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RECURRENCE RELATIONS FOR SUPER-HALLEY'S METHOD WITH HÖLDER CONTINUOUS SECOND DERIVATIVE IN BANACH SPACES

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Abstract

The aim of this paper is to study the semilocal convergence of the Super-Halley's method used for solving nonlinear equations in Banach spaces by using the recurrence relations. This convergence is established under the assumption that the second Frëchet derivative of the involved operator satisfies the Hölder continuity condition which is milder than the Lipschitz continuity condition. A new family of recurrence relations are defined based on two constants which depend on the operator. An existence-uniqueness theorem and a proori error estimates are provided for the solution x^*. The R-order of the method equals to $(2+p)$ for $p \in (0,1]$ is also established. Three numerical examples are worked out to demonstrate the efficacy of our approach. On comparison with the results obtained for the Super-Halley's method in [3] by using majorizing sequence, we observed improved existence and uniqueness regions for the solution x^* by our approach.

1. Introduction

Many scientific and engineering problems can be reduced to solving nonlinear equations. There exists a large number of applications that give rise to thousands of such equations depending on one or more parameters. The boundary value problems appearing in Kinetic theory of gases, elasticity and other applied areas are reduced to solving nonlinear equations. Dynamic systems are mathematically modelled by difference or differential equations and their solutions usually represent the equilibrium states of the systems obtained by solving nonlinear equations. Many optimization problems also lead to solutions of these equations. Generally, iterative methods are used for this purpose under various continuity conditions like Lipschitz, Hölder and ω on first/second Frëchet derivatives of the involved operators. The local, semilocal and global convergence analysis are also established for them by using either majorizing sequences or

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recurrence relations. In the case of majorizing sequences, a sequence generated by an iterative method in Banach spaces is majorized by a sequence generated by the same iterative method applied on a scalar function. But, the main advantage of recurrence relations is that the problem in Banach spaces can easily be reduced to simpler problems with real sequences and functions. The convergence analysis depends on the choice of the distance $\|.\|$, but for a given distance the speed of convergence of the sequence $\{x_n\}$ is characterized by the speed of convergence of the sequence of nonnegative numbers $\|x^* - x_n\|$. Two important measures of the speed of convergence are the Q-order and R-order of convergence. It is well known that a sequence $\{x_n\}$ converges with Q-order at least $\tau > 1$, if there exists a positive constant b such that $\|x_{n+1} - x^*\| \le b\|x_n - x^*\|^\tau$, $n = 0, 1, \ldots$. And it converges with R-order at least $\tau > 1$ if there are constants $C \in (0, \infty)$ and $\gamma \in (0, 1)$ such that $\|x_n - x^*\| \le \gamma^{\tau^n}$, $n = 0, 1, \ldots$. Both these orders of convergence are important to study the convergence of the sequence $\{x_n\}$ derived from an iterative method. But the R-order of convergence is more important because of their differences. That is, if a sequence $\{x_n\}$ converges with Q-order at least $\tau > 1$, then it converges with R-order at least $\tau > 1$, but not vice versa. For further studies of these orders of convergence, one can see the work of [13, 14].

In this paper, we are concerned with the semilocal convergence of a third order Super-Halley's method used for approximating a locally unique solution x^* of nonlinear operator equations

$$F(x) = 0. \tag{1}$$

where, $F : \Omega \subseteq \mathbf{X} \to \mathbf{Y}$ be a nonlinear operator on an open convex subset Ω of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$. The most well known second order iterative methods used to solve (1) are Newton's method and it's variants. The Kantorovich theorem [12, 15] provides sufficient conditions to ensure convergence of these methods. Many researchers [1, 6, 9] have also considered the convergence of one point third order iterative methods such as the Chebyshev's method, the Halley's method and the Super-Halley's method used for solving (1) under the assumption that the second order Frëchet derivative satisfies Lipschitz continuity condition. Under the assumptions that F'' is Lipschitz continuous, Candela and Marquina [1] used recurrence relations and derived a family of four real sequences in order to study the semilocal convergence of the Chebyshev's method in Banach spaces. Gutiérrez and Hernández [10] used recurrence relations and derived two real sequences in order to study the semilocal convergence of the second order derivative free version of the Chebyshev's method. Gutiérrez and Hernández [6] studied convergence of the Super-Halley's method under the assumption that F'' is Lipschitz continuous. Ezquerro and Hernandez [4] established semilocal convergence of second order derivative free version of the Super-Halley's method by using recurrence relations. Their main assumption for the convergence analysis was that the second Fréchet derivative satisfies Lipschitz continuity condition. However, it is not always true as the following example illustrates.

Example. Let us consider the integral equation of Fredholm type [2]:

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 \frac{s}{s+t} x(t)^{2+p}\, dt,$$

with $s \in [0,1]$, $x, f \in C[0,1]$, $p \in (0,1]$ and λ is a real number.

Under the sup-norm on the operator F, the second Fréchet derivative of F satisfies

$$\|F''(x) - F''(y)\| \le |\lambda|(1+p)(2+p)\log 2\|x-y\|^p, \quad x, y \in \Omega.$$

Hence, for $p \in (0,1)$, F'' does not satisfy the Lipschitz continuity condition, but it satisfies the Hölder continuity condition. Hernandez and Salanova [8] studied the convergence of the Chebyshev's method by using recurrence relations under the assumption that F'' satisfies Hölder continuity condition. J. A. Ezquerro et al. [5] discussed the convergence of Super-Halley's method using majorizing sequences under the Lipschitz continuity condition. The convergence of the Chebyshev's method and the Convex Acceleration of Newton's method using majorizing sequences under the Hölder conditions are given in [3, 7].

