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TEICHMÜLLER DISTANCE AND KOBAYASHI DISTANCE ON SUBSPACES OF THE UNIVERSAL TEICHMÜLLER SPACE

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Abstract

It is known that the Teichmüller distance on the universal Teichmüller space T coincides with the Kobayashi distance. For a metric subspace of T having a comparable complex structure with that of T, we can similarly consider whether or not the Teichmüller distance on the subspace coincides with the Kobayashi distance. In this paper, we give a sufficient condition for metric subspaces under which the problem above has a affirmative answer. Moreover, we introduce an example of such subspaces.

1. Introduction

The universal Teichmüller space T is the deformation space of the unit disk $\Delta = \{z \in \mathbf{C} \mid |z| < 1\}$ in the complex plane $\mathbf{C}$. In other words, T is defined to be the quotient space of the family consisting of all normalized quasiconformal self-mappings of Δ by Teichmüller equivalence. There exists a canonical distance on T called the Teichmüller distance, which measures the difference of marked conformal structures of Δ. On the other hand, since T has a complex structure, we can define the Kobayashi pseudo-distance on T, which is defined for complex manifolds as the generalization of the Poincaré distance on Δ.

It is known that the Teichmüller distance on the Teichmüller space of any hyperbolic Riemann surface coincides with the Kobayashi distance. This result was first proved by Royden [11] for the Teichmüller space of any compact Riemann surface of genus greater than 1. The genaral case was proved by Gardiner [6]. Furthermore, Earle, Kra and Krushkal [5] gave a simpler proof by using the Bers embedding and Slodkowski's theorem on the extension of holomorphic motions.

We consider a similar problem in the following setting. Let T' be a complex manifold with a holomorphic embedding ι of T' into T. We regard T' as a subset of T by identifying T' with $\iota(T')$. There exists two natural distances on T': one is the restriction of the Teichmüller distance of T to T' and the other is the Kobayashi pseudo-distance. Then we propose whether or not

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these two distances on T' coincide. In fact, the Teichmüller distance and the Kobayashi distance coincide on the submanifold T_0 consisting of all Teichmüller equivalence classes of T represented by asymptotically comformal maps of Δ onto itself. This was first proved by Earle, Gardiner and Lakic [4]. Hu, Jiang and Wang [9] showed this result more directly. The aim of this paper is to generalize their arguments and to give a sufficient condition of metric subspaces of T under which the Teichmüller distance coincides with the Kobayashi distance.

THEOREM 1.1. *Let T' be a complex manifold with a holomorphic embedding ι of T' into T, and identify T' with $\iota(T')$. If T' satisfies the following three conditions, then the Teichmüller distance on T' coincides with the Kobayashi distance.*

(1) *The set $T' \backslash \{0\}$ is contained in the set of Strebel points of T;*
(2) *For any $\tau \in T'$, the right translation map for τ maps T' onto itself;*
(3) *For every $\tau \in T' \backslash \{0\}$, there exists a representative $\mu \in \tau$ corresponding to a frame mapping such that, for every $\mu' \in \tau$ that coincides with μ outside some compact subset of Δ and for every $t \in \Delta$, $[t\mu']$ is in T'.*

Here 0 denotes the base point of T and $\tau = [\mu]$ is the Teichmüller equivalence class represented by a Beltrami coefficient μ. A Strebel point of T is a Teichmüller equivalence class of T containing a frame mapping, which is a quasiconformal mapping whose dilatation is less than the extremal maximal dilatation of the equivalence class on the outside of some compact subset in Δ.

We have preliminaries in Section 2 and Section 3, and prove Theorem 1.1 in Section 4. The proof is based on that by Hu, Jiang and Wang [9]. They constructed a sequence of holomorphic quadratic differentials converging to a holomorphic quadratic differential ϕ such that the Beltrami coefficient of the extremal mapping can be represented by $k\bar{\phi}/|\phi|$ for $0 < k < 1$ from using Strebel's frame mapping theorem repeatedly. They also showed that ϕ is not identically equal to 0 by the property of asymptotically conformal maps. We prove the same result by a property of frame mappings. Since every asymptotically conformal map is a frame mapping, our result is a generalization of [9].

In Section 5, we apply Theorem 1.1 to introducing an example of the metric subspaces, which consists of all Teichmüller equivalence classes of T containing quasiconformal mappings with square integrable Beltrami coefficients in the Poincaré metric on Δ. This subspace is contractible and characterized by a certain quasiconformal self-mapping of Δ, which is called the Douady-Earle extension (cf. [1]). To prove that the Teichmüller distance on the subspace coincides with the Kobayashi pseudo-distance, it is required to estimate the pull-back of the Poincaré metric of Δ by quasiconformal self-mappings of Δ in terms of the Poincaré metric of Δ. This is difficult because of the variety of quasiconformal mappings, but possible for Douady-Earle extensions. We will give an approximation for normalized Douady-Earle extensions depending only on a constant that bounds their maximal dilatations (Lemma 5.6). We will prove this result by the same argument as the proof of Theorem 2 in [2].

2. The universal Teichmüller space

In this section, we review several standard facts on the universal Teichüller space, the Teichüller distance and the Kobayashi distance.

We consider quasiconformal self-mappings of Δ. Each of these mappings can be extended to a homeomorphic self-mapping of the closure $\overline{\Delta}$ of Δ. By composing a suitable Möbius transformation, we normalize the mappings under the condition that 1, i and -1 are fixed. Let QC be the family of all normalized quasiconformal self-mappings of Δ. Two mappings of QC is equivalent by the definition if they agree on the boundary $\partial\Delta$ of Δ. This equivalence relation is called *Teichmüller equivalence*. The set of all Teichmüller equivalence classes is called the *universal Teichmüller space* T. A point of T represented by $f \in \mathrm{QC}$ is denoted by $[f]$. The point of T determined by the identity mapping on Δ is especially called the *base point* of T and denoted by 0.

There exists another representation of T. Let B denote the open unit ball of the Banach space of all bounded measurable functions on Δ with finite L^∞-norm. Each element of B is called a *Beltrami coefficient* on Δ. For $\mu \in B$, let f^μ be the mapping of QC with Beltrami coefficient μ. This gives a one-to-one correspondence between B and QC by the measurable Riemann mapping theorem. Hence an equivalence relation on B can be defined in the following: Two Beltrami coefficients μ and ν of B are equivalent if f^μ and f^ν agree on $\partial\Delta$. Thus T can be regarded as the set of all equivalence classes of B. A point of T represented by $\mu \in B$ is denoted by $[\mu]$. Hereafter, we use these representations of T properly according to the situation.

