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THE EUCLIDEAN, HYPERBOLIC, AND SPHERICAL SPANS OF AN OPEN RIEMANN SURFACE OF LOW GENUS AND THE RELATED AREA THEOREMS

Dedicated to Professor Nobuyuki Suita on his sixtieth birthday

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Introduction.

It was almost fifty years ago that M. Schiffer [11] introduced the notion of *span* to study the theory of conformal mapping—or the theory of univalent functions if one would like to call it—of multiply connected plane domains along the line of Grötzsch and Grunsky. The notion has since been playing an important role in the theory of conformal mapping of planar Riemann surfaces. To deal with nonplanar Riemann surfaces equally, we shall have to take account of holomorphic *mappings* (into other Riemann surfaces) as well as holomorphic *functions,* and also have to generalize the notion of span. While the study of holomorphic mappings in its full generality is still immature, the theory of conformal embeddings (=injective holomorphic mappings) suffices for our purposes. More specifically, if we confine ourselves to those mappings which embed an open Riemann surface of finite genus into closed ones of the same genus, considerably satisfactory results could be expected. We have shown some of them in the preceding papers [12]-[15], on which the present article is based.

By the phrase *"of low genus"* we mean "either *of genus zero* or *of genus one*". We first consider the case of genus one. To state the preparatory facts briefly and clearly, it is convenient to introduce the term *"an open torus"*, which simply means an open Riemann surface of genus one. Meanwhile we keep the classical terminology *"a torus"* means a closed Riemann surface of genus one as usual. Sometimes the term *"a closed torus"* will be also used for the same purpose. A *compact continuation* of an open torus is, roughly speaking, a conformal embedding of the open torus into a closed torus which induces the prescribed correspondence between their canonical homology bases. We have shown in [13], among other things, that the set of moduli of the compact continuations of an open torus is a closed disk in the upper half plane, and that the diameter of this moduli disk gives a close analogue of Schiffer's span. Although the present work has been motivated by the investigation of open tori, the method does work, in principle, also for planar Riemann surfaces and yields new results

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for these surfaces as well. For instance, we establish a quantitative refinement of the classical area theorems due to Gronwall, Bieberbach, Grötzsch, Grunsky and others (cf. [3], [5], [6], [9], and [10]). See the end of the next paragraph.

We begin the present paper by introducing another span, which is defined to be the *hyperbolic* diameter of the moduli disk. It is reasonable to call it the "hyperbolic span" of the surface, and the former one in [13] the "euclidean span". The hyperbolic span depends on nothing other than the open torus itself. We then prove a generalization of Grunsky's theorem to the case of open tori: Complementary area is maximized by a conformal embedding of the open torus into the (closed) torus whose modulus lies at the euclidean center τ^* of the moduli disk. Actually we prove more. Let τ be a point of the moduli disk, and consider the class of compact continuations of the open torus onto the (closed) torus with modulus τ. Then, there exists a compact continuation which maximizes the complementary area in the class. We show furthermore that the maximum complementary area α_τ for τ depends solely on the distance of τ from τ^* *(i.e.,* α_τ is constant on each euclidean concentric circle), and that $|\alpha_\tau|^{-1}|d\tau|$ is, up to a multiplicative constant, the Poincaré metric of the moduli disk. The corresponding theorem for plane domains also holds, where the first coefficient of the regular part of a univalent meromorphic function plays a similar role as the modulus does for open tori. This is what we announced above.

We then prove the hyperbolic version of our area theorem. If we consider the ratio of the complementary area to the total surface area instead of the complementary area itself, we have a similar theorem: The area ratio is maximized at the *hyperbolic* center of the moduli disk, and the *hyperbolic* concentric circles have the same properties for the area ratio as the euclidean concentric circles do for the complementary area. It would be worth while noting that the corresponding theorem for plane domains does not exist, since the disk of coefficients of univalent functions has no natural structure as a hyperbolic disk and the (euclidean) area of the image domain is always infinite. However, we can regard the coefficient disk as a spherical disk—a disk with respect to the spherical metric—, and this observation gives rise to another span, the "spherical span", of a plane domain and a new extremal problem. The spherical span is defined also for open tori and we have similar results in this case.

We finally discuss some consequences of our results. The first application is an inequality for the univalent meromorphic functions on a minimal slit domain in the sense of Koebe. The inequality is usually proved by other methods—for example, by the method of extremal length ([6]) or by using the Rengel inequality ([17]). The corresponding result for open tori can be also proved. These theorems yield an estimate of Schiffer's and the euclidean spans. As another consequence of our results we prove: If an open torus is realized on a closed torus as a subregion whose area is less than a half of the total area, then the complementary area cannot be maximum. The last fact shows that the conformal embedding obtained by K. Strebel ([16]) cannot be characterized as a compact continuation whose modulus is the euclidean center of the moduli disk.

A word for the naming of theorems and lemmas. "THEOREM X_n" ($n=0$, 1) means that the theorem concerns with plane domains or tori, according as $n=0$ or $n=1$.

1. Preliminaries.

Let R be an open torus (*i.e.*, an open Riemann surface of genus one) and $\chi=\{a, b\}$ a fixed canonical homology basis of R modulo dividing cycles (cf. [1]). Consider a triplet (R', χ', i') consisting of a (closed) torus R', a canonical homology basis $\chi'=\{a',b'\}$ of R' and a conformal embedding i' of R into R' such that $i'(a)$ (resp. $i'(b)$) is homologous to the cycle a' (resp. b'). Two such triplets (R', χ', i') and (R'', χ'', i'') shall be called equivalent if there is a conformal mapping f of R' onto R'' with $f\circ i'=i''$. Each equivalence class is called a *compact continuation* of (R, χ) and denoted by $[R', \chi', i']$. We often say that i' is a conformal mapping of (R, χ) into (R', χ') and write as $i': (R, \chi)\to(R', \chi')$.