This paper is organized as follows. Section 1 is the introduction. In Section 2, some preliminary results are given. Then, two real sequences are generated and their properties are studied. In Section 3, recurrence relations are derived. In Section 4, a convergence theorem is established for the existence and uniqueness regions along with a priori error bounds for the solution. In section 5, three numerical examples are worked out and results obtained are compared with those obtained in [3] to demonstrate the efficacy of our approach. Finally, conclusions are covered in Section 6.

2. Preliminary results

In this section, we shall give some preliminary results in order to establish the convergence of the Super-Halley's method under the assumption that the second Frëchet derivative satisfies the Hölder continuity condition for solving (1) in Banach spaces by using recurrence relations. Let $\Gamma_0 = F'(x_0)^{-1} \in L(Y,X)$ exists for some $x_0 \in \Omega$, where $L(Y,X)$ is the set of bounded linear operator from **Y** into **X**. The Super-Halley's method can now be written in the form for $n = 0, 1, \dots$

$$(2) \qquad \left.\begin{aligned} y_n &= x_n - \Gamma_n F(x_n) \\ x_{n+1} &= y_n + \tfrac{1}{2} L_F(x_n)(I - L_F(x_n))^{-1}(y_n - x_n) \end{aligned}\right\}$$

where, $L_F(x_n) = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)$.

Let the following assumptions hold true

$$\left.\begin{array}{l} 1.\ \|\Gamma_0\| = \|F'(x_0)^{-1}\| \le \beta, \\ 2.\ \|F'(x_0)^{-1}F(x_0)\| \le \eta, \\ 3.\ \|F''(x)\| \le M, \quad \forall x \in \Omega, \\ 4.\ \|F''(x) - F''(y)\| \le K\|x-y\|^p, \quad \forall x, y \in \Omega,\ K > 0,\ p \in (0,1] \end{array}\right\} \tag{3}$$

Let

$$a_0 = M\beta\eta, \quad b_0 = K\beta\eta^{p+1} \tag{4}$$

Now, we define the real sequences

$$a_{n+1} = a_n f(a_n)^2 g(a_n, b_n) \quad \text{and} \quad b_{n+1} = b_n f(a_n)^{p+2} g(a_n, b_n)^{p+1} \tag{5}$$

where

$$f(x) = \frac{2(1-x)}{2-4x+x^2} \tag{6}$$

and

$$g(x,y) = \frac{x^2}{(1-x)} + \frac{y}{(p+1)(p+2)(1-x)} + \frac{x^3}{8(1-x)^2} \tag{7}$$

The following Lemmas will be used to establish some properties of these sequences.

LEMMA 1. *Let the real functions f and g be given by* (6) *and* (7) *respectively and let $p \in (0,1]$, then for $x \in (0, r_0)$ where r_0 be the smallest positive zero of the polynomial* $2x^4 - 9x^3 + 32x^2 - 32x + 8 = 0$, *we get*
(i) *f is a increasing function and $f(x) > 1$ in $(0, r_0]$.*
(ii) *g is a increasing in both arguements for $y > 0$.*
(iii) *$f(\gamma x) < f(x)$ and $g(\gamma x, \gamma^{p+1} y) \le \gamma^{p+1} g(x,y)$.*

Proof. The proof is simple and hence omitted here.

LEMMA 2. *For a fixed $p \in (0,1]$ and two real functions f and g given by* (6) *and* (7) *respectively, define*

$$\Phi_p(x) = \frac{(p+1)(P+2)(2x^4 - 9x^3 + 32x^2 - 32x + 8)}{8(1-x)} \tag{8}$$

Now, for $0 < a_0 \le r_0$ and $0 \le b_0 \le \Phi_p(a_0)$, we get
(i) $f(a_n)^2 g(a_n, b_n) \le 1$.
(ii) *$\{a_n\}$, $\{b_n\}$ are decreasing sequences and $a_n < 1$.*

Proof. From the definitions of functions f and g, one can easily conclude that $f(a_n)^2 g(a_n, b_n) \leq 1$ iff

$$\frac{4(1-a_n)^2}{(2-4a_n+a_n^2)^2}\left[\frac{a_n^2}{(1-a_n)}+\frac{b_n}{(p+1)(p+2)(1-a_n)}+\frac{a_n^3}{8(1-a_n)^2}\right]<1$$

or,

$$b_n \leq \frac{(p+1)(p+2)(2a_n^4-9a_n^3+32a_n^2-32a_n+8)}{8(1-a_n)} = \Phi_p(a_n)$$

This Lemma can be proved by using induction. For $0 < a_0 \leq r_0$ and $0 \leq b_0 \leq \Phi_p(a_0)$, we can easily conclude that $f(a_0)^2 g(a_0, b_0) \leq 1$. From (5), we obtain

$$a_1 = a_0 f(a_0)^2 g(a_0, b_0) \leq a_0 < 1$$

and as $f(x) > 1$ in $x \in (0, r_0]$, we get

$$b_1 = b_0 f(a_0)^{p+2} g(a_0, b_0)^{p+1} < b_0(f(a_0)^2 g(a_0, b_0))(f(a_0)^2 g(a_0, b_0))^p \leq b_0$$

Now, let us assume that the statement hold true for $n = k$. Then, proceeding similarly as above, one can easily prove that $a_{k+1} \leq a_k \leq r_0 < 1$ and $b_{k+1} \leq b_k$. Since f and g are increasing functions, we get

$$f(a_{k+1})^2 g(a_{k+1}, b_{k+1}) \leq f(a_k)^2 g(a_k, b_k) \leq 1$$

Hence, the Lemma 2 holds true for all n.