We next introduce the Teichmüller distance and the Kobayashi distance on T. For a quasiconformal mapping f with Beltrami coefficient μ, let

$$K(f) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}. \tag{2.1}$$

Then $K(f)$ is said to be the *maximal dilatation* of f. The *Teichmüller distance* between the points p and q of T is defined as

$$d_T(p,q) = \frac{1}{2} \inf \log K(g \circ f^{-1}), \tag{2.2}$$

where the infimum is taken over all $f \in p$ and $g \in q$. Formula (2.1) implies another representation of d_T:

$$d_T(p,q) = \frac{1}{2} \inf \log \frac{1 + \|(\mu - \nu)/(1 - \bar{\mu}\nu)\|_\infty}{1 - \|(\mu - \nu)/(1 - \bar{\mu}\nu)\|_\infty}, \tag{2.3}$$

where the infimum is taken over all $\mu \in p$ and $\nu \in q$. Then (T, d_T) is a complete metric space (cf. [10]).

There exists a complex structure on T by a certain embedding of T into a complex Banach space of holomorphic quadratic differentials on Δ. In fact, define a norm for a holomorphic quadratic differential ϕ on Δ as

$$\|\phi\|_{\mathscr{B}} = \sup_{z\in\Delta} |\phi(z)|\rho(z)^{-2}. \tag{2.4}$$

Let $\mathscr{B}$ be the set of all holomorphic quadratic differentials with finite norm (2.4). Then $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ becomes a complex Banach space.

For any $\mu \in B$, set

$$\check{\mu}(z) = \begin{cases} 0 & (z \in \bar{\Delta}) \\ \overline{\mu\left(\dfrac{1}{\bar{z}}\right)}\left(\dfrac{z}{\bar{z}}\right)^2 & (z \in \mathbf{C}\backslash\bar{\Delta}). \end{cases} \tag{2.5}$$

Let f_μ be the quasiconformal self-mapping of $\mathbf{C}$ that fixes 1, i and -1 and whose Beltrami coefficient agrees with $\check{\mu}$. Then $f_\mu|_\Delta$ is conformal, and we can consider the Schwarzian derivative of $f_\mu|_\Delta$. Here the Schwarzian derivative S_f of a holomorphic function f of a domain D in $\mathbf{C}$ is given by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

It turns out that $S_{f_\mu|_\Delta} \in \mathscr{B}$ and $S_{f_\nu|_\Delta} = S_{f_\mu|_\Delta}$ for any $\nu \in [\mu]$. Let

$$\beta([\mu]) = S_{f_\mu|_\Delta} \tag{2.6}$$

for $[\mu] \in T$. Then β is a well-defined injection of T into $\mathscr{B}$. Furthermore, β becomes a homeomorphism from (T, d_T) into $(\mathscr{B}, q)$, where q is the distance induced by norm (2.4). The map β is called the *Bers embedding* of T. Thus T becomes a complex Banach manifold modeled on $\mathscr{B}$. There is another definition of the complex structure of T. This structure is obtained by right translation maps defined in the next paragraph and a local inverse of the Bers embedding. It follows that these two definitions are equivalent. For these results, we refer the reader to Chapter V in [10].

Fix $\tau \in T$ arbitrarily and let g be a representative of τ. Define a map α of T as $\alpha([f]) = [f \circ g^{-1}]$ for $[f] \in T$. Then α can be defined independently of the choice of representatives of τ. Moreover, α is a biholomorphic automorphism of T and maps τ to the base point of T. We rewrite α as α_τ. This map is called the *right translation map for* τ. Condition (2) of Theorem 1.1 means that α_τ is also an biholomorphic automorphism of T' for all $\tau \in T'$.

The family QC becomes a group in terms of the composition of mappings. It follows from the definition of T that T inherits this group structure of QC: If $[f], [g] \in T$, the rule

$$[f] \circ [g] = [f \circ g]$$

defines the group operation in T. Condition (2) of Theorem 1.1 is equivalent to that T' becomes a subgroup of T with respect to this operation.

We introduce the Kobayashi pseudo-distance before the Kobayashi distance. This is determined for complex manifolds. Let N be a complex manifold and let $H(\Delta, N)$ be the set of holomorphic maps from Δ into N. For $p, q \in N$, let

$$d_1(p,q) = \frac{1}{2}\log\frac{1+r}{1-r}, \tag{2.7}$$

where r denotes the infimum of $s \geq 0$ such that there exists $f \in H(\Delta, N)$ satisfying $f(0) = p$ and $f(s) = q$. If no such f exists in $H(\Delta, N)$, then we define $d_1(p,q) = \infty$. Let

$$d_n(p,q) = \inf \sum_{i=1}^{n} d_1(p_{i-1}, p_i), \tag{2.8}$$

where the infimum is taken over all chains of points $p_0 = p, p_1, \dots, p_n = q$ in N. Clearly, $d_{n+1} \leq d_n$ for all $n > 0$. The *Kobayashi pseudo-distance* on N is defined as

$$d_K(p,q) = \lim_{n\to\infty} d_n(p,q). \tag{2.9}$$

If d_K is non-degenerate, i.e. if $d_K(p,q) = 0$ implies $p = q$, then d_K is called the *Kobayashi distance* on N.

The Kobayashi pseudo-distance has an important property concerning the contraction of the distance.

Proposition 2.1. *Let M and N be two complex manifolds and $d_{K,M}$ and $d_{K,N}$ denote the Kobayashi pseudo-distances on M and N, respectively. Then for any holomorphic map F from M into N and any two points $p, q \in M$,*

$$d_{K,N}(F(p), F(q)) \leq d_{K,M}(p,q). \tag{2.10}$$

If F is a biholomorphic map between M and N, then F is an isometry in the Kobayashi pseudo-distance. Furthermore, if both M and N are Δ, then Proposition 2.1 is nothing but the Schwarz-Pick lemma.

It is known that the Teichmüller distance on T coincides with the Kobayashi distance.

Theorem 2.2. *The Teichmüller distance on T coincides with the Kobayashi distance.*

For the proof, we refer the reader to [5], [6] and [7].

3. Extremality of Teichmüller equivalence classes of T

In this section, we summarize without proofs the extremality in Teichmüller equivalence classes of T and the property of Teichmüller mappings.

For any $[f] \in T$, there always exists a mapping of $[f]$ that has the smallest maximal dilatation in $[f]$. This is called an *extremal mapping* of $[f]$. If $[f]$ has the property as in Theorem 3.2 below, then the extremal mapping is uniquely determined and can be represented concretely.