As is well known, each compact continuation $[R', \chi', i']$ carries a unique holomorphic differential ϕ' whose a'-period is 1. It will be called the *normal holomorphic differential* on (R', χ') (or on $[R', \chi', i']$). The differential ϕ' induces a natural metric $|\phi'|$ on R'. We may and do assume that the cycle a' is a geodesic on (R', χ') with respect to the metric.

Denote by $C(R, \chi)$ the set of compact continuations of (R, χ). For $[R',\chi',i'] \in C(R, \chi)$ let $\tau[R', \chi', i']$ denote the modulus of the marked torus (R',χ'), to which we refer as the *modulus* of $[R', \chi', i']$ (see [13]). We denote by $\mathfrak{M}(R,\chi)$ the set of moduli of the compact continuations of (R, χ):

$$\mathfrak{M}(R, \chi)=\{\tau\in \boldsymbol{C}\,|\,\tau=\tau[R', \chi', i'],\ [R', \chi', i']\in C(R, \chi)\}\,.$$

The set $\mathfrak{M}(R, \chi)$ obviously lies in the upper half plane H. We are concerned with euclidean and hyperbolic properties of $\mathfrak{M}(R,\chi)$. We begin with the following theorem, which characterizes the euclidean properties of $\mathfrak{M}(R, \chi)$. For a proof of this theorem, see [13]. Part of the theorem was proved by Grötzsch ([4]).

THEOREM A_1. (I) *$\mathfrak{M}(R,\chi)$ is a closed disk (which may degenerate to a singleton); there exist $\tau^*\in \boldsymbol{H}$ and $\rho_1\in \boldsymbol{R}$ such that $0\leqq\rho_1<\operatorname{Im}\tau^*$ and*

$$\mathfrak{M}(R, \chi)=\{\tau\in \boldsymbol{H}|\ |\tau-\tau^*|\leqq\rho_1\}\,.$$

(II) *To each boundary point of $\mathfrak{M}(R, \chi)$ corresponds a single element of $C(R, \chi)$. Furthermore, if the function*

$$\tau(t)=\tau^*+\rho_1 e^{(t-1/2)\pi i}, \qquad -1<t\leqq 1,$$

parametrizes $\partial\mathfrak{M}(R, \chi)$ *and* $[R_{\tau(t)}, \chi_{\tau(t)}, i_{\tau(t)}]$, $\chi_{\tau(t)}=\{a_{\tau(t)}, b_{\tau(t)}\}$, *denotes the compact continuation corresponding to the point* $\tau(t)$, *then* $R_{\tau(t)}\setminus i_{\tau(t)}(R)$ *is a set of null area which is a union of geodesic parallel segments making an angle* $\pi/2$ *with* $a_{\tau(t)}$

(III) *The euclidean radius* ρ_1 *of* $\mathfrak{M}(R, \chi)$ *vanishes if and only if* R *belongs to* O_{AD}.

Theorem A_1 contains a generalization of Koebe's general uniformization theorem—which is often referred to as the fundamental theorem in the theory of conformal mapping (cf., e.g., [3], [6])—to surfaces of genus one. From the viewpoint of the continuation theory of Riemann surfaces, Theorem A_1 gives a refinement of Heins' theorem (cf. [13]) which states that the principal moduli of the compact continuations are bounded. For general cases of higher genera, see [12]. The uniqueness problem of the compact continuations can be also dealt with by Theorem A_1, (III). For example, we can prove the theorems of Nevanlinna-Mori and of Oikawa: A Riemann surface of finite genus admits a unique (in the sense of Nevanlinna or Oikawa respectively) compact continuation if and only if it belongs to the class O_{AD}. For these topics, see [14] and [15].

For later references, we state here the prototype of Theorem A_1 for planar domains, which is due to Koebe, de Possel, Grötzsch and others. Cf. [6] and [11]. To do so, suppose that a plane domain G and a reference point $\zeta\in G$ are given. We do not lose generality to assume $\zeta\neq\infty$. We consider the set $F(G, \zeta)$ of (normalized) compact continuations of G to $\hat{C}$. More precisely, $F(G, \zeta)$ consists of univalent meromorphic functions f on G which have a single simple pole at ζ with residue 1. Furthermore, we identify two functions in $F(G, \zeta)$ if their difference is constant. According to our earlier work an element of $F(G, \zeta)$ should have been written as $[\hat{C}, \infty, f]$. However, we may and do actually use the simpler notation f instead. Each function $f\in F(G, \zeta)$ has a Laurent expansion

$$f(z)=\frac{1}{z-\zeta}+\kappa_f(z-\zeta)+\cdots \quad \text{about } \zeta.$$

Let $\mathfrak{K}(G, \zeta)$ denote the set of coefficients κ_f:

$$\mathfrak{K}(G, \zeta)=\{\kappa\in C\ \kappa=\kappa_f \text{ for some } f\in F(G, 0)\}\ .$$

We recall that the class $F(G, \zeta)$ corresponds to the class $\Sigma'(G)$ in [6] (see Def. 1.3), which coincides with Σ_0 in [9] (cf. p. 13) if G is the domain $\hat{C}\setminus\{|z|\leqq 1\}$.

THEOREM A_0. (I) $\mathfrak{K}(G, \zeta)$ *is a (possibly degenerate) closed disk*

$$\mathfrak{K}(G, \zeta)=\{\kappa\in C\,|\,|\kappa-\kappa^*|\leqq\rho_0\} \text{ with } \kappa^*\in C \text{ and } \rho_0\geqq 0\,.$$

(II) *To each boundary point of* $\mathfrak{K}(G, \zeta)$ *corresponds a single element of*

$F(G, \zeta)$. Furthermore, if

$$\kappa(t)=\kappa^*+\rho_0 e^{\pi i t}, \qquad -1<t\leqq 1,$$

parametrizes $\partial\mathfrak{K}(G, \zeta)$ and if $[\hat{C}, \infty, f_{\kappa(t)}]$ denotes the compact continuation corresponding to the point $\kappa(t)$, then $\hat{C}\backslash f_{\kappa(t)}(G)$ is a compact plane set of null area which is a union of parallel segments making an angle $\pi t/2$ with the real axis.

