{i}} = u), \\ \zeta(u_0) &= (\zeta_1(u_0), \dots, \zeta_d(u_0))^T, & \zeta_r(u_0) &= \text{tr}\left(\ddot{\beta}_r(u_0) \int_{\tilde{M}} uu^T K(u) du\right), \quad r = 1, \dots, d,\end{aligned}$$

where $\ddot{\beta}_r(u_0)$ is the second derivative of $\beta_r(u)$ at $u = u_0$.

Theorem 2.1. *Assume that Assumptions 1–9 hold and $\Phi(u)$, $\Sigma(u)$ are continuous in some neighborhood of u_0 and $\Phi(u_0)$ is positive definite. If u_0 is an interior point of the support of the design density $f(u)$, then, as $\mathbf{n} \rightarrow \infty$,*

$$\sqrt{\tilde{\mathbf{n}}h_{\mathbf{n}}^k} \left(\hat{\beta}(u_0) - \beta(u_0) - \frac{h_{\mathbf{n}}^2}{2\mu_0} \zeta(u_0) \right) \rightarrow_d N\left(0, \frac{v_0}{f(u_0)\mu_0^2} \Phi^{-1}(u_0) \Sigma(u_0) \Phi^{-1}(u_0)\right),$$

where $\mu_0 = \int_{\tilde{M}} K(u) du$, $v_0 = \int_{\tilde{M}} K^2(u) du$, and \rightarrow_d means convergence in distribution.

Theorem 2.1 gives the asymptotic distribution of the estimator of $\beta(u_0)$ at an interior point. Next, we study the asymptotic behavior of the estimator at the boundary of the support E^0 of $f(u)$. Suppose $u_* = (u_{*1}, \dots, u_{*k})^T$ is a boundary point. Take $u_h = u_* + ch_{\mathbf{n}}$, where $c = (c_1, \dots, c_k)^T$ satisfies that $0 \leq c_l < M_l$, $l = 1, \dots, k$. Let $\tilde{M} = [-c_1, M_1] \times \dots \times [-c_k, M_k]$, $\bar{\zeta}(u_*) = (\bar{\zeta}_1(u_*), \dots, \bar{\zeta}_d(u_*))^T$, $\bar{\zeta}_r(u_*) = \text{tr}(\ddot{\beta}_r(u_*) \int_{\tilde{M}} vv^T K(v) dv)$, $r = 1, \dots, d$, $\bar{\zeta}^{(l)}(u_*) = (\bar{\zeta}_1^{(l)}(u_*), \dots, \bar{\zeta}_d^{(l)}(u_*))^T$, $\bar{\zeta}_r^{(l)}(u_*) = \text{tr}(\ddot{\beta}_r(u_*) \int_{\tilde{M}} v_l v_l^T K(v) dv)$, $l = 1, \dots, k$, and

$$\Delta_c = \begin{pmatrix} \int_{\tilde{M}} K(u) du & \int_{\tilde{M}} u^T K(u) du \\ \int_{\tilde{M}} u K(u) du & \int_{\tilde{M}} uu^T K(u) du \end{pmatrix}, \quad \bar{\Delta}_c = \begin{pmatrix} \int_{\tilde{M}} K^2(u) du & \int_{\tilde{M}} u^T K^2(u) du \\ \int_{\tilde{M}} u K^2(u) du & \int_{\tilde{M}} uu^T K^2(u) du \end{pmatrix}.$$

Theorem 2.2. *Assume that Assumptions 1–9 hold in some right neighborhood of u_* and $\Phi(u)$, $\Sigma(u)$ are continuous in some right neighborhood of u_* and $\Phi(u_*)$ is positive definite. Suppose Δ_c is invertible. Then, as $\mathbf{n} \rightarrow \infty$,*

$$\begin{aligned}\sqrt{\tilde{\mathbf{n}}h_{\mathbf{n}}^k} \left(\hat{\beta}(u_h) - \beta(u_h) - \frac{h_{\mathbf{n}}^2}{2} [\delta_{11} \bar{\zeta}(u_*) + \sum_{l=1}^k \delta_{1(l+1)} \bar{\zeta}^{(l)}(u_*)] \right) \\ \rightarrow_d N\left(0, \frac{\lambda_{11}}{f(u_*)} \Phi^{-1}(u_*) \sum(u_*) \Phi^{-1}(u_*)\right),\end{aligned}$$

where δ_{ij} denotes the (i, j) th entry of Δ_c^{-1} and λ_{11} denotes the $(1, 1)$ th entry of $\Delta_c^{-1} \bar{\Delta}_c \Delta_c^{-1}$.

The proofs of Theorems 2.1 and 2.2 are postponed to [Appendix](#).

Theorem 2.2 shows that for local linear estimator the convergence rate at the points near the boundary is the same as that for interior points. Hence for local linear estimator near the boundary no adjustments are required.

For the mixing coefficient $\varphi(t)$, if it decays at an algebraic rate, that is, $\varphi(t) = O(t^{-\mu})$ for some $\mu > 2(3 + \delta)N/\delta$, we can choose constant a such that $(4 + \delta)N/(2 + \delta) < a < \mu\delta/(2 + \delta) - N$, then, as $l \rightarrow \infty$, it holds that

$$\begin{aligned} l^a \sum_{z=l}^{\infty} z^{N-1} \{\varphi(z)\}^{\delta/(2+\delta)} &\leq Cl^a \sum_{z=l}^{\infty} z^{N-1} z^{-\mu\delta/(2+\delta)} \\ &\leq Cl^a l^{N-\mu\delta/(2+\delta)} = Cl^{-[\mu\delta/(2+\delta)-a-N]} \rightarrow 0, \end{aligned}$$

and so (2.4) holds. Using the similar arguments to those used in the proof of Theorem 3.3 of [14], Assumption 9 can be much simplified, and we have the following corollary.

Corollary 2.1. *Assume that Assumptions 1–7 hold and $\Phi(u), \Sigma(u)$ are continuous in some neighborhood of u_0 and $\Phi(u_0)$ is positive definite. Suppose Δ_c is invertible and $\chi(n', n'') \leq \min(n', n'')$ and $\varphi(t) = O(t^{-\mu})$ for some $\mu > 2(3 + \delta)N/\delta$. Let the sequence of positive integers $q = q_{\mathbf{n}} \rightarrow \infty$ and the bandwidth $h_{\mathbf{n}}$ such that $\tilde{\mathbf{n}}q^{-\mu} \rightarrow 0$, $q = o(\min_{1 \leq i \leq N} (n_i h_{\mathbf{n}}^{k/N})^{1/2})$ and $qh_{\mathbf{n}}^{\delta k/[a(2+\delta)]} > 1$. Then, as $\mathbf{n} \rightarrow \infty$, the conclusions of Theorems 2.1 and 2.2 still hold.*

Further, if the mixing coefficient decays at a geometric rate, that is, $\varphi(t) = O(e^{-\nu t})$ for some $\nu > 0$, then similarly to Theorem 3.4 of [14], Assumptions 8 and 9 can also be simplified and we have the following corollary.

Corollary 2.2. *Assume that Assumptions 1–7 hold and $\Phi(u), \Sigma(u)$ are continuous in some neighborhood of u_0 and $\Phi(u_0)$ is positive definite. Suppose Δ_c is invertible and $\chi(n', n'') \leq \min(n', n'')$ and $\varphi(t) = O(e^{-\nu t})$ for some $\nu > 0$. If*

$$\min_{1 \leq i \leq N} \left\{ (n_i h_{\mathbf{n}}^{k/N})^{1/2} \right\} h_{\mathbf{n}}^{\delta k/[a(2+\delta)]} (\ln \tilde{\mathbf{n}})^{-1} \rightarrow \infty$$

for some constant $a > (4 + \delta)N/(2 + \delta)$, then, as $\mathbf{n} \rightarrow \infty$, the conclusions of Theorems 2.1 and 2.2 still hold.

Remark 1. Another way for \mathbf{n} to tend to infinity is the so called isotropic one, where all components of \mathbf{n} tend to infinity at the same rate. We write $\mathbf{n} \Rightarrow \infty$ if $\mathbf{n} \rightarrow \infty$ and $|n_j/n_l| < C_4$ for some $0 < C_4 < \infty$, $1 \leq j, l \leq N$. Obviously, under Assumptions 1–9, as $\mathbf{n} \Rightarrow \infty$, the conclusions of Theorems 2.1 and 2.2 hold. Furthermore, in Corollary 2.1, the conditions on $q = q_{\mathbf{n}} \rightarrow \infty$ and the bandwidth $h_{\mathbf{n}}$ can be modified as $\tilde{\mathbf{n}}q^{-\mu} \rightarrow 0$, $q = o((\tilde{\mathbf{n}}h_{\mathbf{n}}^k)^{1/2N})$ and $qh_{\mathbf{n}}^{\delta k/[a(2+\delta)]} > 1$ for some $(4 + \delta)N/(2 + \delta) < a < \mu\delta/(2 + \delta) - N$, then, under Assumptions 1–7, the conclusions of Theorems 2.1 and 2.2 still hold. Similarly, in Corollary 2.2, if $(\tilde{\mathbf{n}}h_{\mathbf{n}}^k)^{1/2N} h_{\mathbf{n}}^{\delta k/[a(2+\delta)]} (\ln \tilde{\mathbf{n}})^{-1} \rightarrow \infty$ for $a > (4 + \delta)N/(2 + \delta)$, then, under Assumptions 1–7, the conclusions of Theorems 2.1 and 2.2 still hold.

Remark 2. If $\chi(n', n'') \leq C_5(n' + n'' + 1)^\kappa$ for some $C_5 > 0$ and $\kappa > 1$, let the condition $\tilde{\mathbf{n}}\varphi(q) \rightarrow 0$ in Assumption 9 be replaced by $(\tilde{\mathbf{n}}^{\kappa+1}/\tilde{\mathbf{p}})\varphi(q) \rightarrow 0$ as $\tilde{\mathbf{n}} \rightarrow \infty$, then the conclusions of Theorems 2.1 and 2.2 still hold. In this case, analogues of Corollaries 2.1 and 2.2 and Remark 1 can also be obtained.

3. Quantile regression and robust smoothers with functional coefficients

The general Theorems 2.1 and 2.2 have different applications depending on the choice of $\rho(\cdot)$ function. In this section, we are particularly discussing the spatial regression problems with functional coefficients for the conditional quantiles and robust functionals.