Definition 3.1. For $[f] \in T$, let f_0 be an extremal mapping of $[f]$. An element f_1 of $[f]$ is called a *frame mapping* for $[f]$ if f_1 satisfies the following condition: There exists a compact subset $E \subset \Delta$ such that

$$K(f_1|_{\Delta \setminus E}) < K(f_0). \tag{3.1}$$

If there exists a frame mapping in $[f]$, then $[f]$ is called a *Strebel point*.

The set of Strebel points is open and dense in T (see p. 106 in [8]).

Theorem 3.2 (Strebel's Frame Mapping Theorem, Teichmüller's Uniqueness Theorem). *If a point $[f] \in T$ is a Strebel point, then it has the unique extremal mapping f_0 with Beltrami coefficient of the form*

$$k \frac{\bar{\phi}}{|\phi|} \tag{3.2}$$

where $0 < k < 1$ and ϕ is a holomorphic quadratic differential with $\iint_\Delta |\phi(z)|\, dxdy = 1$.

The proof can be found in [8]. A quasiconformal mapping whose Beltrami coefficient is of form (3.2) is said to be a *Teichmüller mapping*.

The next theorem states that the maximal dilatation of every Teichmüller mapping can be estimated.

Theorem 3.3 (Fundamental Inequality). *Let f_0 be a Teichmüller mapping with Beltrami coefficient $k_0\overline{\phi_0}/|\phi_0|$, where $0 < k_0 < 1$ and ϕ_0 is a holomorphic quadratic differential with $\iint_\Delta |\phi_0(z)|\, dxdy = 1$. Then for any $\nu \in [k_0\overline{\phi_0}/|\phi_0|]$,*

$$K(f_0) \le \iint_\Delta \frac{\left|1 + \nu(z)\dfrac{\phi_0(z)}{|\phi_0(z)|}\right|^2}{1 - |\nu(z)|^2} |\phi_0(z)|\, dxdy. \tag{3.3}$$

The proof can be found in [7] and [8].

In the rest of this section, we prepare two lemmas for the proof of Theorem 1.1. We first deal with the locally uniform convergence of sequences consisting of holomorphic quadratic differentials. For a domain D in $\mathbf{C}$, define a norm for a quadratic differential ϕ as

$$\|\phi\|_D = \iint_D |\phi(z)|\, dxdy. \tag{3.4}$$

Let $A(D)$ be the set of all holomorphic quadratic differentials on D with finite norm (3.4). Then $(A(D), \|\cdot\|_D)$ becomes a complex Banach space. Let $A_1(D)$

be the unit ball of $A(D)$. The next proposition says that $A_1(D)$ is a normal family.

PROPOSITION 3.4. *Let $\{\phi_n\}$ be a sequence of $A_1(D)$. Then there exists a subsequence of $\{\phi_n\}$ that converges locally uniformly to a holomorphic quadratic differential $\phi \in A_1(D)$ in D.*

This proposition can be proved similarly to Proposition 4.4 in [9]. By applying this proposition, we obtain the following lemma.

LEMMA 3.5. *Let $\{E_n\}$ be an increasing sequence of subdomains in a domain $D \subset \mathbf{C}$ satisfying $\bigcup_{n=1}^{\infty} E_n = D$ and ϕ_n be a holomorphic quadratic differential of $A_1(E_n)$ for each n. Then there exists a subsequence of $\{\phi_n\}$ that converges locally uniformly to a holomorphic quadratic differential $\phi \in A_1(D)$ in D.*

Proof. We consider the restriction of ϕ_n to E_1 for each $n \geq 1$. Then it follows from Proposition 3.4 that there exists a subsequence $\{\phi_{n(k)}|_{E_1}\}$ of $\{\phi_n|_{E_1}\}$ that converges locally uniformly to an $\psi_1 \in A_1(E_1)$ in E_1. We write the subsequence $\{\phi_{n(k)}\}$ as $\{\phi_n\}$ again. Similarly, there exists a subsequence of $\{\phi_n|_{E_2}\}$ that converges locally uniformly to an $\psi_2 \in A_1(E_2)$ in E_2. Note that $\psi_1 = \psi_2$ on E_1.

In this way, we compose a sequence $\{\psi_n\}$ inductively where $\psi_k \in A_1(E_k)$ and $\psi_k = \psi_{k+1}$ on E_k for $k \geq 1$. Let $\phi(z) = \psi_k(z)$ for $z \in E_k$. By the definition of $\{\psi_n\}$, ϕ is well-defined. Let Ω denote any compact subset of D, and choose k_0 so sufficiently large that $\Omega \subset E_{k_0}$. For $k \geq k_0$, it follows that $|\psi_k(z) - \phi(z)| = 0$ for all $z \in \Omega$. Hence $\{\psi_k\}$ converges locally uniformly to ϕ in D. By the diagonal method, there exists a subsequence of $\{\phi_n\}$ that converges locally uniformly to ϕ in D. From Fatou's Lemma,

$$\|\phi\|_{E_n} = \iint_{E_n} |\phi| \leq \liminf_{m \to \infty} \iint_{E_n} |\phi_m| \leq \liminf_{m \to \infty} \iint_{E_m} |\phi_m| \leq 1,$$

which implies that $\|\phi\|_{\Delta} \leq 1$ as $n \to \infty$. Hence $\phi \in A_1(\Delta)$. $\square$

The second lemma to prove Theorem 1.1 states the relation between the convergence of Beltrami coefficients and that of Teichmüller equivalence classes of T.

LEMMA 3.6. *Suppose that a sequence $\{[\mu_n]\}$ of T satisfies the following three conditions*:

(1) *The sequence $\{[\mu_n]\}$ converges to a point $[\mu] \in T$ with respect to d_T;*
(2) *There exists a constant $0 < k < 1$ such that $\|\mu_n\|_{\infty} \leq k < 1$ for all n;*
(3) *There exists a Beltrami coefficient $\nu \in B$ such that $\{\mu_n\}$ converges pointwise to ν.*

Then $[\mu] = [\nu]$.

This was proved in Lemma V.3.1 of [10] in the case where $S = \Delta$.

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $d_{T'}$ be the restriction of the Teichmüller distance on T to T' and $d_{K'}$ be the Kobayashi pseudo-distance on T'. Our purpose is to show $d_{T'} = d_{K'}$.

By Proposition 2.1 and Theorem 2.2, we have

$$d_{K'}(p,q) \geq d_K(p,q) = d_T(p,q) = d_{T'}(p,q)$$

for any $p, q \in T'$. Thus $d_{T'} \leq d_{K'}$.