(III) *The radius ρ_0 of $\mathfrak{K}(G, \zeta)$ vanishes if and only if G belongs to O_{AD}.*

No explanation will be needed for the correspondence between Theorems A_0 and A_1. There are, however, some differences between these two theorems; a typical and the most definitive one of them is that $\mathfrak{K}(G, \zeta)$ never lies in the upper half plane H.

In the following sections we aim to study the relationship between the euclidean, hyperbolic, or spherical structure of $\mathfrak{M}(R, I)$ and the complementary area of the embedded surface in its compact continuations. See, in particular, Theorems B_1 and C_1. Theorem B_1 will suggest in turn a new result on planar Riemann surfaces (Theorem B_0 in Section 6).

2. The euclidean, hyperbolic, and spherical spans.

The diameter $2\rho_0$ of the disk $\mathfrak{K}(G, \zeta)$ is called by M. Schiffer (see [11]) the *span* of the plane domain G. It depends on the reference point ζ as well as the domain G. Hence, we write it as $\sigma(G, \zeta)$. Theorem A_0 (III) above indicates one of the characteristic properties of $\sigma(G, \zeta)$. Another characterization of $\sigma(G, \zeta)$ was given by H. Grunsky. See [5] and [8] cf. also Corollary B_0 below. J. A. Jenkins ([7]) used the extremal length method to show other aspects of $\sigma(G, \zeta)$.

We have already pointed out in [13] (see also [14]) that $2\rho_1$ plays the same role in the study of open Riemann surfaces as Schiffer's span does in the classical study of plane domains. The diameter $2\rho_1$ depends not only on the surface R but also on the choice of the canonical homology basis χ of R. Hence, we rewrite $2\rho_1$ as $\sigma_E(R, \chi)$ and call it the *euclidean span* of (R, χ). The name and the subscript suggest the euclidean structure of $\mathfrak{M}(R, X)$. Following this convention, we rewrite τ^* and ρ_1 as $\tau_E^*=\tau_E^*(R, \chi)$ and $\rho_E=\rho_E(R, \chi)$ respectively. Similarly, we rewrite κ^*, ρ_0 and $\sigma(G, \zeta)$ as $\kappa_E^*=\kappa_E^*(G, \zeta)$, $\rho_E=\rho_E(G, \zeta)$ and $\sigma_E(G, \zeta)$ respectively. We observe $\sigma_E(R, \chi)=\mathrm{Im}\,\tau(1)-\mathrm{Im}\,\tau(0)=(1/i)[\tau(1)-\tau(0)]$ and $\sigma_E(G, \zeta)=\kappa(0)-\kappa(1)$.

Now, as we remarked earlier, the set $\mathfrak{M}(R, \chi)$ always lies in $\boldsymbol{H}$, which we may consider in a well known fashion the hyperbolic plane. Furthermore, $\mathfrak{M}(R, T)$ is a hyperbolic disk, and hence it makes sense to refer to the hyperbolic diameter of $\mathfrak{M}(R, T)$. We call the (hyperbolic) diameter of $\mathfrak{M}(R, \chi)$ the *hyperbolic span* of R. The hyperbolic span is determined solely by the surface R it is invariant under any change of canonical homology bases of R. Indeed, if the canonical homology basis $\chi=\{a, b\}$ of $R\,(\mathrm{mod}\,\partial R)$ is replaced with another

$\chi_1=\{a_1, b_1\}$, then a_1, b_1 are represented as

fli$-ma+nb$

$$b_1=m''a+rib$$

with $m, m', n, n'\in \boldsymbol{Z}$ and $mn'-m'n=1$. Thus the moduli disk $\mathfrak{M}(R, \chi_1)$ is the image of the moduli disk $\mathfrak{M}(R,$X) under the unimodular transformation

$$\tau \longmapsto n\tau+m'$$

so that their hyperbolic diameters are equal. The hyperbolic span is more intrinsic than the euclidean. We denote by $\sigma_H(R)$ the hyperbolic span of R.

We denote by $d_H(z_1,z_2)$ the hyperbolic distance of two points $z_1,z_2\in\boldsymbol{H}$. It is given by the formula

$$d_H(z_1, z_2)=\log\frac{|z_1-\bar{z}_2|+|z_1-z_2|}{|z_1-\bar{z}_2|-|z_1-z_2|}.$$

See, [2], p. 130, for instance. Let us denote by $\tau_H^*=\tau_H^*(R,$ X) and $\rho_H=\rho_H(R)$ the hyperbolic center and the hyperbolic radius of $\mathfrak{M}(R,$ X), respectively. We have then

$$\mathfrak{M}(R, \chi)=\{\tau\in\boldsymbol{H}\,|\,d_H(\tau, \tau_H^*)\leqq\rho_H(R)\}$$

and

$$\sigma_H(R)=2\rho_H(R)=\log\frac{\operatorname{Im}\tau(1)}{\operatorname{Im}\tau(0)}.$$

Finally, we observe that ft(G, ζ) and $\mathfrak{M}(R,$ X) are disks with respect to the spherical metric on the Riemann sphere. We use the subscript S to mean that we consider the spherical metric. For example, $\tau_S^*=\tau_S^*(R,$X) and $\kappa_S^*=\kappa_S^*(G,\zeta)$ denote the spherical centers of $\mathfrak{M}(R,$X) and ft(G, ζ) respectively. We call the spherical diameter of ft(G, ζ) (resp. $\mathfrak{M}(R,\chi)$) the *spherical span* of (G, ζ) (resp. (R,χ)) and denote it by $\sigma_S(G,\zeta)$ (resp. $\sigma_S(R,\chi)$). For later use we recall here that the spherical distance between two points z_1,z_2 is given by

$$d_S(z_1, z_2)=2\tan^{-1}\left|\frac{z_1-z_2}{1+z_1\bar{z}_2}\right|.$$

Note that each of our spans—euclidean, or spherical, or hyperbolic—concerns the degeneration of analytic functions more closely than that of harmonic functions, as Theorem A_1 (III) shows. Note also that we need neither reference point nor any fixed local parameter.