LEMMA 3. *Let $0 < a_0 < r_0$ and $0 < b_0 < \Phi_p(a_0)$. Define $\gamma = \frac{a_1}{a_0}$, then for $n \geq 1$ we get*

(i) $a_n \leq \gamma^{(2+p)^{n-1}} a_{n-1} \leq \gamma^{((2+p)^n-1)/(1+p)} a_0$ *for* $n \geq 1$

(ii) $b_n \leq (\gamma^{(2+p)^{n-1}})^{1+p} b_{n-1} \leq \gamma^{(2+p)^n-1} b_0$ *for* $n \geq 1$

(iii) $f(a_n) g(a_n, b_n) \leq \gamma^{(2+p)^n} \dfrac{f(a_0) g(a_0, b_0)}{\gamma} = \dfrac{\gamma^{(2+p)^n}}{f(a_0)}$, $n \geq 0$

Proof. We can prove (i) and (ii) by using induction. Since $a_1 = \gamma a_0$ and $a_1 < a_0$, we get $\gamma < 1$. Using (i) of Lemma 1, we get,

$$b_1 = b_0 f(a_0)^{p+2} g(a_0, b_0)^{p+1} < (f(a_0)^2 g(a_0, b_0))^{1+p} b_0 = \left(\frac{a_1}{a_0}\right)^{1+p} b_0 = \gamma^{1+p} b_0$$

Suppose (i) and (ii) hold for $n = k$, then

$$\begin{aligned} a_{k+1} = a_k f(a_k)^2 g(a_k, b_k) &\leq \gamma^{(p+2)^{k-1}} a_{k-1} f(a_{k-1})^2 g(\gamma^{(2+p)^{k-1}} a_{k-1}, (\gamma^{(2+p)^{k-1}})^{1+p} b_{k-1}) \\ &\leq \gamma^{(p+2)^{k-1}} a_{k-1} f(a_{k-1})^2 (\gamma^{(2+p)^{k-1}})^{1+p} g(a_{k-1}, b_{k-1}) = \gamma^{(2+p)^k} a_k \end{aligned}$$

Hence,

$$a_{k+1} \le \gamma^{(2+p)^k} a_k \le \gamma^{(2+p)^k} \gamma^{(2+p)^{k-1}} a_{k-1} \le \gamma^{(2+p)^k} \gamma^{(2+p)^{k-1}} \cdots \gamma^{(2+p)^0} a_0$$
$$= \gamma^{((p+2)^{k+1}-1)/(p+1)} a_0$$

From $f(x) > 1$ in $(0, r_0]$, we get

$$b_{k+1} = b_k f(a_k)^{p+2} g(a_k, b_k)^{p+1} \le b_k (f(a_k)^2 g(a_k, b_k))^{p+1}$$
$$= b_k \left(\frac{a_{k+1}}{a_k}\right)^{p+1} \le (\gamma^{(p+2)^k})^{p+1} b_k$$

Hence,

$$b_{k+1} = (\gamma^{(p+2)^k})^{p+1} b_k < (\gamma^{(p+2)^k})^{p+1} (\gamma^{(p+2)^{k-1}})^{p+1} \cdots (\gamma^{(p+2)^0})^{p+1} b_0 = \gamma^{(p+2)^{k+1}-1} b_0$$

Thus, (i) and (ii) hold by induction. (iii) follows from

$$f(a_n) g(a_n, b_n) \le f(\gamma^{((p+2)^n-1)/(p+1)} a_0) g(\gamma^{((p+2)^n-1)/(p+1)} a_0, \gamma^{(p+2)^n-1} b_0)$$
$$\le \gamma^{(p+2)^n} \frac{f(a_0) g(a_0, b_0)}{\gamma} = \gamma^{(p+2)^n} / f(a_0)$$

as $\gamma = a_1/a_0 = f(a_0)^2 g(a_0, b_0)$.

3. Recurrence relations

In this section, the recurrence relations will be derived for the Super-Halley's method under the assumptions given in the previous section. Let $\Gamma_0 = F'(x_0)^{-1}$ exists for $x_0 \in \Omega$. Then,

$$\|L_F(x_0)\| \le M \|\Gamma_0\| \, \|\Gamma_0 F(x_0)\| \le a_0$$
$$K \|\Gamma_0\| \, \|\Gamma_0 F(x_0)\|^{p+1} \le K\beta\eta^{p+1}$$

and

$$\|y_0 - x_0\| \le \|\Gamma_0 F(x_0)\| \le \eta$$

we can now prove the following conditions for $n \ge 1$

$$(9) \quad \left.\begin{array}{ll} \text{(I)} & \|\Gamma_n\| = \|F'(x_n)^{-1}\| \le f(a_{n-1}) \|\Gamma_{n-1}\| \\ \text{(II)} & \|\Gamma_n F(x_n)\| \le f(a_{n-1}) g(a_{n-1}, b_{n-1}) \|\Gamma_{n-1} F(x_{n-1})\| \\ \text{(III)} & \|L_F(x_n)\| \le M \|\Gamma_n\| \, \|\Gamma_n F(x_n)\| \le a_n \\ \text{(IV)} & K \|\Gamma_n\| \, \|\Gamma_n F(x_n)\|^{p+1} \le b_n \\ \text{(V)} & \|x_{n+1} - x_n\| \le \left(1 + \dfrac{a_n}{2(1-a_n)}\right) \|\Gamma_n F(x_n)\| \\ \text{(VI)} & y_n, x_{n+1} \in \mathscr{B}(x_0, R\eta) \end{array}\right\}$$

Assuming that $\left(1+\frac{a_0}{2(1-a_0)}\right)a_0 < 1$ and $x_1 \in \Omega$, we get

$$\begin{aligned} \|I - \Gamma_0 F'(x_1)\| &\le \|\Gamma_0\|\, \|F'(x_0) - F'(x_1)\| \\ &\le \|\Gamma_0\| \int_0^1 \|F''(x_0 + t(x_1 - x_0))\|\, dt \|x_1 - x_0\| \\ &\le M\|\Gamma_0\|\, \|x_1 - x_0\| \end{aligned}$$

From

$$\|x_1 - x_0\| \le \left(1 + \frac{a_0}{2(1-a_0)}\right)$$

we get

$$\begin{aligned} \|I - \Gamma_0 F'(x_1)\| &\le M\beta\left(1 + \frac{a_0}{2(1-a_0)}\right)\eta \\ &\le \left(1 + \frac{a_0}{2(1-a_0)}\right)a_0 < 1 \end{aligned}$$