We next show $d_{K'} \leq d_{T'}$. Let $\tau \in T'$. It follows from condition (2) that the right translation map α_τ is biholomorphic from T' onto itself. Hence this map is an isometry in $d_{K'}$. It follows from formula (2.2) that α_τ is also an isometry in $d_{T'}$. Thus it is sufficient to show the inequality

$$d_{K'}(0,\tau) \leq d_{T'}(0,\tau) \tag{4.1}$$

for any non-base point $\tau \in T'$.

The proof will be divided into two steps. The first step is to construct a sequence of maximal dilatations $\{K_n\}$ satisfying

$$d_{K'}(0,\tau) \leq \frac{1}{2}\log K_n$$

for all n. The second step is to find the subsequence of $\{K_n\}$ converging to the maximal dilatation K_0 of the extremal mapping of τ. Since $\frac{1}{2}\log K_0 = d_{T'}(0,\tau)$, inequality (4.1) follows.

From condition (1), τ is a Strebel point. Let f be a frame mapping of τ satisfying condition (3) with Beltrami coefficient μ and $K = K(f)$. It follows from Strebel's frame mapping theorem (Theorem 3.2) that τ has a unique extremal mapping f_0. Note that $1 < K_0 = K(f_0) < K$.

Let D_n denote the open disk centered at 0 and of radius $1 - \dfrac{1}{n}$ for $n \geq 2$. Then $\{D_n\}$ is an increasing sequence satisfying $\bigcup_{n=2}^{\infty} D_n = \Delta$. Since f is a frame mapping, there exists a number $N \in \mathbf{N}$ such that

$$K(f|_{\Delta\setminus\bar{D}_n}) < K_0 \tag{4.2}$$

for each $n \geq N$. For such n, let $h_n : f(D_n) \to D_n$ be a conformal map such that $F_n = h_n \circ f$ fixes $-\left(1 - \dfrac{1}{n}\right)$, $\left(1 - \dfrac{1}{n}\right)i$ and $1 - \dfrac{1}{n}$. The comformal map h_n is determined uniquely because of the Riemann mapping theorem. We consider the Teichmüller space of D_n defined as $T = T(\Delta)$. Let $\tilde{F}_n$ be an extremal mapping in $[F_n] \in T(D_n)$. If we set $\tilde{f}_n = h_n^{-1} \circ \tilde{F}_n$, then $K(\tilde{f}_n) \geq K_0$. Indeed, suppose to the contrary that $K(\tilde{f}_n) < K_0$. Set

$$f_n(z) = \begin{cases} \tilde{f}_n(z) & (z \in D_n) \\ f(z) & (z \in \Delta\setminus D_n). \end{cases} \tag{4.3}$$

Since $\tilde{F}_n$ belongs to $[F_n]$, $\tilde{f}_n$ agrees with f on ∂D_n. Hence f_n is homeomorphism of Δ onto itself. Moreover, f and $\tilde{f}_n$ are quasiconformal and ∂D_n has zero measure. Hence f_n is a quasiconformal self-mapping on Δ and $f_n \in \tau$. Thus we have $K_0 \leq K(f_n)$. However, from inequality (4.2), we have

$$K(f_n) = \max\{K(f|_{\Delta \backslash D_n}), K(\tilde{f}_n)\} < K_0.$$

This contradicts the assumption. Since $\mu_{F_n} = \mu_f$ and $\mu_{\tilde{F}_n} = \mu_{\tilde{f}_n}$ on D_n,

$$\begin{aligned} K(F_n|_{D_n \backslash \bar{D}_N}) &= K(f|_{D_n \backslash \bar{D}_N}) \\ &\leq K(f|_{\Delta \backslash \bar{D}_N}) < K_0 \leq K(\tilde{f}_n) = K(\tilde{F}_n). \end{aligned}$$

Thus F_n is a frame mapping for $[F_n] \in T(D_n)$. By applying Theorem 3.2 again, $\tilde{F}_n$ is the Teichmüller mapping with Beltrami coefficient $k_n \overline{\phi_n}/|\phi_n|$ where $0 < k_n < 1$ and $\|\phi_n\|_\Delta = 1$. Note that $\tilde{f}_n$ has the same Beltrami coefficient as $\tilde{F}_n$. Let μ_n be the Beltrami coefficient of f_n and $K_n = K(f_n)$. Then it is easily seen that

(a) $K_n > K_0$,
(b) $[\mu_n] = [\mu]$

for each n.

Let $g(t) = [t\mu_n/\|\mu_n\|_\infty]$ for $t \in \Delta$. It follows from condition (3) that g maps Δ into T'. Furthermore, g is holomorphic on Δ and $g(0) = 0$, $g(\|\mu_n\|_\infty) = [\mu_n]$. By formula (b) and formulas (2.7)–(2.9),

$$d_{K'}(0, \tau) \leq d_1(0, \tau) = d_1(0, [\mu_n]) \leq \frac{1}{2} \log \frac{1 + \|\mu_n\|_\infty}{1 - \|\mu_n\|_\infty} = \frac{1}{2} \log K_n.$$

Let us show that there exists a subsequence of $\{K_n\}$ tending to K_0. By Lemma 3.5, there exists a subsequence of $\{\phi_n\}$ which converges locally uniformly in Δ to a holomorphic quadratic differential ϕ^* in $A_1(\Delta)$. We write this subsequence as $\{\phi_n\}$ again.

To show $\|\phi^*\|_\Delta > 0$, suppose to the contrary that $\|\phi^*\|_\Delta = 0$. Then $\{\phi_n\}$ converges locally uniformly to 0 in Δ. Take $\varepsilon > 0$ arbitrarily. Since ϕ_n converges uniformly to 0 on $\overline{D}_N$, it follows that $\lim_{n\to\infty} \iint_{\bar{D}_N} |\phi_n| = 0$. Thus there exists a number $N' \in \mathbf{N}$ such that $\iint_{\bar{D}_N} |\phi_n| < \varepsilon$ for $n > N'$. The difference from the proof in [9] is that the domain of integration $\overline{D}_N$ is taken independently of ε. Let $\tilde{N} = \max\{N, N'\}$. For any $n > \tilde{N}$, the fundamental inequality (Theorem 3.3) implies that

$$K(\tilde{f}_n) \leq \iint_{D_n} \frac{\left|1 + \mu \dfrac{\phi_n}{|\phi_n|}\right|^2}{1 - |\mu|^2} |\phi_n| \, dxdy.$$

We estimate the right-hand integral by dividing D_n into $\overline{D}_N$ and $D_n \backslash \overline{D}_N$.