In what follows, we always assume that $\sigma_H(R)>0$ —or equivalently: ff£C#,X)>0 or $\sigma_S(R,$X$)>0$ for every canonical homology basis X. Otherwise, all the theorems below would be trivial. For the same reason the plane domain G is supposed to satisfy $\sigma_E(G,\zeta)>0$, ζ being a point of G.

3. The euclidean span and the Area Theorem.

Let $(R, \mathfrak{X})$ be as before and $[R', \mathrm{r}, i'] \in C(R, \mathfrak{X})$. As we have already remarked, a marked torus $(R', \mathfrak{X}')$ has a natural metric which is induced by the normal holomorphic differential on it. In the rest of this paper we always understand the term *"area"* as the area with respect to this natural metric. For any $[R', \mathfrak{X}', i'] \in C(R, \mathfrak{X})$, we denote by

$$A[R', \mathrm{r}, i']$$

the total area of R', and by

$$\alpha[R', \mathfrak{X}', i']$$

the (outer) area of $R' \backslash i'(R)$. We also consider the ratio

$$S[R', \mathfrak{X}', i'] = \alpha[R', \mathfrak{X}', i'] / A[R', \mathfrak{X}', i'] .$$

Obviously, A, a, and S give rise to mappings of $C(R, 1C)$ into $\boldsymbol{R}_+$, the set of nonnegative real numbers, and satisfy the following inequalities:

$$0 \leqq \alpha[R', \mathfrak{X}', i'] < A[R', \mathfrak{X}', i'], \qquad 0 \leqq S[R', \mathrm{r}, i'] < 1$$

for all $[R', \mathrm{r}, i'] \in C(R, 30.$ In [13] we have proved the boundedness of A on $\mathfrak{M}(R, \mathfrak{X})$; Theorem A_1 actually solves the extremal problem of maximizing and minimizing $A[R', \mathfrak{X}', i']$ in $C(R, \mathfrak{X})$. Now we consider similar extremal problems for a and S. To state our problems more precisely, we set for any $\tau \in \mathfrak{M}(R, I)$

$$C_\tau(R, \mathfrak{X}) = \{[R', \mathrm{r}, i'] \in C(R, \mathfrak{X}) \mid \tau[R', \mathfrak{X}', i'] = \tau\} .$$

In other words, $C_\tau(R,$ 30 stands for the set of all possible conformal embeddings i': $(R, \mathfrak{X}) \to (R', \Gamma)$, where $(R', \mathfrak{X}')$ is the marked torus with modulus τ. We consider the extremal problems of maximizing a and S in $C_\tau(R,$ 30. We note that it makes little sense, because of Theorem A_1 (II), to consider the problem of *minimizing* a in $C(R, \mathfrak{X})$. Actually we can prove that

$$\min \{\alpha[R', \mathfrak{X}', i'] \mid [R', \mathfrak{X}', i'] \in C_\tau(R, \mathfrak{X})\} = 0$$

for each $\tau \in \mathfrak{M}(R, \mathfrak{X})$.

Let

$$\alpha_\tau := \sup \{\alpha[R', \mathfrak{X}', i'] \mid [R', \mathfrak{X}', i'] \in C_\tau(R, \mathfrak{X})\}$$

and

$$S_\tau := \sup \{S[R', \mathfrak{X}', i'] \mid [R', \mathfrak{X}', i'] \in C_\tau(R, \mathfrak{X})\} .$$

Clearly, α_τ and S_τ are functions of $\tau \in \mathfrak{M}(R, \mathfrak{X})$, which vanish on the boundary of $\mathfrak{M}(R, \mathfrak{X})$. We will show that α_τ and S_τ are respectively attained by a unique compact continuation in $C_\tau(R, \mathfrak{X})$. We show furthermore that the function α_τ (resp. S_τ) does not depend on the particular location of τ but depends solely on the euclidean (resp. hyperbolic) distance of the point τ from the euclidean (resp. hyperbolic) center of $\mathfrak{M}(R, T)$. We can also write these functions in closed form, which yields a close relationship among the complementary area of conformal embeddings, the spans of (R, χ), and the geometric structures of the moduli disk $\mathfrak{M}(R, I)$. A classical theorem of Grunsky and its generalizations also follow.

In this and the subsequent two sections we discuss the problem from the euclidean viewpoint. We will also obtain a theorem for planar surfaces (Theorem BO in Section 6), which is a direct, but new in part, generalization of the theorem of Grunsky (see [5] cf. also [6], [10]).

If we observe a similar extremal problem for the function S, we will obtain a new area theorem. This theorem may be called the *absolute area theorem*, since the quantities appearing in the theorem—the area ratio S, the hyperbolic span $\sigma_H(R)$, and the hyperbolic distance—depend nothing other than the surface R. These topics will be studied in Sections 7 and 8.

First, concerning the function α_τ we have

THEOREM B_1. (Area Theorem). (I) *For each* $\tau \in \mathfrak{M}(R, \chi)$ *there is a unique* $[R_\tau, \chi_\tau, i_\tau] \in C_\tau(R,$ 50 *such that* $\alpha[R_\tau, \chi_\tau, i_\tau] = \alpha_\tau$.

(II) α_τ *is a function of a single real variable* $\tau - \tau_E^*|$ *in other words, it is constant on the enclidean concentric circle* $\{\tau \mid |\tau - \tau_E^*| = r_E\}$ *for each ΓE,* $0 \leqq r_E \leqq \rho_E$. *It holds furthermore*

$$\alpha_\tau \quad \frac{\rho_E^2(R, \chi) - r_E^2}{2\rho_E(R, \chi)}.$$

We immediately have the following

COROLLARY B_1. (I) *There is a unique compact continuation in C(R, I) that maximizes* $\alpha[R', \chi'', i']$ *in the whole class* $C(R, \chi)$.