Then, by Banach Lemma, $\Gamma_1 = F'(x_1)^{-1}$ exists and

$$\|\Gamma_1\| \le \frac{\|\Gamma_0\|}{1 - \|\Gamma_0\|\, \|F'(x_0) - F'(x_1)\|} \le \frac{2(1-a_0)}{(2 - 4a_0 + a_0^2)}\|\Gamma_0\| \le f(a_0)\|\Gamma_0\| \tag{10}$$

Using the Taylor's formula, we get

$$\begin{aligned} F(x_{n+1}) &= F(y_n) + F'(y_n)(x_{n+1} - y_n) + \int_{y_n}^{x_{n+1}} F''(x)(x_{n+1} - x)\, dx \\ &= \int_0^1 [F''(x_n + t(y_n - x_n)) - F''(x_n)] \\ &\quad \times (1-t)(I - L_F(x_n))^{-1}(y_n - x_n)^2\, dt \\ &\quad - \int_0^1 F''(x_n + t(y_n - x_n))(I - L_F(x_n))^{-1} L_F(x_n)(1-t)(y_n - x_n)^2\, dt \\ &\quad + \int_0^1 F''(x_n + t(y_n - x_n))\, dt (y_n - x_n)(x_{n+1} - y_n) \\ &\quad + \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)\, dt (x_{n+1} - y_n)^2 \end{aligned} \tag{11}$$

For $n = 0$ and $y_0 \in \Omega$, we get

$$\|F(x_1)\| \le \frac{M\eta^2 a_0}{2(1-a_0)} + \frac{K\eta^{p+2}}{(p+1)(p+2)(1-a_0)} + \frac{M\eta^2 a_0}{2(1-a_0)} + \frac{M\eta^2 a_0^2}{8(1-a_0)^2}$$

This gives

$$\|\Gamma_1 F(x_1)\| \leq \|\Gamma_1\|\,\|F(x_1)\| \leq f(a_0)g(a_0,b_0)\|\Gamma_0 F(x_0)\| \tag{12}$$

and (II) holds for $n = 1$. To prove (III) and (IV) for $n = 1$, notice that

$$\|L_F(x_1)\| = M\|\Gamma_1\|\,\|\Gamma_1 F(x_1)\| \leq a_0 f(a_0)^2 g(a_0,b_0) \leq a_1, \tag{13}$$

and

$$\begin{aligned} K\|\Gamma_1\|\,\|\Gamma_1 F(x_1)\|^{p+1} &\leq Kf(a_0)\|\Gamma_0\|\,\|\Gamma_0 F(x_0)\|^{p+1} f(a_0)^{p+1} g(a_0,b_0)^{p+1} \\ &\leq b_0 f(a_0)^{p+2} g(a_0,b_0)^{p+1} \\ &\leq b_1 \end{aligned} \tag{14}$$

This leads to

$$\begin{aligned} \|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| \\ &\leq \left[f(a_0)g(a_0,b_0) + \left(1 + \frac{a_0}{2(1-a_0)}\right)\right]\eta \\ &\leq \left(1 + \frac{a_0}{2(1-a_0)}\right)[1 + f(a_0)g(a_0,b_0)]\eta \\ &< \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{\eta}{(1-\gamma\Delta)} \end{aligned}$$

Therefore,

$$\|y_1 - x_0\| < R\eta \tag{15}$$

and

$$\|x_2 - x_1\| \leq \left(1 + \frac{a_1}{2(1-a_1)}\right)\|\Gamma_1 F(x_1)\| \tag{16}$$

From,

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| < \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{\eta}{(1-\gamma\Delta)} = \frac{2-a_0}{2(1-a_0)}\eta$$

we get

$$\|x_2 - x_0\| < R\eta \tag{17}$$

Using (10), (12) and (13) to (17) respectively, the conditions (I)–(VI) hold true for $n = 1$. Let us assume that (I) to (VI) hold for $n = k$. Proceeding similarly as above, we can easily prove that these conditions also hold for $n = k + 1$. Hence, by induction (I) to (VI) hold for all n.

4. Convergence theorem

In this section, an existence and uniqueness theorem is established for the solution of nonlinear equations (1) in Banach spaces obtained by the Super-Halley's method by using recurrence relations under the assumption that the second Frëchet derivative satisfies the Hölder continuity condition. The R-order of the method equals to $(2+p)$ is shown along with a priori error bounds. Let us denote $\gamma = \frac{a_1}{a_0}$, $\Delta = 1/f(a_0)$, $R = \frac{2-a_0}{2(1-a_0)}$, $\mathscr{B}(x_0, R\eta) = \{x \in \mathbf{X} : \|x - x_0\| < R\eta\}$ and $\overline{\mathscr{B}}(x_0, R\eta) = \{x \in \mathbf{X} : \|x - x_0\| \leq R\eta\}$.