$$\iint_{\bar{D}_N} \frac{\left|1+\mu\dfrac{\phi_n}{|\phi_n|}\right|^2}{1-|\mu|^2}|\phi_n|\,dxdy \le \iint_{\bar{D}_N} \frac{1+|\mu|}{1-|\mu|}|\phi_n|\,dxdy$$
$$\le \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}\iint_{\bar{D}_N}|\phi_n|\,dxdy < K\varepsilon$$

$$\iint_{D_n\backslash\bar{D}_N} \frac{\left|1+\mu\dfrac{\phi_n}{|\phi_n|}\right|^2}{1-|\mu|^2}|\phi_n|\,dxdy \le \frac{1+\|\mu|_{\Delta\backslash\bar{D}_N}\|_\infty}{1-\|\mu|_{\Delta\backslash\bar{D}_N}\|_\infty}\iint_{D_n\backslash\bar{D}_N}|\phi_n|\,dxdy$$
$$\le K(f|_{\Delta\backslash\bar{D}_N})$$

Recall that $K = K(f) = (1+\|\mu\|_\infty)/(1-\|\mu\|_\infty)$. It follows from these inequalities that $K(\tilde{f}_n) < K\varepsilon + K(f|_{\Delta\backslash\bar{D}_N})$. Taking $\varepsilon \to 0$, we obtain

$$\limsup_{n\to\infty} K(\tilde{f}_n) \le K(f|_{\Delta\backslash\bar{D}_N}).$$

By inequality (4.2), we have

$$\limsup_{n\to\infty} K_n \le \max\left\{\limsup_{n\to\infty} K(\tilde{f}_n), K(f|_{\Delta\backslash\bar{D}_N})\right\} < K_0.$$

However, this contradicts inequality (a). Therefore, $\|\phi^*\|_\Delta > 0$.

Since $0 < k_n < 1$ for all n, $\{k_n\}$ has a convergent subsequence. Let k^* be the limit of this subsequence and $\mu^* = k^*\overline{\phi^*}/|\phi^*|$. Since $\|\phi^*\|_\Delta > 0$, μ^* is well-defined. It follows that $\{\mu_n\}$ converges pointwise to μ^* on Δ. Because $K(f_n) = K_n \le K$ for any n, we have $\|\mu_n\|_\infty \le \|\mu\|_\infty < 1$. By Lemma 3.6, it follows that $[\mu^*] = [\mu]$. From the uniqueness of extremal mappings, we have $k^* = k_0$. Noting that $K_0 = (1+k_0)/(1-k_0)$, this implies that $\{K_n\}$ has a subsequence converging to K_0. Therefore, $d_{T'} = d_{K'}$. □

5. Integrably asymptotic affine classes

In this section, we give a metric subspace of T where the Teichmüller distance coincides with the Kobayashi distance. At first, we explain another metric subspace of T.

Definition 5.1. A quasiconformal mapping f on a domain D of $\mathbf{C}$ is called *asymptotically conformal* on D if for any $\varepsilon > 0$, there exists a compact subset E of D such that

$$K(f|_{D\backslash E}) < 1+\varepsilon. \tag{5.1}$$

Let f be an quasiconformal mapping of a domain D and μ be the Beltrami coefficient of f. It follows immediately that f is asymptotically conformal on D if and only if for any $\varepsilon' > 0$, there exists a compact subset E' of D such that

$$\|\mu|_{D\setminus E'}\|_\infty < \varepsilon'. \tag{5.2}$$

We call an Teichmüller equivalence class $[f]$ of T *asymptotically conformal* if $[f]$ has an asymptotically conformal map of Δ onto itself. Let T_0 be the set of all asymptotically conformal classes of T. Then T_0 becomes a closed submanifold of T. In fact, let $\mathscr{B}_0$ be the set of all holomorphic quadratic differential of $\mathscr{B}$ vanishing at infinity with respect to norm (2.4). In other words, for every $\phi \in \mathscr{B}_0$, there exists an increasing sequence $\{D_n\}$ of Δ satisfying $\bigcup D_n = \Delta$ such that $\|\phi|_{\Delta\setminus D_n}\|_{\mathscr{B}} \to 0$ as $n \to \infty$. Since $\mathscr{B}_0$ is a closed subspace of $\mathscr{B}$, $\mathscr{B}_0$ becomes a complex Banach space. In [3], it is shown that $\beta(T_0) = \beta(T) \cap \mathscr{B}_0$. Thus T_0 has a complex structure modeled on $\mathscr{B}_0$. It is known that the Teichmüller distance on T_0 coincides with the Kobayashi distance, which is also the corollary to Theorem 1.1.

Corollary 5.2. *The Teichmüller distance on T_0 coincides with the Kobayashi distance.*

Proof. By Theorem 1.1, it suffices to show that T_0 satisfies conditions (1)–(3). For any $\tau \in T_0$, there exists an asymptotically conformal map f with Beltrami coefficient μ. Inequality (5.1) implies that f is a frame mapping and condition (1) holds. Take $\mu' \in \tau$ arbitrarily that coincides with μ outside some compact subset of Δ (for example, see formula (4.3)). By the definition of μ', for any $\varepsilon' > 0$, if we take a compact subset E'' of D sufficient largely such that E'' contains E' and inequality (5.2), then for any $t \in \Delta$, $t\mu'$ satisfies inequality (5.2) for E'' clearly. Hence T_0 satisfies condition (3). The composition of two asymptotically conformal maps of Δ onto itself is also asymptotically conformal. Therefore, T_0 satisfies conditions (2). $\square$

Now, we introduce the metric subspace of T in our purpose.

Definition 5.3. An Teichmüller equivalence class $\tau \in T$ is called *integrably asymptotic affine* if there exists a Beltrami coefficient $\mu \in \tau$ such that μ is square integrable with respect to the Poincaré metric on Δ, namely,

$$\iint_\Delta |\mu(z)|^2 \rho(z)^2 \, dxdy < \infty. \tag{5.3}$$

Here $\rho(z) = (1 - |z|^2)^{-1}$ is the Poincaré metric on Δ.

Let T_* be the set of all integrably asymptotic affine classes of T. Define a norm for a holomorphic quadratic differential ϕ on Δ as

$$\|\phi\|_{\mathscr{Q}} = \left(\iint_\Delta |\phi(z)|^2 \rho(z)^{-2} \, dxdy\right)^{1/2}. \tag{5.4}$$

Let $\mathscr{Q}$ be the set of all quadratic differentials with finite norm (5.4). It follows that $(\mathscr{Q}, \|\cdot\|_{\mathscr{Q}})$ is a complex Banach space and $\beta(T_*) = \beta(T) \cap \mathscr{Q}$. Then T_* has a complex structure modeled on $\mathscr{Q}$. Moreover, T_* is contractible in T and contained in T_0. These results are proved by Cui [1].