3.1. Quantile regression

Let $F(\cdot|X, U)$ denote the conditional distribution of Y given X and U . Then the τ th conditional quantile of Y given X and U is $F^{-1}(\tau|X, U)$, for $0 < \tau < 1$. Conditional quantiles have several advantages over conditional means. For example, they can be defined without any moment restrictions on Y . Plotting the 0.25th, 0.5th, and 0.75th conditional quantiles would give us more understanding on the data than plotting just the conditional mean. Quantile regression can also be useful for the estimation of predictive intervals. For example, estimates of $F^{-1}(\tau/2|X, U)$ and $F^{-1}(1 - \tau/2|X, U)$ can be used to obtain a $100(1 - \tau)\%$ nonparametric interval of prediction of the response given X and U . Hallin *et al.* [15] have studied the spatial conditional quantile regression estimation, which may however suffer from curse of dimensionality in general.

We estimate the τ th conditional quantile of Y_i given X_i and U_i , approximated by the functional-coefficient linear structure in (1.2) with $\beta_r(u)$'s defined by minimizing (1.3) with $\rho_\tau(\cdot)$ instead of $\rho(\cdot)$. If $\tau = 1/2$, we estimate the conditional median. Let $\hat{a}_\tau, \hat{b}_\tau$ be the minimizer of (2.2) with $\rho_\tau(\cdot)$ instead of $\rho(\cdot)$. Set $\hat{\beta}_\tau(u_0) = \hat{a}_\tau$. Then the estimator of the τ th conditional quantile of Y given $X = x$ and $U = u_0$, approximated by the functional-coefficient linear structure, is $\hat{Y}_\tau = x^T \hat{\beta}_\tau(u_0)$. To state the asymptotic results, we need the following.

Assumption Q. *There exist positive constants \bar{c}_6, C_6 such that the conditional density function $f_\varepsilon(y|X_i, U_i)$ of ε_i given X_i, U_i satisfies that $|f_\varepsilon(y|X_i, U_i) - f_\varepsilon(0|X_i, U_i)| \leq C_6|y|$ for all $y \in [-\bar{c}_6, \bar{c}_6]$, where ε_i is defined in Assumption 3.*

In this case, since $\psi_\tau(z) = 2\tau I(z > 0) + 2(\tau - 1)I(z < 0)$, it is easy to show that Assumption 4 holds and $E(\psi_\tau^2(\varepsilon_i)|X_i, U_i) = 4\tau(1 - \tau)$. If Assumption Q holds, then Assumption 3 holds with $\phi(X_i, U_i) = 2f_\varepsilon(0|X_i, U_i)$. Let $\Phi_\tau(u) = 2E(f_\varepsilon(0|X_i, U_i)X_iX_i^T|U_i = u)$ and $\Omega(u) = E(X_iX_i^T|U_i = u)$. Applying Theorems 2.1 and 2.2 to quantile regression, we have the following theorem.

Theorem 3.1. (1) *Assume that Assumptions 1, 2, 5–9 and Q hold. Suppose $\Phi_\tau(u)$ and $\Omega(u)$ are continuous in some neighborhood of u_0 and $\Phi_\tau(u_0)$ is positive definite, with u_0 an interior point*

of E_0 . Then, as $\mathbf{n} \rightarrow \infty$,

$$\sqrt{\tilde{\mathbf{n}}h_{\mathbf{n}}^k} \left(\hat{\beta}_{\tau}(u_0) - \beta(u_0) - \frac{h_{\mathbf{n}}^2}{2\mu_0} \zeta(u_0) \right) \rightarrow_d N \left(0, \frac{4\tau(1-\tau)v_0}{f(u_0)\mu_0^2} \Phi_{\tau}^{-1}(u_0)\Omega(u_0)\Phi_{\tau}^{-1}(u_0) \right).$$

(2) Assume that Assumptions 1, 2, 5–9 and Q hold in some right neighborhood of u_* and $\Phi_{\tau}(u)$, $\Omega(u)$ are continuous in some right neighborhood of u_* . Suppose Δ_c is invertible and $\Phi_{\tau}(u_*)$ is positive definite. Then, as $\mathbf{n} \rightarrow \infty$,

$$\begin{aligned} & \sqrt{\tilde{\mathbf{n}}h_{\mathbf{n}}^k} \left(\hat{\beta}_{\tau}(u_h) - \beta(u_h) - \frac{h_{\mathbf{n}}^2}{2} \left[\delta_{11} \bar{\zeta}(u_*) + \sum_{l=1}^k \delta_{1(l+1)} \bar{\zeta}^{(l)}(u_*) \right] \right) \\ & \rightarrow_d N \left(0, \frac{4\tau(1-\tau)\lambda_{11}}{f(u_*)} \Phi_{\tau}^{-1}(u_*)\Omega(u_*)\Phi_{\tau}^{-1}(u_*) \right). \end{aligned}$$

3.2. Robust smoothers

It is known that the mean is sensitive to outliers, see [16] and [21]. Since the local average estimator is basically a mean type estimator, it is also sensitive to outliers. To robustify this procedure, it is suggested that the function $\rho(\cdot)$ be chosen so that its first derivative is given by

$$\psi_c(z) = \max\{-1, \min\{z/c, 1\}\}, \quad c > 0,$$

see [18] for interesting discussions. We estimate the conditional robust smoother of Y_i given X_i and U_i , approximated by (1.2) with $\beta_r(u)$'s defined by minimizing (1.3), with the $\rho(z)$ that has the derivative $\psi_c(z)$.

Assumption R. The conditional density function $f_{\varepsilon}(y|X_i, U_i)$ of ε_i given X_i, U_i is symmetric about 0. There is a positive constant C_7 such that $f_{\varepsilon}(y|X_i, U_i) \leq C_7$.

Let \hat{a}_c, \hat{b}_c be the minimizer of (2.2) with the $\rho(\cdot)$ satisfying that $\rho'(z) = \psi_c(z), c > 0$. Set $\hat{\beta}_c(u_0) = \hat{a}_c$. In this case, Assumption 4 holds automatically. If Assumption R holds, then Assumption 3 holds with $\phi(X_i, U_i) = P\{|\varepsilon_i| \leq c|X_i, U_i\}/c$. Let $\Phi_c(u) = E(P\{|\varepsilon_i| \leq c|X_i, U_i\}X_iX_i^T|U_i = u)/c$. An application of Theorems 2.1 and 2.2 yields

Theorem 3.2. Assume that Assumptions 1, 2, 5–9 and R hold. Suppose $\Phi_c(u)$ and $\Sigma(u)$ are continuous in some neighborhood of u_0 and $\Phi_c(u_0)$ is positive definite. Then, as $\mathbf{n} \rightarrow \infty$, the conclusions of Theorems 2.1 and 2.2 hold with $\Phi(u)$ replaced by $\Phi_c(u)$.

Remark 3. Analogues of Theorems 3.1 and 3.2 can also be obtained under the conditions that $\chi(n', n'') \leq C_5(n' + n'' + 1)^{\kappa}$ and (or) $\mathbf{n} \Rightarrow \infty$ and (or) $\varphi(t) = O(t^{-\mu})$ for some $\mu > 2(3 + \delta)N/\delta$ or $\varphi(t) = O(e^{-\nu t})$ for some $\nu > 0$, details are omitted for the sake of brevity.

4. Random fields with a spatial trend

In Section 2, the stationary process $\{Y_i, X_i, U_i\}$ was assumed to be observed. This assumption may often be violated in practice. As a reasonable alternative, we can assume that nonstationarity is due to the presence of a spatial trend, as done in [15], and that, instead, we actually observe $\{\tilde{Y}_i, \tilde{X}_i, \tilde{U}_i\}$, with

$$\tilde{Y}_i = \alpha_Y(s_i) + Y_i, \quad \tilde{X}_i = \alpha_X(s_i) + X_i, \quad \tilde{U}_i = \alpha_U(s_i) + U_i, \tag{4.1}$$

where $s_i = (s_{i_1}, \dots, s_{i_N}) := (i_1/n_1, \dots, i_N/n_N)$ and $s \in [0, 1]^N \rightarrow (\alpha_Y(s), \alpha_X(s), \alpha_U(s))$ is some deterministic but unknown trend function.

For the sake of simplicity, we assume throughout this section that $N = 2$, which is the most frequent case in practice. Since $(Y_i, X_i, U_i) = (\tilde{Y}_i - \alpha_Y(s_i), \tilde{X}_i - \alpha_X(s_i), \tilde{U}_i - \alpha_U(s_i))$ is unobservable, the analysis proceeds in two steps. First, obtain an estimation of the spatial trend $(\alpha_Y(s_i), \alpha_X(s_i), \alpha_U(s_i))$ via kernel smoothing method. In the second step, the detrended data is supposed to satisfy the stationarity assumption, yielding the estimated coefficient function $\check{\beta}_r(u), r = 1, \dots, d$ with the detrended Y_i 's, X_i 's and U_i 's.

Let

$$w(s_i, s) = \frac{W((s_i - s)/g_n)}{\sum_{j \in G_N} W((s_j - s)/g_n)},$$

where g_n is a bandwidth tending to 0 and $W(\cdot)$ is a chosen kernel function. Then the kernel estimators of $\alpha_Y(s), \alpha_X(s)$ and $\alpha_U(s)$ are

$$\hat{\alpha}_Y(s) = \sum_{i \in G_N} \tilde{Y}_i w(s_i, s), \quad \hat{\alpha}_X(s) = \sum_{i \in G_N} \tilde{X}_i w(s_i, s), \quad \hat{\alpha}_U(s) = \sum_{i \in G_N} \tilde{U}_i w(s_i, s). \tag{4.2}$$

Let $\hat{Y}_i = \tilde{Y}_i - \hat{\alpha}_Y(s_i), \hat{X}_i = \tilde{X}_i - \hat{\alpha}_X(s_i)$ and $\hat{U}_i = \tilde{U}_i - \hat{\alpha}_U(s_i)$. Based on the estimated spatial data $\{(\hat{Y}_i, \hat{X}_i, \hat{U}_i) : i \in G_N\}$, we solve the following minimization problem

$$\min_{a_r, b_r, r=1, \dots, d} \sum_{i \in G_N} \rho \left(\hat{Y}_i - \sum_{r=1}^d [a_r + (\hat{U}_i - u_0)^T b_r] \hat{X}_{ir} \right) K \left(\frac{\hat{U}_i - u_0}{h_n} \right). \tag{4.3}$$

Let \check{a}_r, \check{b}_r be the minimizer of (4.3). Set $\check{a} = (\check{a}_1, \dots, \check{a}_d)^T$. Then the M-estimator of $\beta(u_0) = (\beta_1(u_0), \dots, \beta_d(u_0))^T$ is

$$\check{\beta}(u_0) = (\check{\beta}_1(u_0), \dots, \check{\beta}_d(u_0))^T = \check{a}.$$

To study the asymptotic behavior of the new estimators, we need the following additional conditions similar to those in [15].