THEOREM 1. *Let $F : \Omega \subseteq \mathbf{X} \to \mathbf{Y}$ be a nonlinear twice Frëchet differentiable on a non-empty open convex subset Ω of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$. Assume that $\Gamma_0 = F'(x_0)^{-1} \in L(\mathbf{Y}, \mathbf{X})$ exists for some $x_0 \in \Omega$. Under the assumptions given in* (3) *and $\overline{\mathscr{B}}(x_0, R\eta) \subseteq \Omega$, the method* (2) *starting from x_0 generates a sequence of iterates $\{x_n\}$ converging to the root x^* of* (1), *which is unique in $\mathscr{B}\left(x_0, \frac{2}{M\beta} - R\eta\right) \cap \Omega$. Furthermore the error bounds on x^* is given by*

$$\|x^* - x_n\| \leq \frac{(2 - a_0\gamma^{((p+2)^n-1)/(p+1)})}{2(1 - a_0\gamma^{((p+2)^n-1)/(p+1)})} \frac{\gamma^{((p+2)^n-1)/(p+1)}\Delta^n}{1-\gamma^{(p+2)^n}\Delta} \tag{18}$$

Proof. In order to establish the convergence of $\{x_n\}$, it is sufficient to show that the sequence $\{x_n\}$ is a Cauchy sequence. From the condition (IV) of (9), we can conclude that

$$\begin{aligned} \|y_n - x_n\| &\leq f(a_{n-1})g(a_{n-1}, b_{n-1})\|y_{n-1} - x_{n-1}\| \\ &\leq \cdots \left(\prod_{k=0}^{n-1} f(a_k)g(a_k, b_k)\right)\|y_0 - x_0\| \\ &\leq \left(\prod_{k=0}^{n-1} f(a_k)g(a_k, b_k)\right)\eta \end{aligned} \tag{19}$$

and

$$\begin{aligned} \|x_{m+n} - x_m\| &\leq \|x_{m+n} - x_{m+n-1}\| + \cdots \|x_{m+1} - x_m\| \\ &\leq \frac{(2 - a_{m+n-1})}{2(1 - a_{m+n-1})}\|y_{m+n-1} - x_{m+n-1}\| + \cdots \frac{(2 - a_m)}{2(1 - a_m)}\|y_m - x_m\| \\ &\leq \frac{(2 - a_m)}{2(1 - a_m)}\left[\prod_{k=0}^{m+n-2} f(a_k)g(a_k, b_k) + \cdots \prod_{k=0}^{m-1} f(a_k)g(a_k, b_k)\right]\eta \end{aligned} \tag{20}$$

Now, for $a_0 = r_0$, we get $b_0 = \Phi_p(a_0) = 0$. Hence, from Lemma 2, we get

$$f(a_0)^2 g(a_0, b_0) = 1,$$
$$a_n = a_{n-1} = a_{n-2} = \cdots a_1 = a_0$$

and

$$b_n = b_{n-1} = b_{n-2} = \cdots b_0 = 0$$

Thus,

$$\|y_n - x_n\| \le (f(a_0)g(a_0, b_0))^n \eta = \Delta^n \eta$$

and

$$\|x_{n+1} - x_n\| \le \frac{(2 - a_0)}{2(1 - a_0)} \Delta^n \eta$$

From this, we get

$$\|x_{m+n} - x_m\| \le \frac{(2 - a_0)}{2(1 - a_0)} [\Delta^{m+n-1} + \cdots \Delta^m]\eta = \frac{(2 - a_0)\Delta^m}{2(1 - a_0)} \left(\frac{1 - \Delta^n}{1 - \Delta}\right)\eta \tag{21}$$

If we take $m = 0$ then we get

$$\|x_n - x_0\| \le \frac{(2 - a_0)}{2(1 - a_0)} \left(\frac{1 - \Delta^n}{1 - \Delta}\right)\eta \tag{22}$$

Thus, $x_n \in \overline{\mathscr{B}}(x_0, R\eta)$. Similarly, we can prove that $y_n \in \overline{\mathscr{B}}(x_0, R\eta)$. As $\Delta < 1$, from (21), we conclude that $\{x_n\}$ is a Cauchy sequence. Let $0 < a_0 < r_0$ and $0 < b_0 < \Phi_p(a_0)$ then from the (19) and Lemma 3 (iii), for $n \ge 1$

$$\|y_n - x_n\| \le \left(\prod_{k=0}^{n-1} f(a_k)g(a_k, b_k)\right)\eta < \prod_{k=0}^{n-1} (\gamma^{(p+2)^k}\Delta)\eta = \gamma^{((p+2)^n - 1)/(p+1)}\Delta^n \eta \tag{23}$$

Hence, from (20), we obtain

$$\begin{aligned}
\|x_{m+n} - x_m\| &\le \frac{2 - a_m}{2(1 - a_m)} \left[\prod_{k=0}^{m+n-2} f(a_k)g(a_k, b_k) + \cdots \prod_{k=0}^{m-1} f(a_k)g(a_k, b_k)\right]\eta \\
&\le \frac{2 - a_m}{2(1 - a_m)} [\gamma^{((p+2)^{m+n-1} - 1)/(p+1)}\Delta^{m+n-1} + \cdots \gamma^{((p+2)^m - 1)/(p+1)}\Delta^m]\eta \\
&\le \frac{2 - a_m}{2(1 - a_m)} \Delta^m [\gamma^{((p+2)^{m+n-1} - 1)/(p+1)}\Delta^{n-1} + \cdots \gamma^{((p+2)^m - 1)/(p+1)}]\eta \\
&\le \frac{2 - \gamma^{((p+2)^m - 1)/(p+1)}a_0}{2(1 - \gamma^{((p+2)^m - 1)/(p+1)}a_0)} \Delta^m \gamma^{((p+2)^m - 1)/(p+1)} \\
&\quad [\gamma^{(p+2)^m((p+2)^{n-1} - 1)/(p+1)}\Delta^{n-1} + \cdots + \gamma^{(p+2)^m((p+2)^{n-1} - 1)/(p+1)}\Delta^{n-1} + 1]
\end{aligned}$$

By Bernoulli's inequality, for every real number $x > -1$ and every integer $k \geq 0$, we have $(1+x)^k - 1 \geq kx$. Thus, we get

$$\|x_{m+n} - x_m\| \leq \frac{(2-\gamma^{((p+2)^m-1)/(p+1)}a_0)}{2(1-\gamma^{((p+2)^m-1)/(p+1)}a_0)}\Delta^m\gamma^{((p+2)^m-1)/(p+1)} \frac{1-\gamma^{(p+2)^m}n\Delta^n}{1-\gamma^{(p+2)^m}\Delta} \tag{24}$$