Proposition 5.4 ([1]). $T_* \subset T_0$.

Proof. For the sake of convenience, we write the outline of the proof here. It follows that $|\phi(z)|\rho(z)^{-2} \to 0$ as $|z| \to 1-0$ for every $\phi \in \mathscr{Q}$ (Lemma 2 in [1]), which implies that $\mathscr{Q} \subset \mathscr{B}_0$. Since $\beta(T_0) = \beta(T) \cap \mathscr{B}_0$ and $\beta(T_*) = \beta(T) \cap \mathscr{Q}$, we have $T_* \subset T_0$. $\square$

The metric subspace T_* can be characterized by quasiconformal mappings with a special property, which is called the Douady-Earle extension. We first introduce the Douady-Earle extension.

Let $[\mu] \in T$ and $h = f^{\mu}|_{\partial\Delta}$. Define

$$F(z,w) = F_h(z,w) = \frac{1}{2\pi}\iint_{\partial\Delta} \frac{h(t)-w}{1-\bar{w}h(t)} \frac{1-|z|^2}{|z-t|^2}|dt| \tag{5.5}$$

for $z, w \in \Delta$. Then the Jacobian of F with respect to w is positive. It follows from the implicit function theorem that the equation $F(z,w) = 0$ has the unique solution $w = \tilde{h}(z)$. Furthermore, $\tilde{h}$ becomes a quasiconformal mapping on Δ such that $\tilde{h}|_{\partial\Delta} = h$. The quasiconformal mapping $\tilde{h}$ is called the *Douady-Earle extension* of h. From the definition, the Douady-Earle extension has the following property:

Proposition 5.5 ([2]). *If A and B denote arbitrary Möbius transformations preserving Δ, then the Douady-Earle extension of $A \circ h \circ B$ coincides with $A \circ \tilde{h} \circ B$.*

The proof can be also found in [13].

We will look more closely at the Douady-Earle extension. Every Möbius transformation g preserving Δ is an isometry with respect to the Poincaré metric of Δ. Namely, for any $z \in \Delta$,

$$\frac{|g'(z)|}{1-|g(z)|^2} = \frac{1}{1-|z|^2}.$$

This is not true for any quasiconformal self-mapping of Δ. However, we have a certain approximation for normalized Douady-Earle extensions. Recall that a quasiconformal self-mapping of Δ is normalized if it fixes 1, i and -1.

Lemma 5.6. *For $K \geq 1$, let $\mathrm{DE}(K)$ be the set of all normalized Douady-Earle extensions that are K-quasiconformal. Then there exist constants $C_1, C_2 > 0$ depending only on K such that for any $(f,z) \in \mathrm{DE}(K) \times \Delta$,*

$$\frac{C_1}{1-|z|^2} \leq \frac{|\partial f(z)| - |\bar{\partial} f(z)|}{1-|f(z)|^2} \leq \frac{|\partial f(z)| + |\bar{\partial} f(z)|}{1-|f(z)|^2} \leq \frac{C_2}{1-|z|^2}. \tag{5.6}$$

Proof. It is sufficient to show the first and third inequalities in inequality (5.6). Suppose that for any $C_1 > 0$, there exists $(f, z) \in \mathrm{DE}(K) \times \Delta$ such that

$$G_f(z) = \frac{1 - |z|^2}{1 - |f(z)|^2}(|\partial f(z)| - |\bar{\partial} f(z)|) < C_1.$$

In other words, there exists a sequence $\{(f_n, a_n)\}$ of $\mathrm{DE}(K) \times \Delta$ such that $G_{f_n}(a_n) \to 0$ as $n \to \infty$. Set

$$A_n(z) = \frac{z - f_n(a_n)}{1 - \overline{f_n(a_n)}z}, \quad B_n(z) = \frac{z + a_n}{1 + \overline{a_n}z}.$$

Then A_n and B_n are Möbius transformations preserving Δ and $g_n = A_n \circ f_n \circ B_n$ fixes 0. By the left composition of a suitable rotation with g_n, we may assume that g_n also fixes 1. It follows that

$$|\partial g_n(0)| = \frac{1 - |a_n|^2}{1 - |f_n(a_n)|^2}|\partial f_n(a_n)|,$$

$$|\bar{\partial} g_n(0)| = \frac{1 - |a_n|^2}{1 - |f_n(a_n)|^2}|\bar{\partial} f_n(a_n)|.$$

Thus we obtain

$$\begin{aligned} |\partial g_n(0)| - |\bar{\partial} g_n(0)| &= \frac{1 - |a_n|^2}{1 - |f_n(a_n)|^2}(|\partial f_n(a_n)| - |\bar{\partial} f_n(a_n)|) \\ &= G_{f_n}(a_n). \end{aligned} \tag{5.7}$$

Since $\overline{\Delta}$ is compact, there exists a subsequence of $\{a_n\}$ that converges to an $a \in \overline{\Delta}$ as $n \to \infty$. We may assume that $\{a_n\}$ tends itself to a. Because f_n is K-quasiconformal and fixes 1, i and -1 for each $n \geq 1$, there exists a K-quasiconformal self-mapping f of Δ such that $\{f_n\}$ converges to f locally uniformly in Δ. As before, set

$$A(z) = \frac{z - f(a)}{1 - \overline{f(a)}z}, \quad B(z) = \frac{z + a}{1 + \bar{a}z}.$$

Then $\{g_n\}$ converges to $g = A \circ f \circ B$ locally uniformly in Δ. If $a \in \partial\Delta$, then B is a constant function. This implies that g becomes a constant function. However, since g_n fixes 0 and 1, it contradicts the assumption that $a \in \partial\Delta$. Thus $a \in \Delta$ and g is K-quasiconformal.

From Proposition 5.5, g_n is the Douady-Earle extension of $h_n = A_n \circ f_n|_{\partial\Delta} \circ B_n$. Then it follows from the definition of the Douady-Earle extension that $F_{h_n}(z, g_n(z)) = 0$ for all $z \in \Delta$. The absolute value of the integrand in (5.5) is equal to $(1 - |z|^2)/(|z - t|^2)$, which is integrable with respect to t over

$\partial\Delta$. Let h be the restriction of g to $\partial\Delta$. Note that $h(z) = \lim_{n\to\infty} h_n(z)$ for all $z \in \partial\Delta$. By Lebesgue's dominated convergence theorem,

$$\begin{aligned}
0 &= \lim_{n\to\infty} F_{h_n}(z, g_n(z)) \\
&= \frac{1}{2\pi}\int_{\partial\Delta} \lim_{n\to\infty} \frac{h_n(t) - g_n(z)}{1 - \overline{g_n(z)}h_n(t)} \frac{1 - |z|^2}{|z - t|^2}|dt| = F_h(z, g(z))
\end{aligned}$$

for all $z \in \Delta$. Hence g is the Douady-Earle extension of h. It follows that g is a diffeomorphism of Δ. Thus there exist $\partial g(0)$ and $\bar{\partial} g(0)$. Since g is sense-preserving, we have $|\partial g(0)| - |\bar{\partial} g(0)| > 0$.