- (B1) $E|Y_i|^{2+\delta} < \infty, E\|X_i\|^{2+\delta} < \infty$ and $E\|U_i\|^{2+\delta} < \infty$ for some $\delta > 0$ and $\varphi(z)$ in (2.1) satisfies that $\varphi(z) < \tilde{C}_0 z^{-\beta}$, where $0 < \tilde{C}_0 < \infty$ and $\beta > (1 + (1 + \delta)(1 + N))/\delta$.
- (B2) For $\varrho = (\beta - 1 - N - (1 + \beta)/(1 + \delta))/(\beta + 3 - N - (1 + \beta)/(1 + \delta))$, $\ln \tilde{n}/(\tilde{n}^\varrho g_n^N) = o(1)$.

- (B3) $s \rightarrow \alpha_Y(s)$, $s \rightarrow \alpha_X(s)$ and $s \rightarrow \alpha_U(s)$ are m times differentiable with bounded derivatives on $S := [0, 1]^2$, where m is some positive integer.
- (B4) There exists a continuous sampling intensity (density) function \tilde{f} defined on S and constants \tilde{c}_0 and \tilde{c}_1 such that $0 < \tilde{c}_0 \leq \tilde{f}(s) \leq \tilde{c}_1 < \infty$ for any $s \in S$ and $\tilde{\mathbf{n}}^{-1} \sum_{\mathbf{i} \in G_N} I(s_{\mathbf{i}} \in A) \rightarrow \int_A \tilde{f}(s) ds$ for any measurable set $A \subset S$, as $\tilde{\mathbf{n}} \rightarrow \infty$.
- (B5) The kernel $W(s)$, defined on R^2 , has bounded support with Lipschitz property, that is $|W(s) - W(s')| \leq \tilde{C}_1 \|s - s'\|$ for all $s, s' \in R^2$, where \tilde{C}_0 is a generic positive constant, and satisfies ($s^{\otimes i}$ stands for the i th Kronecker power of s)

$$\int W(s) ds = 1, \quad \int s^{\otimes i} W(s) ds = 0, \quad i = 1, \dots, m-1,$$

$$\int s^{\otimes m} W(s) ds \neq 0.$$

Assumptions (B1) and (B2) are technical conditions for deriving the convergence of this kernel smoothing; see [17] for similar assumptions. Assumption (B4) is mentioned for the sake of generality, and is trivially satisfied in the case of a regular grid. Assumptions (B3) and (B5) are standard assumptions on the smoothness of spatial trend functions and a higher order kernel function, respectively, which ensure that the bias term of the spatial trend estimators is of order $O(g_{\mathbf{n}}^m)$ (which can also be achieved by a local polynomial fitting of order $(m-1)$).

We further need to strengthen Assumptions 4–7 as the follows.

Assumption 4'. Let $\mathcal{L}_p(\mathcal{F})$ denote the class of \mathcal{F} -measurable random variable ξ satisfying $\|\xi\|_p = (E|\xi|^p)^{1/p} < \infty$. The function $\psi(\cdot)$ satisfies that $E(|\psi(\eta_{\mathbf{i}} + \xi) - \psi(\eta_{\mathbf{i}})| | X_{\mathbf{i}}, U_{\mathbf{i}}) \leq \tilde{C}_1 \epsilon$ for $\eta_{\mathbf{i}} \in \mathcal{L}_1(\mathcal{B}(\{\mathbf{i}\}))$ and $\xi \in \mathcal{L}_1(\mathcal{B}(G_N))$ such that $|\xi| < \epsilon$, and that $|\psi(v+s) - \psi(v)| \leq \tilde{C}_1$ for any $|s| \leq c_2$ and $v \in R^1$, where \tilde{C}_1, ϵ and c_2 are some positive constants.

Assumption 5'. The bandwidths $h_{\mathbf{n}}$ and $g_{\mathbf{n}}$ satisfy that $h_{\mathbf{n}} \leq \tilde{C}_2 \tilde{\mathbf{n}}^{-1/(k+4)}$ for some positive constant \tilde{C}_2 , $g_{\mathbf{n}}^m / h_{\mathbf{n}} \rightarrow 0$, $\tilde{\mathbf{n}} h_{\mathbf{n}}^k g_{\mathbf{n}}^{2m} \rightarrow 0$, $h_{\mathbf{n}}^k \ln \tilde{\mathbf{n}} / g_{\mathbf{n}}^2 \rightarrow 0$ and $\ln \tilde{\mathbf{n}} / (\tilde{\mathbf{n}} g_{\mathbf{n}}^2 h_{\mathbf{n}}^2) \rightarrow 0$.

Assumption 6'. $\max_{\mathbf{i} \in G_N} \|X_{\mathbf{i}}\| = O_p(1)$ and $E\|X_{\mathbf{i}}\|^{4+2\delta} < \infty$.

Assumption 7'. The kernel function $K(\cdot) \geq 0$ is a bounded symmetric function with a compact support \tilde{M} and is continuously differentiable in $\tilde{M} = (-M_1, M_1) \times \dots \times (-M_k, M_k)$ and $\int_{\tilde{M}} uu^T K(u) du$ is positive definite.

Assumption 4' is easily checked. For example, it holds when $\rho(z) = \rho_{\tau}(z)$ or $\psi(z) = \psi_c(z)$ and the conditional density of $\eta_{\mathbf{i}}$ given $X_{\mathbf{i}}$ and $U_{\mathbf{i}}$ is bounded on $[-\epsilon, \epsilon]$. Assumption 5' on the bandwidths $h_{\mathbf{n}}$ and $g_{\mathbf{n}}$ is easily satisfied, and can be weakened as: $\tilde{\mathbf{n}}^{2/(k+4)} g_{\mathbf{n}}^m \rightarrow 0$ and $\ln \tilde{\mathbf{n}} / (\tilde{\mathbf{n}}^{k/(k+4)} g_{\mathbf{n}}^2) \rightarrow 0$ if we take the optimal $h_{\mathbf{n}} = h_0 \tilde{\mathbf{n}}^{-1/(k+4)}$ for some $h_0 > 0$. The condition, $\max_{\mathbf{i} \in G_N} \|X_{\mathbf{i}}\| = O_p(1)$, in Assumption 6' is only a technical condition, and can also be weakened with $h_{\mathbf{n}}$ and $g_{\mathbf{n}}$ properly chosen.

We state the asymptotic distribution of the estimators $\check{\beta}_r(u_0)$, $r = 1, \dots, d$, as follows.

Theorem 4.1. Assume that Assumptions 1–3, 4'–7', 8–9 and (B1)–(B5) hold and $\Phi(u)$, $\Sigma(u)$ are continuous in some neighborhood of u_0 and $\Phi(u_0)$ is positive definite. If u_0 is an interior point of the support of the design density $f(u)$, then, as $\mathbf{n} \rightarrow \infty$,

$$\sqrt{\mathbf{\bar{n}}h_{\mathbf{n}}^k} \left(\check{\beta}(u_0) - \beta(u_0) - \frac{h_{\mathbf{n}}^2}{2\mu_0} \zeta(u_0) \right) \rightarrow_d N \left(0, \frac{\nu_0}{f(u_0)\mu_0^2} \Phi^{-1}(u_0) \Sigma(u_0) \Phi^{-1}(u_0) \right).$$

With $\rho_{\tau}(z)$ instead of $\rho(z)$ and Assumption 4' replaced by Assumption Q, we have the following theorem.

Theorem 4.2. Assume that Assumptions 1–2, 5'–7', 8–9, (B1)–(B5) and Q hold. Suppose $\Phi_{\tau}(u)$ and $\Omega(u)$ are continuous in some neighborhood of u_0 and $\Phi_{\tau}(u_0)$ is positive definite and $f_{\varepsilon}(0|X_{\mathbf{i}}, U_{\mathbf{i}}) \leq C$ for some $C > 0$. If u_0 is an interior point of the support of the design density $f(u)$, then, as $\mathbf{n} \rightarrow \infty$,

$$\sqrt{\mathbf{\bar{n}}h_{\mathbf{n}}^k} \left(\check{\beta}_{\tau}(u_0) - \beta(u_0) - \frac{h_{\mathbf{n}}^2}{2\mu_0} \zeta(u_0) \right) \rightarrow_d N \left(0, \frac{4\tau(1-\tau)\nu_0}{f(u_0)\mu_0^2} \Phi_{\tau}^{-1}(u_0) \Omega(u_0) \Phi_{\tau}^{-1}(u_0) \right).$$

Similarly, with $\psi_c(z) = \max\{-1, \min\{z/c, 1\}\}$, $c > 0$, it is easy to check that Assumption 4' holds. In this case, we have the following.

Theorem 4.3. Assume that Assumptions 1–2, 5'–7', 8–9, (B1)–(B5) and R hold. Suppose $\Phi_c(u)$ and $\Sigma(u)$ are continuous in some neighborhood of u_0 and $\Phi_c(u_0)$ is positive definite. Then the conclusions of Theorem 4.1 hold with $\Phi(u)$ replaced by $\Phi_c(u)$.

The proofs of Theorems 4.1–4.3 are postponed to [Appendix](#).

5. An application to soil data analysis

We are analysing a spatial soil data set, soil250, in R package GeoR, which consists in uniformity trial with 250 undisturbed soil samples collected at 25 cm soil depth of spacing of 5 meters, resulting on a regular grid of 25×10 points. The data frame is with 250 observations on the 22 variables about several soil chemistry properties measured on the grid. In this analysis, for simplicity, we only consider 10 variables, which are Linha (x-coordinate), Coluna (y-coordinate), pHKCl (soil pH by KCl), Ca (calcium content), Mg (magnesium content), K (potassio content), Al (aluminium content), C (carbon content), N (nitrogen content), and CTC (cation exchange capability). Zheng *et al.* [42] recently analysed the spatial spectral density for the CTC variable. Our objective here is to analyse the impacts of the soil chemistry properties of Ca, Mg, K, Al, C and N as well as the soil chemistry property index pHKCl on the CTC, an important soil property for soil reservation concerned with in agriculture science. In the original data, there seem to be some spatial trends for all variables, so we apply `sm.regression` in R package `sm` to remove the spatial trends. The resulting spatial data of these variables, denoted by prefix “res.” standing for residual, are plotted in panel (a) of Figure 1, which appear more stationary. We hence analyse the relationship of these variables based on the residual data, the kernel density