(II) // $[R_E, \chi_E, i_E]$ *is the compact continuation of (R, I) that maximizes a, then the modulus of* $[R_E, \chi_E, i_E]$ *is the euclidean center* τ_E^* *of* $\mathfrak{M}(R, \chi)$.

(III) $\alpha[R_E, \chi_E, i_E] = (1/4)\sigma_E(R, T)$.

(IV) $\sigma_E(R, \chi) \geqq 4\alpha[R', \chi', i']$ *for all* $[R', \chi', i'] \in C(R, \chi)$.

Remark. The expression of α_τ is noteworthy. It shows that the maximum complementary area is essentially the reciprocal of the Poincaré metric of the "hyperbolic space" $\mathfrak{M}(R, I)$.

4. Some lemmas.

For the proof of Theorem B_1 we need several lemmas. To state them, take a boundary point r of $\mathfrak{M}(R, T)$ and let t be a real number for which r— $\tau(t) = \tau_E^* + \rho_E e^{(t-1/2)\pi i}$holds. For convenience' sake we extend the domain of definition from the interval $(-1, 1]$ onto the whole real numbers by the periodicity. The number t is thus determined up to an additive constant $2n$, $n\in\boldsymbol{Z}$; $\tau(t')=\tau(t'')$ if and only if $t'\equiv t''(\mathrm{mod}\,2)$. Let $[R_{\tau(t)}, \chi_{\tau(t)}, i_{\tau(t)}]$ be the compact continuation of *(R,* 30 corresponding to $\tau(t)$. Cf. Theorem A_1 (II). Let $\psi_{\tau(t)}$ be the normal holomorphic differential on $[R_{\tau(t)}, \chi_{\tau(t)}, i_{\tau(t)}]$ and $\phi_{\tau(t)}$ the pullback of $\psi_{\tau(t)}$ by $i_{\tau(t)}: (R, \chi)\to(R_{\tau(t)}, \chi_{\tau(t)})$.

LEMMA 1 ([12] cf. also [13], Lemma 4 and Theorem 1′). *For any boundary point* $\tau(t)=\tau_E^*+\rho_E e^{(t-1/2)\pi i}$ *of* $\mathfrak{M}(R,$ 30 *the harmonic differential* $\mathrm{Im}[e^{-\pi i t/2}\phi_{\tau(t)}]$ *is distinguished in the sense of Ahlfors* [1].

Remark. Some other important characterizations of the differentials ϕ_t can be found in [12] and [13].

On the other hand, for the interior points of $\mathfrak{M}(R,$ 30, we have

LEMMA 2. *Let* τ *be an interior point of* $\mathfrak{M}(R,$ 30. *Then*

(i) *there exist diametrically opposite points* $\tau(t)$ *and* $\tau(t+1)$ *on the circle* $\partial\mathfrak{M}(R, \chi)$ *and a real number* ξ *with* $0<\xi\leqq 1/2$ *such that*

$$\tau=\xi\tau(t)+(1-\xi)\tau(t+1);$$

(ii) *there exists a compact continuation* $[R_\tau, \chi_\tau, i_\tau]\in C_\tau(R, \chi)$ *such that*

$$\phi_\tau:=\xi\phi_{\tau(t)}+(1-\xi)\phi_{\tau(t+1)}$$

is the pullback of the normal holomorphic differential on $[R_\tau, \chi_\tau, i_\tau]$

Proof. The second assertion was proved in Section 6 of [13], while the first is obvious by elementary geometrical considerations. □

The following lemma shows how to compute the total area $A[R', \chi', i']$ of $[R', T, i']\in C(R, T)$ and the complementary area $\alpha[R', \chi', i']$ of the embedded surface $i'(R)$ in the torus (R', χ'). It plays a similar role in the present work as the classical area principle (cf. [3], [9]) does in the theory of univalent functions. See Lemma 3_0 in Section 5, too. The proof is not difficult, and is hence omitted.

LEMMA 3_1. *Let* $[R', \Gamma, i']\in C(R,$ 30 *with* $\chi'=\{a', b'\}$. *Let* ψ' *be the normal holomorphic differential on* (R', χ') *and* ϕ' *its pullback to (R, T) by* $i': (R, \chi)\to(R', \chi')$. *Then the following identities hold.*

$$A[R', \chi', i']=\mathrm{Im}\,\tau[R', \Gamma, i']=\mathrm{Im}\int_{b'}\psi'=\mathrm{Im}\int_b\phi'.$$

$$\alpha[R', \chi', i']=\mathrm{Im}\int_{b'}\psi'-\frac{1}{2}\|\psi'\|_{i'(R)}^2=\mathrm{Im}\int_b\phi'-\frac{1}{2}\|\phi'\|_R^2.$$

LEMMA 4_1 (cf. Lemma 5 in [13]). *Let ω_1, ω_2 be square integrable holomorphic differentials on R. Suppose furthermore that the a-period of ω_1 vanishes and* $\mathrm{Im}[e^{-\pi it/2}\omega_2]$ *is distinguished. Then*

$$(\omega_1, \omega_2)_R = -2\int_b e^{-\pi it/2}\omega_1 \int_a \mathrm{Im}[e^{-\pi it/2}\omega_2].$$

LEMMA 5. *For any $t \in \boldsymbol{R}$*

$$\phi_{\tau(t)} = \frac{1}{2}(\phi_{\tau(0)} + \phi_{\tau(1)}) - \frac{1}{2}e^{\pi it}(\phi_{\tau(1)} - \phi_{\tau(0)})$$

holds.