In order to prove the lemma, it is sufficient to show that $|\partial g_n(0)|$ and $|\bar{\partial} g_n(0)|$ converge to $|\partial g(0)|$ and $|\bar{\partial} g(0)|$, respectively. The partial derivatives of F_{h_n} at $(0, 0)$ are

$$\frac{\partial F_{h_n}}{\partial z}(0,0) = \frac{1}{2\pi}\int_{\partial\Delta} \bar{t} h_n(t)|dt| = \alpha_1(n),$$

$$\frac{\partial F_{h_n}}{\partial \bar{z}}(0,0) = \frac{1}{2\pi}\int_{\partial\Delta} t h_n(t)|dt| = \alpha_2(n),$$

$$\frac{\partial F_{h_n}}{\partial w}(0,0) = -1,$$

$$\frac{\partial F_{h_n}}{\partial \overline{w}}(0,0) = \frac{1}{2\pi}\int_{\partial\Delta} h_n(t)^2|dt| = \alpha_3(n).$$

For each $j = 1, 2, 3$, the absolute value of the integrand in $\alpha_j(n)$ is equal to 1. By applying Lebesgue's dominated convergence theorem again, we have

$$\lim_{n\to\infty} \alpha_1(n) = \frac{1}{2\pi}\int_{\partial\Delta} \bar{t} h(t)|dt| = \frac{\partial F_h}{\partial z}(0,0) = \alpha_1,$$

$$\lim_{n\to\infty} \alpha_2(n) = \frac{1}{2\pi}\int_{\partial\Delta} t h(t)|dt| = \frac{\partial F_h}{\partial \bar{z}}(0,0) = \alpha_2,$$

$$\lim_{n\to\infty} \alpha_3(n) = \frac{1}{2\pi}\int_{\partial\Delta} h(t)^2|dt| = \frac{\partial F_h}{\partial \overline{w}}(0,0) = \alpha_3.$$

It follows from these formulas and the implicit function theorem that

$$\lim_{n\to\infty} \partial g_n(0) = \lim_{n\to\infty} \frac{\alpha_1(n) - \overline{\alpha_2(n)}\alpha_3(n)}{1 - |\alpha_3(n)|^2} = \frac{\alpha_1 - \overline{\alpha_2}\alpha_3}{1 - |\alpha_3|^2} = \partial g(0).$$

We also have $\lim_{n\to\infty} \bar{\partial} g_n(0) = \bar{\partial} g(0)$ in the same way. These results imply $\lim_{n\to\infty}(|\partial g_n(0)| - |\bar{\partial} g_n(0)|) = |\partial g(0)| - |\bar{\partial} g(0)| > 0$. From formula (5.7), this contradicts that $|\partial g_n(0)| - |\bar{\partial} g_n(0)| = G_{f_n}(a_n)$ converges to 0 as $n \to \infty$. Therefore, there exists a constant $C_1 > 0$ satisfying the first inequality in inequality (5.6).

The third inequality in inequality (5.6) is shown similarly. For simplicity, we will use the same notations as above in places. Suppose that for any $C_2 > 0$, there exists $(f, z) \in \mathrm{DE}(K) \times \Delta$ such that

$$H_f(z) = \frac{1-|z|^2}{1-|f(z)|^2}(|\partial f(z)| + |\bar{\partial} f(z)|) > C_2.$$

In other words, there exists a sequence $\{(f_n, a_n)\}$ of $\mathrm{DE}(K) \times \Delta$ such that $H_{f_n}(a_n) \to \infty$ as $n \to \infty$. We can compose a function g_n similarly as above. It follows that g_n converges to a K-quasiconformal self-mapping g of Δ locally uniformly in Δ and g is the Douady-Earle extension of the limit function of the boundary functions of $\{f_n\}$. Hence there exist $\partial g(0)$ and $\bar{\partial} g(0)$ and $|\partial g(0)| + |\bar{\partial} g(0)| < \infty$. Furthermore, $|\partial g_n(0)|$ and $|\bar{\partial} g_n(0)|$ converge to $|\partial g(0)|$ and $|\bar{\partial} g(0)|$, respectively. This contradicts that $|\partial g_n(0)| + |\bar{\partial} g_n(0)| = H_{f_n}(a_n)$ diverges as $n \to \infty$. Therefore, there exists a constant $C_2 > 0$ satisfying the third inequality in inequality (5.6). □

It follows immediately from Lemma 5.6 that there exists a constant $C > 0$ depending only on K such that for any $(f, z) \in \mathrm{DE}(K) \times \Delta$,

$$\frac{J_f(z)}{(1-|f(z)|^2)^2} \le \frac{C}{(1-|z|^2)^2}, \tag{5.8}$$

where J_f is the Jacobian of f. Similarly, it follows that there exists a constant $C' > 0$ depending only on K such that for any $(f, w) \in \mathrm{DE}(K) \times \Delta$,

$$\frac{J_{f^{-1}}(w)}{(1-|f^{-1}(w)|^2)^2} \le \frac{C'}{(1-|w|^2)^2}. \tag{5.9}$$

In fact, let $C_1, C_2 > 0$ be constants in inequality (5.6). Then

$$\frac{C_1}{1-|z|^2} \le \frac{|\partial f(z)|}{1-|f(z)|^2}.$$

If we set $w = f(z)$, then

$$\frac{1}{C_1}\frac{1}{1-|w|^2} \ge \frac{1}{1-|f^{-1}(w)|^2}\frac{1}{|\partial f(f^{-1}(w))|} = \frac{|\partial(f^{-1})(w)|}{1-|f^{-1}(w)|^2}.$$

By the similar computation,

$$\frac{2}{C_2 - C_1}\frac{1}{1-|w|^2} \le \frac{|\bar{\partial}(f^{-1})(w)|}{1-|f^{-1}(w)|^2}.$$

We can take C_2 sufficient largely such that $C_2 > 3C_1$. Thus

$$\frac{J_{f^{-1}}(w)}{(1-|f^{-1}(w)|^2)^2} \le \left(\frac{1}{C_1^2} - \frac{4}{(C_2 - C_1)^2}\right)\frac{1}{(1-|w|^2)^2}.$$

We obtain inequality (5.9) if we set $C' = \dfrac{1}{C_1^2} - \dfrac{4}{(C_2 - C_1)^2}$.