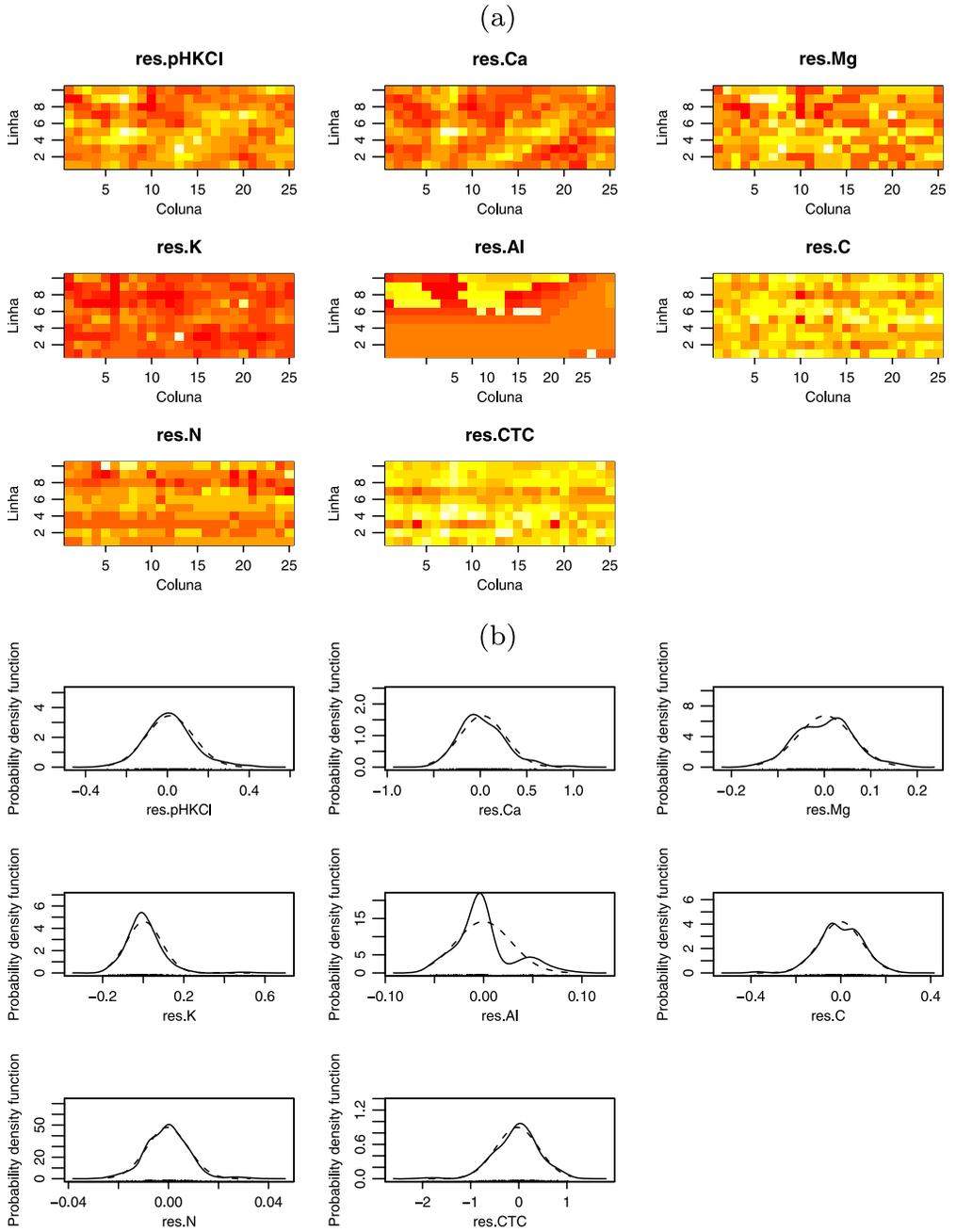


Figure 1. Soil data: (a) The images of 8 soil properties variables after spatial trend removal by sm.regression, plotted over space (Linha, Coluna); (b) The kernel density estimates (solid line) of the 8 soil properties variables after spatial trend removal by sm.regression, where the dashed line is for the Gaussian density with the same mean and variance, respectively, for each variable.

estimates (in solid line) of which are also plotted in Figure 1(b), with the dashed line for the Gaussian density of the same mean and variance. It is clear that the distribution of the response, res.CTC , is quite close to normal, indicating that mean and median regression analyses are similar (only median regression is provided below). Further, considering the spatial neighbouring effects, we also include the nearest neighbour variables of the CTC, denoted by res.CTCw , res.CTCe , res.CTCn and res.CTCs for the west, east, north and south nearest neighbours. Thus we have 11 covariates, including res.pHKCl and the soil chemistry property variables (res.Ca , res.Mg , res.K , res.Al , res.C and res.N) as well as the four neighbouring variables of the CTC. It is impossible to apply general nonparametric quantile analysis of the impacts of these covariates on the response as done in [15] as it suffers from severe ‘‘curse of dimensionality’’.

To have a preliminary understanding of the possible relationship, we made a simple nonparametric regression analysis of the CTC on each covariate by applying sm.regression in R package sm (the results not reported here to save space). It appears that the response res.CTC is basically linearly related with each of the individual covariates, suggesting we may consider regressing $Y_{ij} = \text{res.CTC}_{ij}$ at a grid (i, j) on the covariates in a linear form

$$\begin{aligned} & a_0(\text{res.pHKCl}_{ij}) + a_1(\text{res.pHKCl}_{ij})\text{res.Ca}_{ij} + a_2(\text{res.pHKCl}_{ij})\text{res.Mg}_{ij} \\ & + a_3(\text{res.pHKCl}_{ij})\text{res.K}_{ij} + a_4(\text{res.pHKCl}_{ij})\text{res.Al}_{ij} + a_5(\text{res.pHKCl}_{ij})\text{res.C}_{ij} \\ & + a_6(\text{res.pHKCl}_{ij})\text{res.N}_{ij} + a_7(\text{res.pHKCl}_{ij})\text{res.CTCw}_{ij} \\ & + a_8(\text{res.pHKCl}_{ij})\text{res.CTCe}_{ij} + a_9(\text{res.pHKCl}_{ij})\text{res.CTCn}_{ij} \\ & + a_{10}(\text{res.pHKCl}_{ij})\text{res.CTCs}_{ij} \end{aligned} \quad (5.1)$$

for $1 \leq i \leq 25, 1 \leq j \leq 10$, where we take the chemical property index variable, res.pHKCl , as a regime variable U and are interested in the effects of this index variable in the coefficient functions $a_1(\cdot), \dots, a_{10}(\cdot)$ of the components of X denoted for the vector of other variables, for example, whether these coefficient functions are constant or not. Here $a_0(\cdot)$ is the baseline effect from the index variable.

We here suggest selecting the required bandwidth h in (2.2) by applying an empirical rule of Fan *et al.* [6] with cross-validation (CV) of Stone [35] using the check function $\rho(z)$ in (2.2). In the time series context, the argument for cross-validation as an appropriate method for the bandwidth can be found in [22,32] and [39], among others. This empirical rule of bandwidth selection procedure is computationally efficient [6]; see also [28], Section 2.3, in the least squares setting. We first examine the median regression under $\tau = 0.5$, with the range of h taken between 0.15 and 0.3 (partitioned into q small intervals of length 0.01). The spatial quantile estimates of these coefficient functions under $\tau = 0.5$ are provided in Figure 2 in solid lines, for the selected bandwidth of $h = 0.263$ by a leave-one-out CV with $\rho(z) = |z|$ in (2.2). In order to take into account the dependence in the observations, we also applied a leave-five-out CV for the selection of bandwidth with $h = 0.235$ selected, by which the estimated median regression coefficient functions are very similar to those with leave-one-out CV, and are therefore omitted in Figure 2. It seems that many of the functional coefficients, such as the baseline function $a_0(\cdot)$, are nearly linear. We hence also made median regression analysis with the coefficient functions of a linear form, reported in dashed lines in Figure 2. In order to examine the impacts of the covariates on the high or low CTC variable, we also made similar analysis of spatial quantile regression of

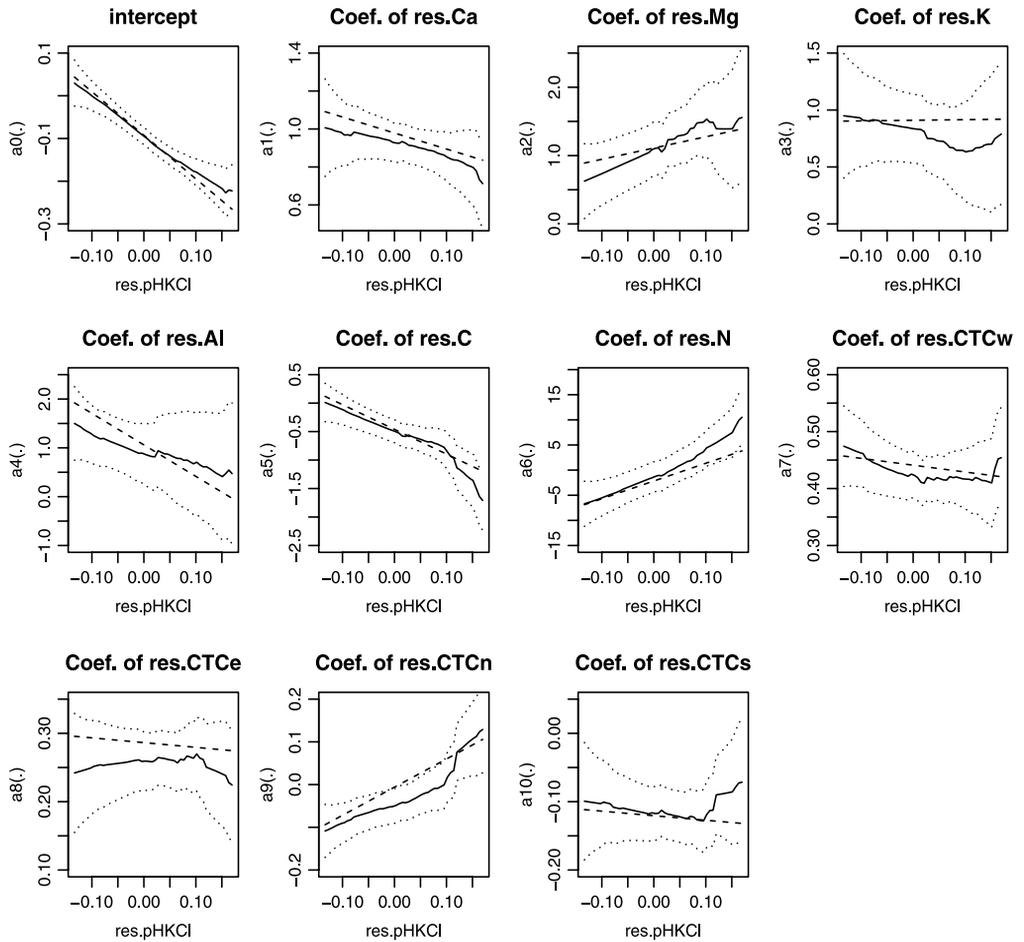


Figure 2. Soil data: The median regression estimate ($\tau = 0.5$) of the functional coefficients in (5.1). The solid (—) line is for the quantile regression with nonparametric functional coefficients in this paper by using the selected bandwidth of 0.263 by the leave-one-out CV, the dashed (---) line is for the functional coefficient of parametric linear function, and the dotted (···) lines are for the 95% confidence intervals constructed by asymptotic normality.