Proof. We set $\omega = (\phi_{\tau(0)} + \phi_{\tau(1)})/2 - e^{\pi it}(\phi_{\tau(1)} - \phi_{\tau(0)})/2$. The period of ω along the cycle a is obviously 1. Furthermore,

$$\mathrm{Im}[e^{-\pi it/2}\omega] = \mathrm{Im}\left[\frac{e^{\pi it/2} + e^{-\pi it/2}}{2}\phi_{\tau(0)} - \frac{e^{\pi it/2} - e^{-\pi it/2}}{2}\phi_{\tau(1)}\right]$$

$$= \cos\frac{\pi}{2}t \ \mathrm{Im}[\phi_{\tau(0)}] - \sin\frac{\pi}{2}t \cdot \mathrm{Im}[i\phi_{\tau(1)}].$$

Since $\mathrm{Im}[\phi_{\tau(0)}]$ and $\mathrm{Im}[i\phi_{\tau(1)}]$ are distinguished harmonic differentials by Lemma 1, so is the differential $\mathrm{Im}[e^{-\pi it/2}\omega]$. Hence, by a uniqueness theorem (cf. Lemma 4 in [13]) we know that $\omega = \phi_{\tau(t)}$, which is to be proved. □

5. Proof of the Area Theorem.

We first show (I). Since $C_\tau(R, X)$ consists of a single element if $\tau \in \partial\mathfrak{M}(R, 30$, it suffices to assume that τ is an interior point of $\mathfrak{M}(R, T)$. The existence and the uniqueness of a compact continuation of (R, X) which maximizes a in the class $C_\tau(R, X)$ follow immediately. In fact, let $t, \xi \in \boldsymbol{R}$, $[R_\tau, \chi_\tau, i_\tau] \in C_\tau(R, X)$, and $\phi_\tau = \xi\ \phi_{\tau(t)} + (1-\xi)\ \phi_{\tau(1+t)}$ be as in Lemma 2. For any $[R', \chi', i'] \in C_\tau(R, \chi)$, let φ' be the pullback of ϕ', the normal holomorphic differential on $[R', \chi', i']$, to (R, X) via $i' : (R, \chi) \to (R', \chi')$. By a well known general property of the Dirichlet norm we have then

$$0 \leqq \|\phi' - \phi_\tau\|_R^2 = \|\phi'\|_R^2 - \|\phi_\tau\|_R^2 - 2\mathrm{Re}(\phi' - \phi_\tau, \phi_\tau)_R = \|\phi'\|_R^2 - \|\phi_\tau\|_R^2,$$

since by Lemmas 1, 2 and 4

$$\begin{aligned}\mathrm{Re}(\phi' - \phi_\tau, \phi_\tau)_R &= \xi \cdot \mathrm{Re}(\phi' - \phi_\tau, \phi_{\tau(t)})_R + (1-\xi) \cdot \mathrm{Re}(\phi' - \phi_\tau, \phi_{\tau(1+t)})_R \\ &= -2\xi \cdot \mathrm{Re}\left[e^{-\pi it/2}\int_b (\phi' - \phi_\tau)\right] \cdot \mathrm{Im}\left[e^{-\pi it/2}\int_a \phi_{\tau(t)}\right] \\ &\quad + 2(1-\xi) \cdot \mathrm{Im}\left[e^{-\pi it/2}\int_b (\phi' - \phi_\tau)\right] \cdot \mathrm{Re}\left[e^{-\pi it/2}\int_a \phi_{\tau(t+1)}\right] \\ &= 0.\end{aligned}$$

Lemma 3 now yields

$$\alpha[R_\tau, \chi_\tau, i_\tau] \geqq \alpha[R', \chi', i'] \quad \text{for all} \quad [R', \chi', i'] \in C_\tau(R, \chi),$$

so that $\alpha | C_\tau(R, I)$ attains its maximum for the compact continuation $[R_\tau, \chi_\tau, i_\tau]$ Moreover, the equality holds if and only if $\phi' = \phi_\tau$ on R, and this implies the uniqueness of a compact continuation $[R', \chi', i']$ that maximizes α; if $\phi' = \phi_\tau$ on R, there exists a conformal mapping / of (R', χ') onto (R_τ, χ_τ) with $f \circ i' = i_\tau$.

To prove (II), we use Lemma 5 and rewrite ϕ_τ as

$$\phi_\tau = \left(\frac{1}{2} - \frac{1-2\xi}{2} e^{\pi i t}\right) \phi_{\tau(0)} + \left(\frac{1}{2} + \frac{1-2\xi}{2} e^{\pi i t}\right) \phi_{\tau(1)}.$$

We use Lemma 4 again to obtain

$$\begin{aligned} \|\phi_\tau\|_R^2 &= \{(2\xi^2 - 2\xi + 1) + (1 - 2\xi) \cos \pi t\} A_1 - \{(2\xi^2 - 2\xi - 1) + (1 - 2\xi) \cos \pi t\} A_0 \\ &= (A_1 + A_0) + 2\left(\xi^2 - \xi + \frac{1-2\xi}{2} \cos \pi t\right)(A_1 - A_0), \end{aligned}$$

where we set $A_0 = A[R_{\tau(0)}, \chi_{\tau(0)}, i_{\tau(0)}]$ and $A_1 = A[R_{\tau(1)}, \chi_{\tau(1)}, i_{\tau(1)}]$ for simplicity. Since the a- and b-periods of ϕ_τ are respectively 1 and

$$\tau = \left(\frac{1}{2} - \frac{1-2\xi}{2} e^{\pi i t}\right) \tau(0) + \left(\frac{1}{2} + \frac{1-2\xi}{2} e^{\pi i t}\right) \tau(1)$$

and $\operatorname{Re} \tau(0) = \operatorname{Re} \tau(1)$, the total surface area $A[R_\tau, \chi_\tau, i_\tau]$ of $[R_\tau, \chi_\tau, i_\tau]$ is equal to

$$\begin{aligned} \operatorname{Im} \tau &= \left(\frac{1}{2} - \frac{1-2\xi}{2} \cos \pi t\right) A_0 + \left(\frac{1}{2} + \frac{1-2\xi}{2} \cos \pi t\right) A_1 \\ &= \frac{1}{2}(A_0 + A_1) + \frac{1-2\xi}{2} \cos \pi t \cdot (A_1 - A_0). \end{aligned}$$