Remark. In order to proceed the following discussion, it is enough to take a constant C such that inequalities (5.8) and (5.9) hold for any $z \in \Delta$ when the normalized Douady-Earle extension f is fixed. We have shown, however, that the inequality is valid in the more general situation for the application in the future.

For $\tau \in T$, let $E(\tau)$ be the Douady-Earle extension of the boundary function determined by τ. By the definition of T, $E(\tau)$ is well-defined. Inequalities (5.8) and (5.9) imply the following lemma:

Lemma 5.7. *For any $\tau \in T_*$, the following conditions are equivalent*:
(1) *The Beltrami coefficient μ of $E(\tau)$ is square integrable with respect to the Poincaré metric on Δ.*
(2) *The Beltrami coefficient ν of $E(\tau)^{-1}$ is square integrable with respect to the Poincaré metric on Δ.*

Proof. If condition (1) holds, set $w = E(\tau)(z)$. From inequality (5.8), there exists a constant $C > 0$ such that

$$\begin{aligned}\iint_\Delta |\nu(w)|^2 \rho(w)^2\, dudv &= \iint_\Delta |\nu(E(\tau)(z))|^2 \rho(E(\tau)(z))^2 J_{E(\tau)}(z)\, dxdy \\ &\le C \iint_\Delta |\mu(z)|^2 \rho(z)^2\, dxdy < \infty.\end{aligned}$$

Thus condition (2) follows. By inequality (5.9) and the similar computation, condition (2) induces condition (1). □

Let us characterize T_* by the Douady-Earle extension. Cui characterized T_* by the inverse map of the Douady-Earle extension in [1]. We will modify this result a little to make it more useful.

Proposition 5.8. *For any $\tau \in T$, the following conditions are equivalent*:
(1) $\tau \in T_*$;
(2) $\tau^{-1} \in T_*$;
(3) *The Beltrami coefficient μ of $E(\tau)$ is square integrable with respect to the Poincaré metric on Δ.*

Here τ^{-1} is the Teichmüller equivalence class defined by the inverse map of $f^{\mu'}$ where $\mu' \in \tau$.

Proof. Condition (3) implies condition (1) clearly.

If condition (1) holds, then by Theorem 1 in [1], the Beltrami coefficient of $E(\tau^{-1})^{-1}$ is square integrable with respect to the Poincaré metric on Δ. It follows from Lemma 5.7 that the Beltrami coefficient of $E(\tau^{-1})$ is square integrable with respect to the Poincaré metric on Δ. This implies that $\tau^{-1} \in T_*$ and condition (2).

We will show that condition (2) induces condition (3) by the similar way above. Using Theorem 1 in [1] again, the Beltrami coefficient of $E(\tau)^{-1}$ is square integrable with respect to the Poincaré metric on Δ. It follows from Lemma 5.7 that the Beltrami coefficient of $E(\tau)$ is square integrable with respect to the Poincaré metric on Δ. Thus condition (3) holds. □

It was shown in [1] and [12] that T_* becomes a subgroup of T. We will give this result more clearly and simply.

PROPOSITION 5.9. *The metric subspace T_* becomes a subgroup of T.*

Proof. For any $q \in T_*$, let ν be the Beltrami coefficient of $E(q)$. By Proposition 5.8, ν is square integrable with respect to the Poincaré metric on Δ. For any $p \in T_*$, there exists a representative $\mu \in p$ satisfying condition (5.3). Let η be the Beltrami coefficient of $f^\mu \circ (f^\nu)^{-1}$. Then for $\zeta \in \Delta$,

$$\begin{aligned} |\eta(f^\nu(\zeta))|^2 &= \left|\frac{\mu(\zeta) - \nu(\zeta)}{1 - \mu(\zeta)\overline{\nu(\zeta)}}\right|^2 \\ &\le \frac{|\mu(\zeta) - \nu(\zeta)|^2}{(1 - \|\mu\|_\infty \|\nu\|_\infty)^2} \le \frac{2(|\mu(\zeta)|^2 + |\nu(\zeta)|^2)}{(1 - \|\mu\|_\infty \|\nu\|_\infty)^2}. \end{aligned}$$

By combining this formula and Lemma 5.6,

$$\begin{aligned} \iint_\Delta |\eta(z)|^2 \rho(z)^2 \, dxdy &= \iint_\Delta |\eta(f^\nu(\zeta))|^2 \rho(f^\nu(\zeta))^2 J_{f^\nu}(\zeta) \, d\xi d\eta \\ &\le C \iint_\Delta \frac{2(|\mu(\zeta)|^2 + |\nu(\zeta)|^2)}{(1 - \|\mu\|_\infty \|\nu\|_\infty)^2} \rho(\zeta)^2 \, d\xi d\eta < \infty. \end{aligned}$$

This implies that $[f^\mu \circ (f^\nu)^{-1}] \in T_*$. Thus T_* becomes a subgroup of T. □

Now, let us prove the main theorem in this section.

THEOREM 5.10. *The Teichmüller distance on T_* coincides with the Kobayashi distance.*

Proof. We will show that T_* satisfies conditions (1)–(3) in Theorem 1.1.
From Proposition 5.4, it follows immediately that condition (1) holds.
From Proposition 5.9, condition (2) holds.

For any $[\mu] \in T_*$, let μ be a Beltrami coefficient of $[\mu]$ satisfying condition (5.3). Take a representative $\mu' \in [\mu]$ arbitrarily that coincides with μ outside some compact subset E of Δ. Then for any $t \in \Delta$,

$$\begin{aligned}
&\iint_\Delta |t\mu'(z)|^2 \rho(z)^2 \, dxdy \\
&\quad = |t|^2 \left(\iint_\Delta |\mu(z)|^2 \rho(z)^2 \, dxdy + \iint_E (|\mu'(z)|^2 - |\mu(z)|^2) \rho(z)^2 \, dxdy \right) \\
&\quad < \iint_\Delta |\mu(z)|^2 \rho(z)^2 \, dxdy + \iint_E (|\mu'(z)|^2 - |\mu(z)|^2) \rho(z)^2 \, dxdy < \infty.
\end{aligned}$$

Thus $[t\mu'] \in T_*$ and condition (3) holds. □

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