Now, for $m = 0$, we get

$$\|x_n - x_0\| \leq \frac{(2-a_0)}{2(1-a_0)}\frac{(1-\gamma^n\Delta^n)}{(1-\gamma\Delta)}\eta \leq R\eta \tag{25}$$

Hence, $x_n \in \overline{\mathscr{B}}(x_0, R\eta)$. From

$$\begin{aligned} \|y_{n+1} - x_0\| &\leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \cdots \|x_1 - x_0\| \\ &\leq \|y_{n+1} - x_{n+1}\| + \frac{2-a_n}{2(1-a_n)}\|y_n - x_n\| \\ &\quad + \cdots + \frac{2-a_0}{2(1-a_0)}\|y_0 - x_0\| \\ &\leq \frac{2-a_{n+1}}{2(1-a_{n+1})}\|y_{n+1} - x_{n+1}\| + \frac{2-a_n}{2(1-a_n)}\|y_n - x_n\| \\ &\quad + \cdots + \frac{2-a_0}{2(1-a_0)}\|y_0 - x_0\| \\ &\leq \cdots \leq \frac{2-a_0}{2(1-a_0)}\frac{1-\gamma^{n+1}\Delta^{n+1}}{1-\gamma\Delta}\eta = R\eta \end{aligned} \tag{26}$$

we get $y_n \in \overline{\mathscr{B}}(x_0, R\eta)$. On taking the limit as $n \to \infty$ in (22) and (25), we get $x^* \in \overline{\mathscr{B}}(x_0, R\eta)$. To show that x^* is a solution of $F(x) = 0$, we use $\|F(x_n)\| \leq \|F'(x_n)\|\,\|\Gamma_n F(x_n)\|$ and since, the sequence $\{\|F'(x_n)\|\}$ is bounded as

$$\|F'(x_n)\| \leq \|F'(x_0)\| + M\|x_n - x_0\| < \|F'(x_0)\| + MR\eta$$

and F is continuous, by taking the limit as $n \to \infty$, we get $F(x^*) = 0$.

To prove the uniqueness of the solution, let y^* be the another solution of (1) in $\mathscr{B}(x_0, 2/((M\beta) - R\eta)) \cap \Omega$. From,

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\, dt(y^* - x^*)$$

Clearly, $y^* = x^*$, if $\int_0^1 F'(x^* + t(y^* - x^*))\, dt$ is invertible. This follows from

$$\begin{aligned}
\|\Gamma_0\| & \left\| \int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_0)]\, dt \right\| \\
& \leq M\beta \int_0^1 \|x^* + t(y^* - x^*) - x_{\alpha,0}\|\, dt \\
& \leq M\beta \int_0^1 (1-t)\|x^* - x_0\| + t\|y^* - x_0\|\, dt \\
& \leq \frac{M\beta}{2}\left(R\eta + \frac{2}{M\beta} - R\eta\right) = 1
\end{aligned}$$

and by Banach's theorem. Thus, $y^* = x^*$.

5. Numerical results

Example 1 ([4]). Let $\mathbf{X} = C[0,1]$ be the space of all continuous functions on the interval $[0,1]$ and consider the integral equation $F(x) = 0$, where

$$F(x)(s) = x(s) - s + \frac{1}{2}\int_0^1 s\cos(x(t))\, dt \tag{27}$$

If we choose $x_0 = x_0(s) = s$ and the norm $\|x\| = \max_{s\in[0,1]}|x(s)|$, then we get

$$F(x_0)(s) \leq \frac{1}{2}\sin 1s$$

$$F'(x)u(s) = u(s) - \frac{1}{2}\int_0^1 s\sin(x(t))u(t)\, dt$$

and

$$F''(x)uv(s) = \frac{-1}{2}\int_0^1 s\cos(x(t))u(t)v(t)\, dt$$

Taking $u(s) = [F'(x)]^{-1}v(s)$, we get

$$v(s) = u(s) - \frac{s}{2}\int_0^1 u(t)\sin(x(t))\, dt \tag{28}$$

Now, multiplying (28) by $\int_0^1 \sin(x(s))\, ds$, we get

$$\int_0^1 v(s)\sin(x(s))\, ds = \int_0^1 u(s)\sin(x(s))\, ds - \int_0^1 \frac{s}{2}\sin(x(s))\left[\int_0^1 u(t)\sin(x(t))\, dt\right] ds$$

This gives

$$\int_0^1 u(s)\sin(x(s))\, ds = \frac{\displaystyle\int_0^1 v(s)\sin(x(s))}{1 - \displaystyle\int_0^1 \frac{s}{2}\sin(x(s))\, ds} \tag{29}$$

Consequently,

$$u(s) = [F'(x)]^{-1}v(s) = v(s) + \frac{\displaystyle\int_0^1 v(s)\,\sin(x(s))\,ds}{1 - \displaystyle\int_0^1 \frac{s}{2}\,\sin(x(s))\,ds}$$

From this, we get

$$\Gamma_0 v(s) = [F'(x_0)]^{-1}v(s) = v(s) + \frac{\displaystyle\int_0^1 v(s)\,\sin(x(s))}{1 - \displaystyle\int_0^1 \frac{s}{2}\,\sin(x(s))\,ds} \tag{30}$$

This gives

$$\|[F'(x_0)]^{-1}\| \le \frac{3 - \sin 1}{2 - \sin 1 + \cos 1}$$

$$\|F'(x_0)^{-1}F(x_0)\| \le \frac{\sin 1}{2 - \sin 1 + \cos 1} = \eta$$

$$\|F''(x)\| \le \frac{1}{2} = M$$

and

$$\begin{aligned}\|F''(x) - F''(y)\|uv(s) &\le \frac{1}{2}\int_0^1 |\cos x(t) - \cos y(t)|uv(t)\,dt \\ &\le \frac{1}{2}\|x - y\|\end{aligned}$$

This implies that $K = \frac{1}{2}$, $p = 1$, $\beta = 1.2705964$ and $\eta = 0.4953234$. Hence, we get $a_0 = M\beta\eta = 0.314678 < r_0 = 0.380778$ and $b_0 = K\beta\eta^2 = 0.155867 < \Phi_p(a_0) = 0.191729$. Hence, the conditions of theorem 1 are satisfied and the solution of (27) exists in the ball $\overline{\mathscr{B}}(x_0, 0.883997)$. and is unique in the ball $\mathscr{B}(x_0, 2.26413)$. However, solving (27) by using majorizing sequence [3], we find that the solution exists in the ball $\overline{\mathscr{B}}(x_0, 0.609569) \subseteq \Omega$ and is unique in $\mathscr{B}(x_0, 1.70991)$. This clearly improves the existence and uniqueness regions of solution by our approach.