(5.1) under $\tau = 0.85$ and $\tau = 0.15$, plotted in Figure 3 and Figure 4, respectively. In view of the sparsity of extreme data, the range of h was taken a bit larger between 0.25 and 0.6 (with refined partition of q small intervals of length 0.001), with the leave-one-out CV-selected bandwidths equal to 0.5 and 0.487 for $\tau = 0.85$ and $\tau = 0.15$, respectively. Again the estimated coefficient functions based on leave-five-out CV, which are omitted here, are similar to those with leave-one-out CV.

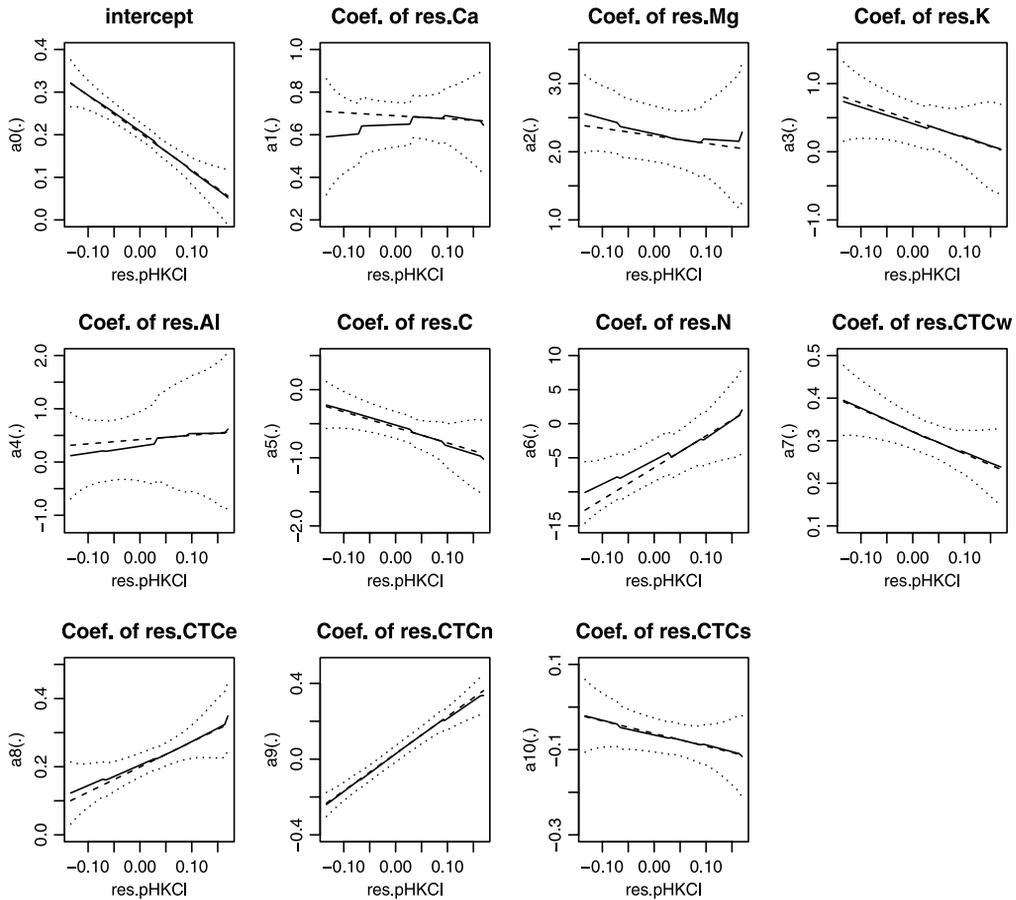


Figure 3. Soil data: The quantile regression estimate ($\tau = 0.85$) of the functional coefficients in (5.1). The solid (—) line is for the quantile regression with nonparametric functional coefficients in this paper by using the selected bandwidth of 0.263 by the leave-one-out CV, the dashed (---) line is for the functional coefficient of parametric linear function, and the dotted (···) lines are for the 95% confidence intervals constructed by asymptotic normality.

As the information on how variable the estimates are would be interesting for statistical inference, we have also provided pointwise bands, that is, a collection of confidence intervals, for the quantile coefficient estimates on the basis of the asymptotic theorem (Theorem 3.1), which are plotted in dotted lines in Figures 2–4. Here the key difficulty in doing so is the estimation of $f_\varepsilon(0|X_i, U_i)$ associated with $\Phi_\tau(u)$ in the asymptotic variance of Theorem 3.1. Note that we cannot simply assume ε_i and (X_i, U_i) are independent as in the traditional varying-coefficient analysis in the literature. Therefore, the estimation suffers from severe curse of dimensionality (note that the dimension of (X_i, U_i) is equal to 11) at a first glance. Fortunately, however, by applying indepTest in the R package “copula” with the independence test method of Genest

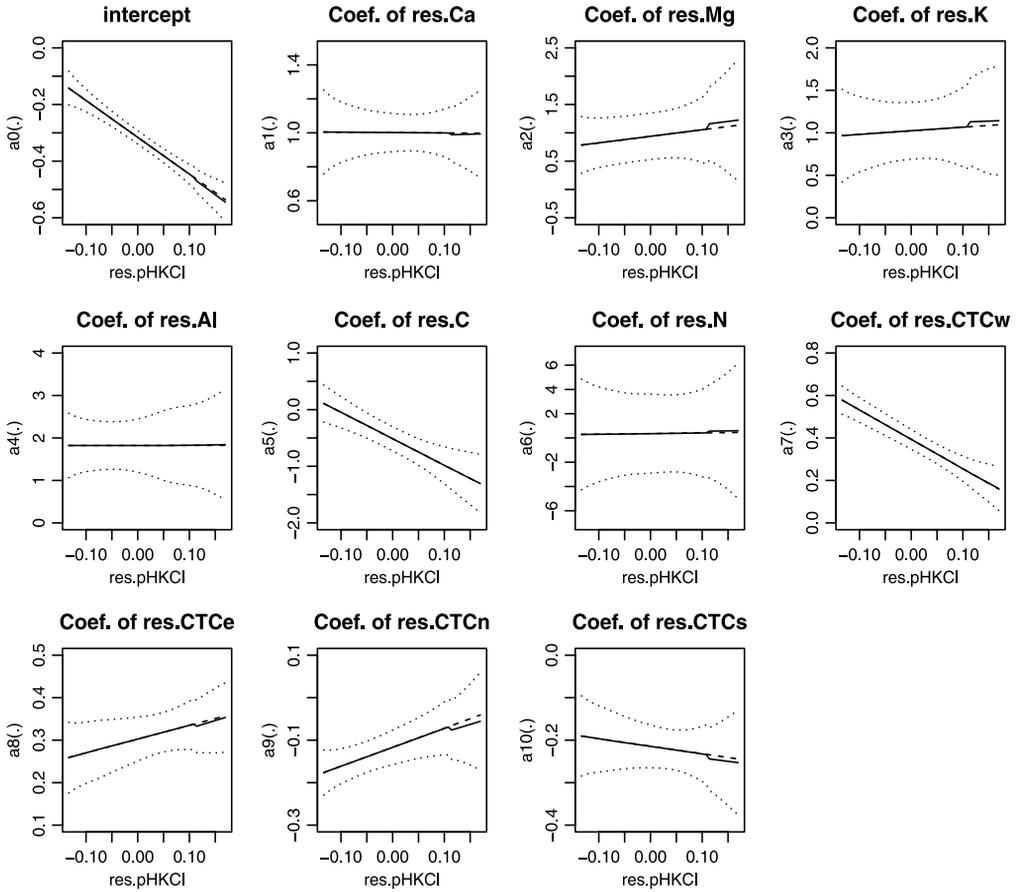


Figure 4. Soil data: The quantile regression estimate ($\tau = 0.15$) of the functional coefficients in (5.1). The solid (—) line is for the quantile regression with nonparametric functional coefficients in this paper by using the selected bandwidth of 0.263 by the leave-one-out CV, the dashed (---) line is for the functional coefficient of parametric linear function, and the dotted (···) lines are for the 95% confidence intervals constructed by asymptotic normality.

and Rémillard [11], we find at 5% significance level that the estimated ε_i is only dependent on $(X_{i,8}, X_{i,9}) = (\text{res.CTCw}, \text{res.CTCe})$ at $\tau = 0.85$, and on $(X_{i,10}, X_{i,11}) = (\text{res.CTCn}, \text{res.CTCs})$ at $\tau = 0.15$, while the estimated ε_i and (X_i, U_i) are independent at $\tau = 0.5$. Hence, we can easily estimate the conditional density function $f_\varepsilon(0|X_i, U_i)$ by applying `npcdens` in the R package “np” with the method of Hall *et al.* [13]. The asymptotic variance of Theorem 3.1 can thus be calculated, by which the confidence intervals are constructed.

Let us first examine Figure 4 with $\tau = 0.15$. We can see in this figure that the coefficient functions are close to linear lines, which, except $a_6(\cdot)$, are significant from zero at 5% significance level. For Figure 3 with $\tau = 0.85$, though the coefficient functions are also close to

linear, the magnitudes of the effects of different covariates through the regime index, res.pHKCl , appear quite significantly different from those in Figure 4. These findings are also different from that in Figure 2 with $\tau = 0.5$, meaning that the effects of the different covariates through the regime index, res.pHKCl , on CTC perform differently at low, median and high values of CTC:

- (1) Different covariate effects: The covariate effects under different τ 's appear quite different in magnitude, but mostly are of the same signs. Here res.Ca , res.Mg , res.K and res.Al have nonnegative effects for which we cannot reject their constancy, while res.C has a negative effect decreasing with res.pHKCl , at 5% level of significance. However, for the covariate res.N , it seems clear at 5% significance level that $a_6(\cdot)$ is not significant from zero under $\tau = 0.15$, but it is an increasing function that is negative (turning to positive values) when the regime, res.pHKCl , is less than the thresholds, 0.05 and 0.10, under $\tau = 0.5$ and $\tau = 0.85$, respectively. It looks that the chemistry properties of N (nitrogen content) may play a significantly different role with the regime in the adjustment of the high/low CTC in the soil. These findings are beyond the traditional median or mean regression analysis.
- (2) Different neighbouring effects: The neighbouring effects in quantile analysis under different τ 's also appear different in magnitude, and mostly are of the same sign in the coefficients (positively correlated with west and east neighbours but negatively with south). However, it looks at 5% significance level that the CTC in the soil has a negative correlation with its north neighbour res.CTCn (note the coefficient $a_9(\cdot)$ is negative) under $\tau = 0.15$, but becomes positively correlated with its north neighbour when the regime, res.pHKCl , is over the thresholds 0.10 and 0.05 under $\tau = 0.5$ and $\tau = 0.85$, respectively.
- (3) Different regime effects: The regime effects of res.pHKCl seem more involved under $\tau = 0.5$, in particular in the coefficients of res.C , res.N and res.CTCn , which appear marginally nonlinear at 5% significance level. For high ($\tau = 0.85$) and low ($\tau = 0.15$) quantiles, the regime effects of res.pHKCl appear linear or constant.