Hence, by Lemma 3, we have

$$\alpha[R_\tau, \chi_\tau, i_\tau] = \operatorname{Im} \tau - \frac{1}{2} \|\phi_\tau\|_R^2 = (\xi - \xi^2)(A_1 - A_0).$$

Setting

$$r_E = |\tau - \tau_E^*|,$$

we have

$$\xi = \frac{\rho_E(R, \chi) - r_E}{2\rho_E(R, \chi)}.$$

Since $A_1 - A_0 = 2\rho_E(R, \chi)$, we finally have

$$\alpha[R_\tau, \chi_\tau, i_\tau] = \frac{\rho_E^2(R, \chi) - r_E^2}{2\rho_E(R, \chi)} = \frac{\sigma_E^2(R, \chi) - 4r_E^2}{4\sigma_E(R, \chi)},$$

and the theorem is proved. □

Remark. It seems interesting to prove Theorem B_1 (II) more directly—*i. e.* without employing the differentials ϕ_τ. We would then have a new insight into the Poincaré metric.

6. Planar Riemann surfaces.

Now we look back to the case of planar surfaces. As in Section 1, suppose that G is a plane domain and ζ is a fixed point of G. (The necessary modifications for general planar surfaces are trivial.) As before, we also assume that $\zeta\neq\infty$. If this is the case, each $f\in F(G, \zeta)$ embeds the domain G into the Riemann sphere $\hat{C}$ and the image domain G' has an infinite euclidean area, so that it makes no sense to consider the euclidean area function which corresponds to A. However, the complementary area of $G'=f(G)$—*i. e.*, the *outer* area of $\hat{C}\backslash G'$—is always finite, which we denote by $\delta[f]=\delta[\hat{C},\infty, /]$. We consider, for any fixed $\kappa\in\mathfrak{K}(G, \zeta)$, the class

$$F_\kappa(G, \zeta):=\left\{f\in F(G, \zeta)\mid f(z)=\frac{1}{z-\zeta}+\kappa(z-\zeta)+ \quad \text{about } \zeta\right\}$$

and

$$\delta_\kappa:=\sup\{\delta[f]\mid[\hat{C}, \infty, f]\in F_\kappa(G, \zeta)\} .$$

THEOREM B_0. (I) *For any $\kappa\in\mathfrak{K}(G, \zeta)$ there exists a unique element f_κ in $F_\kappa(G, \zeta)$ which maximizes $\delta[f]$ in the class $F_\kappa(G, \zeta)$.*

(II) *The maximum δ_κ is a function of a single variable $|\kappa-\kappa_E^*$. That is, it ιs constant on each concentric circle*

$$\{\kappa\in C\mid|\kappa-\kappa_E^*|=r_E\}$$

and is equal to

$$\pi\frac{\rho_E^2-r_E^2}{pE},$$

where $0\leqq r_E\leqq\rho_E=\sigma_E(G,0/2$.

We can immediately deduce the following classical theorem of Grunsky:

COROLLARY B_0 ([5], [6], [8]). (I) *There exists a unique element* $[\hat{C}$, oo, $f_E]$ *which maximizes* $\delta[f]$ *in* $F(G, \zeta)$.

(II) *The Laurent expansion of the extremal function f_E about ζ is of the form*

$$\frac{1}{z-\zeta}+\kappa_E^*(z-\zeta)+\cdots .$$

(III) $\delta[f_E]=(\pi/2)\sigma_E(G, \zeta)$.

(IV) $\sigma_E(G,\zeta)\geqq(2/\pi)\delta[f]$ *for all* $f\in F(G, \zeta)$.

Part (I) of Theorem B_0 is already known, provided that κ lies on the horizontal diameter of $\mathfrak{K}(G, \zeta)$. See [8], p. 367. Part (II) is, on the contrary, new. We will later give a simple application of (II). The proof of Theorem B_0 is quite similar to that of Theorem B_1. Indeed, the prototypes of Lemmas 1, 2 and 5 can be found in any standard books (see [6], for example), and the following well known lemma will substitute for Lemma 3_1.

LEMMA 3_0 (cf., e.g., [5]). *For any* $f \in F(G, \zeta)$ *the identity*

$$\delta[f]=\frac{1}{2i}\int_{\partial G} f\, d\bar{f}$$

holds. (The right hand side is of course defined as the limit of integrals over ∂G_n*, where* $\{G_n\}_{n=1}^{\infty}$ *is an exhaustion of* G *by subdomains with regularly embedded boundary.)*

Furthermore, Lemma 4_1 should be replaced with the following lemma, whose proof is purely computational.

LEMMA 4_0. *Let* $f(z)=1/(z-\zeta)+a_1(z-\zeta)+a_2(z-\zeta)^2+\cdots\in F(G, \zeta)$*, and suppose that* $\mathrm{Im}[e^{-\pi i t/2} df]$ *is distinguished. Let* $g(z)=b_0+b_1(z-\zeta)+b_2(z-\zeta)^2+\cdots$ *be a holomorphic function on* G *with a finite Dirichlet integral. Let* $\varepsilon>0$ *be so small that* $D_\varepsilon:=\{z\in \boldsymbol{C} \mid |z-\zeta|\leqq\varepsilon\}\subset G$*, Then*

$$(dg,\ df)_{G\setminus D_\varepsilon}=2\pi e^{-\pi i t} b_1-2i\sum_{m=1}^{\infty} m\overline{a_m} b_m \varepsilon^{2m}\ .$$

Now Theorem B_0 follows at once if we apply Lemmas 1, 2, 3_0, 4_0 and 5_0 to compute the Dirichlet integral $\|df-df_\kappa\|_{G\setminus D_\varepsilon}$ and let $\varepsilon\to 0$, / being a generic element of $F_\kappa(G, \zeta)$. D

Remark. One of the important aspects of Theorem B_0 is, just like that of Theorem B_1 the reciprocity relationship between the maximum complementary area and the Poincaré density (at each point of the coefficients disk ff(G, ζ)). Also, as in the case of tori, it would be interesting to give a more diret proof of Theorem B_0 (II).