Example 2. Let $\mathbf{X}$ be the space of all continuous functions on $[a, b]$ and consider the integral equation $F(x) = 0$ on $\mathbf{X}$, where

$$F(x)(s) = x(s) - f(s) - \lambda\int_0^1 \frac{s}{s+t}x(t)^{2+p}\,dt \tag{31}$$

with $s \in [0, 1]$, $x, f \in \mathbf{X}$, $p \in (0, 1]$ and λ is a real number.

The given integral equation is called Fredholm-type integral equations studied by [2]. Here norm is taken as the sup-norm.

Now it is easy to find that the first and the second order Frëchet derivatives of F as

$$F'(x)u(s) = u(s) - \lambda(2+p)\int_0^1 \frac{s}{s+t}x(t)^{1+p}u(t)\,dt, \quad u \in \Omega$$

$$F''(x)uv(s) = -\lambda(2+p)(1+p)\int_0^1 \frac{s}{s+t}x(t)^{p}(uv)(t)\,dt, \quad u, v \in \Omega$$

Here F'' does not satisfy the Lipschitz condition as, for $p \in (0,1)$ and for all $x, y \in \Omega$

$$\begin{aligned}\|F''(x) - F''(y)\| &= \left\|\lambda(2+p)(1+p)\int_0^1 \frac{s}{s+t}[x(t)^p - y(t)^p]\,dt\right\| \\ &\le |\lambda|(2+p)(1+p)\max_{s\in[0,1]}\left|\int_0^1 \frac{s}{s+t}\,dt\right| \|x(t)^p - y(t)^p\| \\ &\le |\lambda|(2+p)(1+p)\log 2\|x-y\|^p\end{aligned}$$

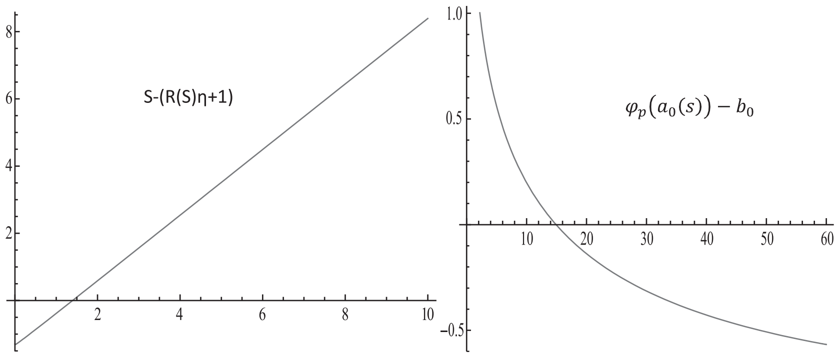

FIGURE 1. Conditions on the parameter S.

But it satisfies the Holder continuity condition as $p \in (0,1]$. Hence $K = |\lambda|(2+p)(1+p)\log 2$. Now, we need to compute the parameters M, η, β. Clearly,

$$\|F(x_0)\| \le \|x_0 - f\| + |\lambda|\log 2\|x_0\|^{2+p}$$

Also note that

$$\|I - F'(x_0)\| \le |\lambda|(2+p)\log 2\|x_0\|^{1+p}$$

and, if $|\lambda|(2+p)\log 2\|x_0\|^{1+p} < 1$, then by Banach theorem, we obtain

$$\|\Gamma_0\| = \|F'(x_0)^{-1}\| \leq \frac{1}{1-|\lambda|(2+p)\log 2\|x_0\|^{1+p}} = \beta$$

and

$$\|F''(x)\| \leq |\lambda|(2+p)(1+p)\log 2\|x\|^p = M(x)$$

Hence

$$\|\Gamma_0 F(x_0)\| \leq \frac{\|x_0 - f\| + |\lambda|\log 2\|x_0\|^p}{1-|\lambda|(2+p)\log 2\|x_0\|^{1+p}} = \eta$$

Now, for $\lambda = 1/4$, $p = 1/5$, $f(s) = 1$ and $x_0 = x_0(s) = 1$ in the interval $[0,1]$, we have $\|\Gamma_0\| \leq \beta = 1.61611$, $\|\Gamma_0 F(x_0)\| \leq \eta = 0.280051$, $K = 0.457477$, and $b_0 = K\beta\eta^{1+p} = 0.160518$. Now we find the domain in the form of $\Omega = \mathscr{B}(x_0, S)$ such that $\Omega = \mathscr{B}(x_0, S) \subseteq C[0,1] = \mathbf{X}$. So $M = M(S) = 0.457477S^p$, $a_0 = a_0(S) = M(S)\beta\eta = 0.207051S^p$. In this situation, from Theorem 1, it is necessary that $\overline{\mathscr{B}}(x_0, R\eta) \subseteq \Omega$. For this it is sufficient to check $S - (R(S)\eta + 1) > 0$ and $\Phi_p(a_0) - b_0 > 0$. This requires that $S \in (1.39475, 14.9209)$ as is evident from Fig. 1. Also, $a_0(S) \leq r_0 = 0.380778$ if and only if $S < 21.0365$. Taking $S = 14$, we get $\Omega = \mathscr{B}(1, 14)$, $M = 0.775523$, $a_0 = 0.350997$, and $b_0 = 0.160518 < 0.178796 = \Phi_p(a_0)$. Thus, all the conditions of the Theorem 1 are satisfied. Hence a solution of (31) exists in $\overline{\mathscr{B}}(x_0, 0.76485)$. and is unique in the ball $\mathscr{B}(1, 0.830898)$. But, by using majorizing sequences [3], we find that the solution exists in the ball $\overline{\mathscr{B}}(x_0, 0.377703). \subseteq \Omega$ and is unique in $\mathscr{B}(x_0, 0.958765)$. From this result, one can easily conclude that our existence and uniqueness regions of solution improved the existence and uniqueness regions obtained by majorizing sequences.