To sum up, although the above analysis is illustrative only, it seems apparent that the functional-coefficient spatial quantile regression proposed in this paper is helpful to uncover and understand the underlying relationship of the soil chemistry properties with CTC (cation exchange capability) through the regime index pHKCl . These properties are interesting and important topics in soil reservation and management.

Appendix: Proofs

In this section, we only sketch the proof of the main theorems with the necessary lemmas listed. The detail of the proof of the lemmas is much more complicated and we describe it in detail in the supplementary material [29].

Let C denote a generic positive constants not depending on \mathbf{n} , which may take on different values at each appearance. Under Assumption 2, by Taylor expansion, for $U_{\mathbf{i}} = (U_{i1}, \dots, U_{ik})^T$ such that $|U_{i\ell} - u_{0\ell}| \leq Mh_{\mathbf{n}}$, $1 \leq \ell \leq k$, we have $\beta_r(U_{\mathbf{i}}) = \beta_r(u_0) + \dot{\beta}_r(u_0)^T(U_{\mathbf{i}} - u_0) + \frac{1}{2}(U_{\mathbf{i}} - u_0)^T \ddot{\beta}_r(\xi_{\mathbf{i}r})(U_{\mathbf{i}} - u_0)$, where $\dot{\beta}_r(u_0) = (\dot{\beta}_{r1}(u_0), \dots, \dot{\beta}_{rk}(u_0))^T$ stands for the gradient of $\beta_r(u)$ with respect to u at $u = u_0$, and $\xi_{\mathbf{i}r} = (\xi_{i r 1}, \dots, \xi_{i r k})^T$ satisfies that $|\xi_{i r \ell} - u_{0\ell}| < |U_{i\ell} - u_{0\ell}|$ for $1 \leq \ell \leq k$. Denote $\beta^*(\xi_{\mathbf{i}}) = ((U_{\mathbf{i}} - u_0)^T \dot{\beta}_1(\xi_{i1})(U_{\mathbf{i}} - u_0), \dots, (U_{\mathbf{i}} - u_0)^T \ddot{\beta}_d(\xi_{id})(U_{\mathbf{i}} - u_0))^T$, and

$e_i = \frac{1}{2}\beta^*(\xi_i)^T X_i$. Let $\tilde{b}_l = (b_{1l}, \dots, b_{dl})^T$, $\tilde{\beta}_l(u_0) = (\dot{\beta}_{1l}(u_0), \dots, \dot{\beta}_{dl}(u_0))^T$, $\tilde{a} = (a_1, \dots, a_d)^T$, $\tilde{b} = (\tilde{b}_1^T, \dots, \tilde{b}_k^T)^T$, $\dot{\beta}(u_0) = (\tilde{\beta}_1(u_0)^T, \dots, \tilde{\beta}_k(u_0)^T)^T$. Set $Z_i = (\tilde{\mathbf{n}}h_{\mathbf{n}}^k)^{-1/2}(1, h_{\mathbf{n}}^{-1}(U_i - u_0)^T)^T \otimes X_i$ and $t = (\tilde{\mathbf{n}}h_{\mathbf{n}}^k)^{1/2}((\tilde{a} - \beta(u_0))^T, h_{\mathbf{n}}(\tilde{b} - \dot{\beta}(u_0))^T)^T$, where \otimes is the Kronecker product. Then we have the following new optimization problem

$$\hat{t} = \underset{t}{\text{Argmin}} \sum_{i \in G_N} [\rho(\varepsilon_i + e_i - t^T Z_i) - \rho(\varepsilon_i + e_i)] K\left(\frac{U_i - u_0}{h_{\mathbf{n}}}\right). \quad (\text{A.1})$$

Clearly

$$\hat{t} = (\tilde{\mathbf{n}}h_{\mathbf{n}}^k)^{1/2}((\hat{a} - \beta(u_0))^T, h_{\mathbf{n}}(\hat{b} - \dot{\beta}(u_0))^T)^T. \quad (\text{A.2})$$

Denote the objective function in (A.1) by $S_{\mathbf{n}}(t)$ and set

$$\Gamma_{\mathbf{n}}(t) = \sum_{i \in G_N} E([\rho(\varepsilon_i + e_i - t^T Z_i) - \rho(\varepsilon_i + e_i)] | X_i, U_i) K\left(\frac{U_i - u_0}{h_{\mathbf{n}}}\right).$$

Let $\Upsilon_{\mathbf{n}}(t) = \sum_{i \in G_N} t^T Z_i \psi(\varepsilon_i) K\left(\frac{U_i - u_0}{h_{\mathbf{n}}}\right)$ and $R_{\mathbf{n}}(t) = S_{\mathbf{n}}(t) - \Gamma_{\mathbf{n}}(t) + \Upsilon_{\mathbf{n}}(t)$. Then

$$S_{\mathbf{n}}(t) = \Gamma_{\mathbf{n}}(t) - \Upsilon_{\mathbf{n}}(t) + R_{\mathbf{n}}(t). \quad (\text{A.3})$$

We first present several lemmas that are necessary to prove the theorems.

Lemma A.1. *Under the Assumptions 1 and 3–8, if $n_k h_{\mathbf{n}}^{\delta k/[a(2+\delta)]} > 1$, then for any fixed t , as $\mathbf{n} \rightarrow \infty$, it holds that*

$$R_{\mathbf{n}}(t) = o_{\mathbf{p}}(1).$$

Lemma A.2. *Assume that Assumptions 1, 3 and 5–8 hold and $\Phi(u)$ is continuous in some neighborhood of u_0 . If $n_k h_{\mathbf{n}}^{\delta k/[a(2+\delta)]} > 1$, then, as $\mathbf{n} \rightarrow \infty$, it holds that*

$$\Gamma_{\mathbf{n}}(t) = \frac{1}{2} f(u_0) t^T (\Delta \otimes \Phi(u_0)) t - \frac{1}{2} \tilde{\mathbf{n}}^{1/2} h_{\mathbf{n}}^{(k+4)/2} f(u_0) t^T \Lambda \otimes (\Phi(u_0) \zeta(u_0)) + o_{\mathbf{p}}(1),$$

where $\Delta = \text{diag}(\int_{\tilde{M}} K(u) du, \int_{\tilde{M}} uu^T K(u) du)$ and $\Lambda = (1, O^T)^T$, where O is a $k \times 1$ vector with entries zero.

Lemma A.3. *Assume that Assumptions 1, 3 and 5–8 hold and $\Sigma(u)$ is continuous in some neighborhood of u_0 . If $n_k h_{\mathbf{n}}^{\delta k/[a(2+\delta)]} > 1$, then, as $\mathbf{n} \rightarrow \infty$, it holds that*

$$D(\Upsilon_{\mathbf{n}}(t)) = f(u_0) t^T (\tilde{\Delta} \otimes \Sigma(u_0)) t + o(1),$$

where $\tilde{\Delta} = \text{diag}(\int_{\tilde{M}} K^2(u) du, \int_{\tilde{M}} uu^T K^2(u) du)$.

Lemma A.4. Let $K_h(U_i) = K\left(\frac{U_i - u_0}{h_n}\right)$ and

$$A_n(t) = \sum_{i \in G_N} \psi(\varepsilon_i + e_i - t^T Z_i) t^T [(1, h_n^{-1}(U_i - u_0))^T]^T \otimes (\hat{X}_i - X_i) K_h(U_i),$$

$$B_n(t) = \sum_{i \in G_N} \psi(\varepsilon_i + e_i - t^T Z_i) t^T [(0, h_n^{-1}(\hat{U}_i - U_i))^T]^T \otimes X_i K_h(U_i).$$

Under the assumptions of Theorem 4.1, for any fixed t , as $\mathbf{n} \rightarrow \infty$, it holds that

$$(\tilde{\mathbf{n}} h_n^k)^{-1/2} A_n(t) = o_p(1), \quad (\tilde{\mathbf{n}} h_n^k)^{-1/2} B_n(t) = o_p(1).$$

Lemma A.5. Under the assumptions of Theorem 4.1, for any fixed t , as $\mathbf{n} \rightarrow \infty$, it holds that

$$\sum_{i \in G_N} [\rho(\varepsilon_i + e_i - t^T Z_i) - \rho(\varepsilon_i + e_i)] [K_h(\hat{U}_i) - K_h(U_i)] = o_p(1).$$

Proof of Theorem 2.1. Let $\Gamma(t) = \frac{1}{2} f(u_0) t^T (\Delta \otimes \Phi(u_0)) t$. By Lemmas A.1 and A.2 and (A.3), for fixed t we have

$$S_n(t) = \Gamma(t) - \frac{1}{2} \tilde{\mathbf{n}}^{1/2} h_n^{(k+4)/2} f(u_0) t^T \Lambda \otimes (\Phi(u_0) \zeta(u_0)) - \Upsilon_n(t) + \tilde{R}_n(t), \quad (\text{A.4})$$

where $\tilde{R}_n(t) = R_n(t) + o_p(1) = o_p(1)$, and hence

$$S_n(t) + \frac{1}{2} \tilde{\mathbf{n}}^{1/2} h_n^{(k+4)/2} f(u_0) t^T \Lambda \otimes (\Phi(u_0) \zeta(u_0)) + \Upsilon_n(t) = \Gamma(t) + \tilde{R}_n(t).$$

By Lemma A.3, $\Upsilon_n(t)$ is bounded in probability. Thus, the random convex function $S_n(t) + \tilde{\Gamma}_n(t) + \Upsilon_n(t)$, for fixed t converges in probability to the function $\Gamma(t)$. According to the convexity lemma [31], we conclude that for any compact set K

$$\sup_{t \in K} |\tilde{R}_n(t)| = o_p(1). \quad (\text{A.5})$$

Let $t^* = \frac{1}{2} \tilde{\mathbf{n}}^{1/2} h_n^{(k+4)/2} (\Delta^{-1} \Lambda) \otimes \zeta(u_0)$ and

$$\tilde{t} = t^* + \frac{1}{f(u_0)} (\Delta^{-1} \otimes \Phi^{-1}(u_0)) \sum_{i \in G_N} Z_i \psi(\varepsilon_i) K\left(\frac{U_i - u_0}{h_n}\right). \quad (\text{A.6})$$