if and only if $S \leq 82.1227$. Hence if we choose $S = 14$, then we have $\Omega = \mathscr{B}(1, 14)$, $M = 0.775523$, $a_0 = 0.350997$, and $b_0 = 0.160518 < 0.178796 = \Phi_p(a_0)$. Thus the conditions of the Theorem corresponding paper satisfied. Hence the solution of Eq. (31) exist in $\overline{\mathscr{B}}(1, 0.420745)$ and the solution unique in the ball $\mathscr{B}(1, 1.175) \cap \Omega$

Example 3. Consider the differential equation

$$y'' + y' - y^3 = 0, \quad y(0) = y(1) = 0 \tag{32}$$

We divided the interval $[0,1]$ into n subintervals and we set $h = \frac{1}{n}$. Let $\{z_k\}$ be the points of the subdivision with

$$0 = z_0 < z_1 < z_2 < \cdots < z_n = 1$$

and corresponding values of the function

$$y_0 = y(z_0), y_1 = y(z_1), \ldots, y_n = y(z_n)$$

Standard approximations for the first and second derivatives are given respectively by

$$y_i' = (y_{i+1} - y_i)/h, y_i'' = (y_{i-1} - 2y_i + y_{i+1})/h^2, \quad i = 1, 2, \dots n-1$$

Noting that $y_0 = 0 = y_n$, define the operator $F : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ by

$$F(y) = G(y) + hJ(y) - 2h^2 g(y),$$

where

$$G = \begin{pmatrix} -4 & 2 & 0 & \cdots & 0 \\ 2 & -4 & 2 & \cdots & 0 \\ 0 & 2 & -4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & -4 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

$$g(y) = \begin{pmatrix} y_1^3 \\ y_2^3 \\ \vdots \\ y_{n-1}^3 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix},$$

Then, we get

$$F'(y) = G + hJ - 6h^2 \begin{pmatrix} y_1^2 & 0 & 0 & \cdots & 0 \\ 0 & y_2^2 & 0 & \cdots & 0 \\ 0 & 0 & y_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & y_{n-1}^2 \end{pmatrix},$$

$$F''(y) = -12h^2 \begin{pmatrix} y_1 & 0 & 0 & \cdots & 0 \\ 0 & y_2 & 0 & \cdots & 0 \\ 0 & 0 & y_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & y_{n-1} \end{pmatrix},$$

Let $x \in \mathbf{R}^{n-1}$, $A \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$, and define the norms of x and A by

$$\|x\| = \max_{1 \le i \le n-1} |x_i|, \quad \|A\| = \max_{1 \le i \le n-1} \sum_{k=1}^{n-1} |a_{ik}|$$

For $n = 10$ and for all $x, y \in R^{n-1}$, we now get

$$\|F''(x) - F''(y)\| \le 0.12\|x - y\|$$

As the solution should vanish at the end points and be positive in the interior, a reasonable choice of initial approximation seems to be $\exp(\pi x)/100$. This gives the following vector:

$$x_0 = \begin{pmatrix} 0.0136911 \\ 0.0187446 \\ 0.0256633 \\ 0.0351359 \\ 0.0481048 \\ 0.0658606 \\ 0.0901703 \\ 0.1234530 \\ 0.1690200 \end{pmatrix},$$

We now get the following results for our method:

$\|\Gamma_0\| \le \beta = 6.11998638$, $\|\Gamma_0 F(x_0)\| \le \eta = 0.168893$, $\|F''(x)\| \le M = 0.0202824$, $N = 0.12$, $a_0 = M\beta\eta = 0.02096435$, $b_0 = 0.0209486$, $r_0 = 0.380778$ and $\Phi_p(a_0) = 0.0401631$. This implies that $a_0 < r_0 = 0.380778$, and $b_0 < \Phi_p(a_0)$. Thus, all the conditions of Theorem 1 are satisfied. This implies that the solution of equation (32) exists in the ball $\overline{\mathscr{B}}(1, 0.171405)$ and unique in the ball $\mathscr{B}(1, 15.9411) \cap \Omega$. Solving (32) by using majorizing sequence, we obtained that the solution exists in $\overline{\mathscr{B}}(1, 0.17017)$ and unique in $\mathscr{B}(1, 2.67306) \cap \Omega$. From this, we can easily conclude that the existence and uniqueness regions of solution are improved by our approach.

6. Conclusions

The semilocal convergence of Super-Halley's method used for solving nonlinear equations in Banach spaces by using the recurrence relations is established in this paper. This is done under the assumption that the second Frĕchet derivative of the involved operator satisfies the Hölder continuity condition which is milder than the Lipschitz continuity condition. A new family of recurrence relations are defined based on the two constants which depend on the operator F. An existence and uniqueness theorem along with a priori error bounds for the solution x^* is given. The R-order of the method equals to $(2 + p)$ for $p \in (0, 1]$ is also established. Three numerical examples are worked out to demonstrate the efficacy of our approach. On comparison with the method described in [3], we observed the improved existence and uniqueness regions of solution.

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