In the following, we will prove that for any sufficient small $\epsilon > 0$,

$$P\{\|\hat{t} - \tilde{t}\| < \epsilon\} \rightarrow 1. \quad (\text{A.7})$$

According to (A.1) and Lemma A.3 and using the convexity of ρ , to prove (A.7), we need only to show that for any sufficient large $L^* > 0$,

$$P\left(\left\{\inf_{\|t - \tilde{t}\| = \epsilon} (S_n(t) - S_n(\tilde{t})) > 0\right\} \cap \{\|\tilde{t}\| \leq L^*\}\right) \rightarrow 1. \quad (\text{A.8})$$

By (A.4) and (A.6), we get

$$S_{\mathbf{n}}(t) = \frac{1}{2}f(u_0)t^T(\Delta \otimes \Phi(u_0))t - f(u_0)t^T(\Delta \otimes \Phi(u_0))\tilde{t} + \tilde{R}_{\mathbf{n}}(t).$$

Since

$$t^T(\Delta \otimes \Phi(u_0))\tilde{t} = \frac{1}{2}[t^T(\Delta \otimes \Phi(u_0))t + \tilde{t}^T(\Delta \otimes \Phi(u_0))\tilde{t} - (t - \tilde{t})^T(\Delta \otimes \Phi(u_0))(t - \tilde{t})].$$

Hence,

$$S_{\mathbf{n}}(t) = \frac{1}{2}f(u_0)(t - \tilde{t})^T(\Delta \otimes \Phi(u_0))(t - \tilde{t}) - \frac{1}{2}f(u_0)\tilde{t}^T(\Delta \otimes \Phi(u_0))\tilde{t} + \tilde{R}_{\mathbf{n}}(t).$$

Using the above, for \tilde{t} satisfying that $\|\tilde{t}\| \leq L^*$, it holds that

$$S_{\mathbf{n}}(\tilde{t}) = -\frac{1}{2}f(u_0)\tilde{t}^T(\Delta \otimes \Phi(u_0))\tilde{t} + \tilde{R}_{\mathbf{n}}(\tilde{t}).$$

Note that $\|t - \tilde{t}\| = \epsilon$, we conclude that

$$S_{\mathbf{n}}(t) - S_{\mathbf{n}}(\tilde{t}) \geq \frac{1}{2}f(u_0)\lambda_{\min, \Delta}\lambda_{\min}(u_0)\epsilon^2 - 2 \sup_{\|t\| \leq L^* + \epsilon} |\tilde{R}_{\mathbf{n}}(t)|,$$

where $\lambda_{\min, \Delta}$ and $\lambda_{\min}(u_0)$ are the minimum eigenvalue of Δ and $\Phi(u_0)$ respectively. Therefore, (A.8) follows from (A.5) and the above. Consequently, (A.7) holds. Under assumptions of Theorem 2.1, using arguments similar to those used in the proof of Lemma 3.1 of [14] and Lemma A.3, we can show that

$$\sum_{\mathbf{i} \in G_N} Z_{\mathbf{i}}\psi(\varepsilon_{\mathbf{i}})K\left(\frac{U_{\mathbf{i}} - u_0}{h_{\mathbf{n}}}\right) \rightarrow_d N(0, f(u_0)\tilde{\Delta} \otimes \Sigma(u_0)).$$

Now the conclusion of Theorem 2.1 follows from (A.2), (A.7), (A.6) and the above, and the proof of Theorem 2.1 is finished. \square

Proof of Theorem 2.2. Let $\Pi_0(u_0) = \Phi(u_0)\bar{\zeta}(u_0)$, $\Pi_l(u_0) = \Phi(u_0)\bar{\zeta}^{(l)}(u_0)$, $l = 1, \dots, k$, $\Pi(u_0) = (\Pi_0(u_0)^T, \Pi_1(u_0)^T, \dots, \Pi_k(u_0)^T)^T$ and

$$t^* = \frac{1}{2}\tilde{\mathbf{n}}^{1/2}h_{\mathbf{n}}^{(k+4)/2}(\Delta_c^{-1} \otimes \Phi^{-1}(u_0))\Pi(u_0),$$

$$\tilde{t} = t^* + \frac{1}{f(u_*)}(\Delta_c^{-1} \otimes \Phi^{-1}(u_0)) \sum_{\mathbf{i} \in G_N} Z_{\mathbf{i}}\psi(\varepsilon_{\mathbf{i}})K\left(\frac{U_{\mathbf{i}} - u_h}{h_{\mathbf{n}}}\right).$$

Using the arguments similar to those in the proof of Theorem 2.1, we can finish the proof of Theorem 2.2. \square

Proof of Theorem 4.1. Recall that $N = 2$ has been assumed throughout this section. Following [17], $Y(s)$, $X(s)$ and $U(s)$ satisfy that $\sup_{s \in [0, 1]^2} |\hat{\alpha}_Y(s) - \alpha_Y(s)| = O_p(\epsilon_{\mathbf{n}})$, $\sup_{s \in [0, 1]^2} \|\hat{\alpha}_X(s) - \alpha_X(s)\| = O_p(\epsilon_{\mathbf{n}})$ and $\sup_{s \in [0, 1]^2} \|\hat{\alpha}_U(s) - \alpha_U(s)\| = O_p(\epsilon_{\mathbf{n}})$ with $\epsilon_{\mathbf{n}} = (\ln \tilde{\mathbf{n}}/(\tilde{\mathbf{n}}g_{\mathbf{n}}^2))^{1/2} + g_{\mathbf{n}}^m =:$

$\epsilon_{\mathbf{n}}^{(1)} + \epsilon_{\mathbf{n}}^{(2)}$, where $\epsilon_{\mathbf{n}}^{(1)}$ is obtained as in the proof of Theorem 2 of [17] under Assumptions (B1)–(B3), (B5) and 8, while $\epsilon_{\mathbf{n}}^{(2)}$ readily follows from Assumptions (B3) and (B5). Therefore, we have

$$\max_{\mathbf{i}} |\hat{Y}_{\mathbf{i}} - Y_{\mathbf{i}}| = O_p(\epsilon_{\mathbf{n}}), \quad \max_{\mathbf{i}} \|\hat{X}_{\mathbf{i}} - X_{\mathbf{i}}\| = O_p(\epsilon_{\mathbf{n}}), \quad \max_{\mathbf{i}} \|\hat{U}_{\mathbf{i}} - U_{\mathbf{i}}\| = O_p(\epsilon_{\mathbf{n}}). \quad (\text{A.9})$$

Let $\hat{\epsilon}_{\mathbf{i}} = \hat{Y}_{\mathbf{i}} - \hat{X}_{\mathbf{i}}^T \beta(\hat{U}_{\mathbf{i}})$, $\hat{e}_{\mathbf{i}} = \frac{1}{2} \beta^*(\bar{\xi}_{\mathbf{i}})^T \hat{X}_{\mathbf{i}}$, $\hat{Z}_{\mathbf{i}} = (\tilde{\mathbf{n}} h_{\mathbf{n}}^k)^{-1/2} (1, h^{-1}(\hat{U}_{\mathbf{i}} - u_0)^T)^T \otimes \hat{X}_{\mathbf{i}}$ and $\check{t} = (\tilde{\mathbf{n}} h_{\mathbf{n}}^k)^{1/2} ((\check{a} - \beta(u_0))^T, h_{\mathbf{n}}(\check{b} - \dot{\beta}(u_0))^T)^T$. Then

$$\check{t} = \underset{t}{\text{Argmin}} \sum_{\mathbf{i} \in G_N} [\rho(\hat{\epsilon}_{\mathbf{i}} + \hat{e}_{\mathbf{i}} - t^T \hat{Z}_{\mathbf{i}}) - \rho(\hat{\epsilon}_{\mathbf{i}} + \hat{e}_{\mathbf{i}})] K_h(\hat{U}_{\mathbf{i}}),$$

where $K_h(\hat{U}_{\mathbf{i}}) = K((\hat{U}_{\mathbf{i}} - u_0)/h_{\mathbf{n}})$. Let $\hat{S}_{\mathbf{n}}(t) = \sum_{\mathbf{i} \in G_N} [\rho(\hat{\epsilon}_{\mathbf{i}} + \hat{e}_{\mathbf{i}} - t^T \hat{Z}_{\mathbf{i}}) - \rho(\hat{\epsilon}_{\mathbf{i}} + \hat{e}_{\mathbf{i}})] K_h(\hat{U}_{\mathbf{i}})$. According to (A.4) and the proof of Theorem 2.1, to finish the proof of Theorem 4.1, we need only show that for fixed t , it holds that

$$\hat{S}_{\mathbf{n}}(t) - S_{\mathbf{n}}(t) = o_p(1). \quad (\text{A.10})$$

Let $\theta_{\mathbf{i}} = \epsilon_{\mathbf{i}} + e_{\mathbf{i}}$, $\hat{\theta}_{\mathbf{i}} = \hat{\epsilon}_{\mathbf{i}} + \hat{e}_{\mathbf{i}}$, $V_{\mathbf{n}1} = \sum_{\mathbf{i} \in G_N} [(\rho(\hat{\theta}_{\mathbf{i}} - t^T \hat{Z}_{\mathbf{i}}) - \rho(\hat{\theta}_{\mathbf{i}})) - (\rho(\theta_{\mathbf{i}} - t^T Z_{\mathbf{i}}) - \rho(\theta_{\mathbf{i}}))] K_h(\hat{U}_{\mathbf{i}})$, and $V_{\mathbf{n}2} = \sum_{\mathbf{i} \in G_N} [\rho(\theta_{\mathbf{i}} - t^T Z_{\mathbf{i}}) - \rho(\theta_{\mathbf{i}})] [K_h(\hat{U}_{\mathbf{i}}) - K_h(U_{\mathbf{i}})]$. Then

$$\hat{S}_{\mathbf{n}}(t) - S_{\mathbf{n}}(t) = V_{\mathbf{n}1} + V_{\mathbf{n}2}. \quad (\text{A.11})$$

Let $V_{11} = |\psi(\hat{\theta}_{\mathbf{i}} - t^T \hat{Z}_{\mathbf{i}}) - \psi(\theta_{\mathbf{i}} - t^T Z_{\mathbf{i}})| \cdot |(\hat{\theta}_{\mathbf{i}} - \theta_{\mathbf{i}}) - t^T (\hat{Z}_{\mathbf{i}} - Z_{\mathbf{i}})|$, $V_{12} = |\psi(\hat{\theta}_{\mathbf{i}}) - \psi(\theta_{\mathbf{i}})| \cdot |(\hat{\theta}_{\mathbf{i}} - \theta_{\mathbf{i}})|$ and $V_{13} = |\psi(\theta_{\mathbf{i}} - t^T Z_{\mathbf{i}}) - \psi(\theta_{\mathbf{i}})| \cdot |(\hat{\theta}_{\mathbf{i}} - \theta_{\mathbf{i}})|$. By the convexity of $\rho(\cdot)$, it holds that

$$|\rho(\hat{\theta}_{\mathbf{i}} - t^T \hat{Z}_{\mathbf{i}}) - \rho(\theta_{\mathbf{i}} - t^T Z_{\mathbf{i}}) - \psi(\theta_{\mathbf{i}} - t^T Z_{\mathbf{i}})[(\hat{\theta}_{\mathbf{i}} - \theta_{\mathbf{i}}) - t^T (\hat{Z}_{\mathbf{i}} - Z_{\mathbf{i}})]| \leq V_{11}$$

and $|\rho(\hat{\theta}_{\mathbf{i}}) - \rho(\theta_{\mathbf{i}}) - \psi(\theta_{\mathbf{i}})(\hat{\theta}_{\mathbf{i}} - \theta_{\mathbf{i}})| \leq V_{12}$. Hence

$$V_{\mathbf{n}1} \leq \sum_{\mathbf{i} \in G_N} (V_{11} + V_{12} + V_{13}) K_h(\hat{U}_{\mathbf{i}}) + |V_{\mathbf{n}3}|, \quad (\text{A.12})$$

where

$$V_{\mathbf{n}3} = \sum_{\mathbf{i} \in G_N} \psi(\theta_{\mathbf{i}} - t^T Z_{\mathbf{i}}) t^T (\hat{Z}_{\mathbf{i}} - Z_{\mathbf{i}}) K_h(\hat{U}_{\mathbf{i}}). \quad (\text{A.13})$$

Since $\theta_{\mathbf{i}} = \epsilon_{\mathbf{i}} + e_{\mathbf{i}} = Y_{\mathbf{i}} - X_{\mathbf{i}}^T \beta(u_0) - \sum_{r=1}^d (U_{\mathbf{i}} - u_0)^T \dot{\beta}_r(u_0) X_{\mathbf{i}r}$ and $\hat{\theta}_{\mathbf{i}} = \hat{Y}_{\mathbf{i}} - \hat{X}_{\mathbf{i}}^T \beta(u_0) - \sum_{r=1}^d (\hat{U}_{\mathbf{i}} - u_0)^T \dot{\beta}_r(u_0) \hat{X}_{\mathbf{i}r}$, by (A.9) and Assumption 6', it is easy to prove that $\max_{\mathbf{i}} |\hat{\theta}_{\mathbf{i}} - \theta_{\mathbf{i}}| = O_p(\epsilon_{\mathbf{n}})$. On the other hand,

$$t^T (\hat{Z}_{\mathbf{i}} - Z_{\mathbf{i}}) = (\tilde{\mathbf{n}} h_{\mathbf{n}}^k)^{-1/2} \left[\left(t_0 + \sum_{l=1}^k \frac{\hat{U}_{\mathbf{i}l} - u_{0l}}{h_{\mathbf{n}}} t_l \right)^T (\hat{X}_{\mathbf{i}} - X_{\mathbf{i}}) + \left(\sum_{l=1}^k \frac{\hat{U}_{\mathbf{i}l} - U_{\mathbf{i}l}}{h_{\mathbf{n}}} t_l \right)^T X_{\mathbf{i}} \right].$$

By (A.9) and Assumption 6', we have $\max_i |t^T(\hat{Z}_i - Z_i)| = O_p((\tilde{\mathbf{n}}h_n^k)^{-1/2}h_n^{-1}\epsilon_n)$. Hence

$$\max_i (|\hat{\theta}_i - \theta_i| + |t^T(\hat{Z}_i - Z_i)|) = O_p(\epsilon_n + (\tilde{\mathbf{n}}h_n^k)^{-1/2}h_n^{-1}\epsilon_n) = O_p(\tilde{\epsilon}_n), \tag{A.14}$$

where $\tilde{\epsilon}_n = \epsilon_n + (\tilde{\mathbf{n}}h_n^k)^{-1/2}h_n^{-1}\epsilon_n$. By Assumption 7', we get

$$K_h(\hat{U}_i) = K_h(U_i) + h_n^{-1}(\hat{U}_i - U_i)^T \dot{K}_h(U_i)[1 + o_p(1)] = K_h(U_i) + o_p(1). \tag{A.15}$$

Therefore,

$$\begin{aligned} \sum_{i \in G_N} V_{i1} K_h(\hat{U}_i) &= [1 + o_p(1)] \sum_{i \in G_N} V_{i1} K_h(U_i) \\ &= O_p(\tilde{\epsilon}_n) \sum_{i \in G_N} |\psi(\hat{\theta}_i - t^T \hat{Z}_i) - \psi(\theta_i - t^T Z_i)| K_h(U_i). \end{aligned} \tag{A.16}$$

According to (A.14), we can assume that, with probability arbitrarily close to one, $\max_i (|\hat{\theta}_i - \theta_i| + |t^T(\hat{Z}_i - Z_i)|) \leq C\tilde{\epsilon}_n$ for some C and \mathbf{n} sufficiently large. Then by Assumption 4', it holds that $\sum_{i \in G_N} E(E(|\psi(\hat{\theta}_i - t^T \hat{Z}_i) - \psi(\theta_i - t^T Z_i)| | U_i) K_h(U_i)) = O(\tilde{\mathbf{n}}h_n^k \tilde{\epsilon}_n)$. Therefore, by Assumption 5', it holds that

$$\sum_{i \in G_N} V_{i1} K_h(\hat{U}_i) = O_p(\tilde{\mathbf{n}}h_n^k \tilde{\epsilon}_n^2) = O_p(\tilde{\mathbf{n}}h_n^k \epsilon_n^2 + h_n^{-2} \epsilon_n^2) = o_p(1). \tag{A.17}$$

Similarly

$$\sum_{i \in G_N} V_{i2} K_h(\hat{U}_i) = O_p(\tilde{\mathbf{n}}h_n^k \epsilon_n^2) = o_p(1) \tag{A.18}$$

and

$$\sum_{i \in G_N} V_{i3} K_h(\hat{U}_i) = O_p\left(\tilde{\mathbf{n}}h_n^k \max_i |t^T Z_i| \epsilon_n\right) = O_p((\tilde{\mathbf{n}}h_n^k)^{1/2} \epsilon_n) = o_p(1). \tag{A.19}$$

By (A.13), (A.15) and Lemma A.4, we obtain

$$\begin{aligned} V_{n3} &= [1 + o_p(1)] \sum_{i \in G_N} \psi(\theta_i - t^T Z_i) t^T (\hat{Z}_i - Z_i) K_h(U_i) \\ &= [1 + o_p(1)] [(\tilde{\mathbf{n}}h_n^k)^{-1/2} A_n(t) + (\tilde{\mathbf{n}}h_n^k)^{-1/2} B_n(t)] = o_p(1). \end{aligned} \tag{A.20}$$

Combining (A.12) and (A.17)–(A.20), we conclude that $V_{n1} = o_p(1)$. By Lemma A.5, it holds that $V_{n2} = o_p(1)$. Therefore, by (A.11), (A.10) holds and the proof of Theorem 4.1 is finished. \square

Proof of Theorem 4.2. The proof of Theorem 4.2 is similar to that of Theorem 4.1 except proof of (A.17). Let $\vartheta_i = \hat{\theta}_i - \theta_i - t^T(\hat{Z}_i - Z_i)$. Since $\psi(z) = 2\tau I(z > 0) + 2(\tau - 1)I(z < 0)$, it holds that

$$|\psi(\hat{\theta}_i - t^T \hat{Z}_i) - \psi(\theta_i - t^T Z_i)| \leq 2I_{\{|\theta_i - t^T Z_i| \leq |\vartheta_i|\}} \leq 2I_{\{|\epsilon_i| \leq |\epsilon_i| + |t^T Z_i| + |\vartheta_i|\}}.$$

By Assumptions 5' and 6' and (A.14), we have $\max_i (|e_i| + |t^T Z_i| + |\vartheta_i|) = O_p((\tilde{n}h_n^k)^{-1/2} + \tilde{\epsilon}_n)$. Thus we can assume that, with probability arbitrarily close to one, $\max_i (|e_i| + |t^T Z_i| + |\vartheta_i|) \leq C((\tilde{n}h_n^k)^{-1/2} + \tilde{\epsilon}_n)$ for some C and \mathbf{n} sufficiently large. By Assumption Q and the fact that $f_\varepsilon(0|X_i, U_i) \leq C$ for some $C > 0$, we get that $E I_{\{|e_i| \leq C((\tilde{n}h_n^k)^{-1/2} + \tilde{\epsilon}_n)\}} K_h(U_i) = O((\tilde{n}h_n^k)^{-1/2} + \tilde{\epsilon}_n)h_n^k$. Therefore

$$\begin{aligned} & \sum_{i \in G_N} |\psi(\hat{\theta}_i - t^T \hat{Z}_i) - \psi(\theta_i - t^T Z_i)| K_h(U_i) \\ & \leq [1 + o_p(1)] \sum_{i \in G_N} E I_{\{|e_i| \leq C((\tilde{n}h_n^k)^{-1/2} + \tilde{\epsilon}_n)\}} K_h(U_i) = O((\tilde{n}h_n^k)^{1/2} + \tilde{n}h_n^k \tilde{\epsilon}_n). \end{aligned} \tag{A.21}$$

Hence by (A.16), (A.21) and Assumption 5', we obtain

$$\sum_{i \in G_N} V_{i1} K_h(\hat{U}_i) = O((\tilde{n}h_n^k)^{1/2} \tilde{\epsilon}_n + \tilde{n}h_n^k \tilde{\epsilon}_n^2) = o_p(1).$$

Therefore, (A.17) holds and the proof of Theorem 4.2 is finished. □

Proof of Theorem 4.3. The proof of Theorem 4.3 can be done similarly as in that for Theorem 4.2, and the detail is omitted. □

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Supplementary Material

Supplement: Proofs of the lemmas in Appendix (DOI: [10.3150/12-BEJ480SUPP](https://doi.org/10.3150/12-BEJ480SUPP); .pdf). We collect the proofs for the necessary lemmas used in the above in this supplementary material [29].

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