7. The hyperbolic span and the Absolute Area Theorem.

Going back to the case of genus one, we now consider another extremal problem. We try to maximize the area ratio S first of all in each set $C_\tau(R, \chi)$, $\tau\in\mathfrak{M}(R, \chi)$, and then in the whole set $C(R, \chi)$. We will obtain a theorem analogous to Theorem B_1 in the present case, however, we use the hyperbolic geometry of the moduli set $\mathfrak{M}(R, \chi)$ instead of the euclidean geometry. There is no counterpart of the theorem in the planar case because of the following two reasons. Firstly the function A is no longer finite on $F(G, \zeta)$, so that the function S has no obvious meaning secondly $\mathfrak{K}(G, \zeta)$ is not a hyperbolic disk, since it never lies in the upper half plane. We can of course regard $\mathfrak{K}(G, \zeta)$ as a disk with respect to the spherical metric on the extended κ-plane. Then we have a spherical version of Theorem B_0. See Theorem D_0 in Section 9.

Now the solution of the extremal problem above is as follows.

THEOREM C_1 (Absolute Area Theorem). (I) *For any fixed* $\tau\in\mathfrak{M}(R, X)$ *there exists a unique element in* $C_\tau(R, T)$ *which maximizes the area ratio* $S[R', \chi', i']$ *in the class.*

(II) *The maximum is constant on each hyperbolic concentric circle*

$$\{\tau \mid d_H(\tau, \tau_H^*)=r_H\}$$

and the constant is equal to

$$\frac{\cosh\rho_H-\cosh r_H}{\sinh\rho_H},$$

where $0\leqq r_H\leqq\rho_H=\sigma_H(R)/2$.

COROLLARY C_1. (I) *There is a unique element in* $C(R, \chi)$ *which maximizes* $S[R', \chi', i']$.

(II) *//* $[R_H, \chi_H, i_H]$ *denotes the compact continuation of (R, T) that maximizes the function S in C(R, T), then* $\tau[R_H, \chi_H, i_H]=\tau_H^*$.

(III) $S[R_H, \chi_H, i_H]=\tanh(\sigma_H(R)/4)$.

(III′) $\alpha[R_H, \chi_H, i_H]=\operatorname{Im}\tau_H^*\ \tanh(\sigma_H(R)/4)$

(IV) $\sigma_H(R)\geqq 4\tanh^{-1} S[R', \chi', i']$ *for all* $[R', \chi', i']\in C(R, X)$.

We call Theorem d the Absolute Area Theorem, since it concerns with the ratio of two areas, $A[R', \chi', i']$ and $\alpha[R', \chi', i']$, which is independent of the particular choice of the canonical homology basis. The other quantities such as $\sigma_H(R)$ and r_H are also independent of χ.

8. Proof of the Absolute Area Theorem and the corollary.

Let $\tau\in\mathfrak{M}(R, I)$ and let $r_H=d_H(\tau, \tau_H^*)$. In virtue of Theorem B_1, we know that the maximum S_τ of $S[R', \chi', i']$ in the class $C_\tau(R, T)$ is attained again by $[R_\tau, \chi_\tau, i_\tau]$, since every $[R', \chi', i']\in C_\tau(R, I)$ has the same area

$$\operatorname{Im}\tau=:A.$$

Hence we have only to compute the ratio α_τ/A to know the value S_τ. For simplicity we set

$$A_j=\operatorname{Im}\tau_j\ (j=0,\ 1),\quad A_E=\operatorname{Im}\tau_E^*,\quad A_H=\operatorname{Im}\tau_H^*$$

and

$$q=\left|\frac{\tau-\tau_H^*}{\tau-\overline{\tau_H^*}}\right|.$$

Obviously

$$A_1>A_E>A_H>A_0, \qquad 0\leqq q<1,$$

and it is well known (cf. [2], p. 130, for example) that

$$q=\tanh\frac{r_H}{2}.$$

We also see

$$A_E^2-A_H^2=\frac{1}{4}(A_1-A_0)^2=\rho_E^2,$$

since $A_E=(A_1+A_0)/2$ and $A_H=\sqrt{A_1A_0}$ On the other hand, setting

$$l=\mathrm{Re}\,(\tau-\tau_E^*),$$

and noting that $\mathrm{Re}\,\tau_E^*=\mathrm{Re}\,\tau_H^*$, we have by simple geometric considerations

$$r_E^2=(A-A_E)^2+l^2$$

$$|\tau-\tau_H^*|^2=(A-A_H)^2+l^2$$

and

$$|\tau-\overline{\tau_H^*}|^2=(A+A_H)^2+l^2,$$

from which we immediately have

$$r_E^2=\frac{1}{1-q^2}[q^2\{(A+A_H)^2-(A-A_E)^2\}-\{(A-A_H)^2-(A-A_E)^2\}].$$

Consequently

$$\rho_E^2-r_E^2=2A\frac{A_E-A_H-q^2(A_E+A_H)}{1-q^2},$$

so that we have by Theorem B_1 (II)

$$\alpha_\tau=\frac{\rho_E^2-r_E^2}{2\rho_E}=A\frac{A_E-A_H-q^2(A_E+A_H)}{(1-q^2)\sqrt{A_E^2-A_H^2}}.$$

This equation shows that $S_\tau=\alpha_\tau/A$ depends only on q, or equivalently, only on the hyperbolic distance r_H

To obtain an expression of S_τ in terms of r_H and ρ_H, we use the following lemma.

LEMMA 6.

$$\frac{A_E-A_H}{A_E+A_H}=\tanh^2\frac{\rho_H}{2}.$$

Proof. Direct computations. □

Now we continue the proof of Theorem C_1. Lemma 6 yields

$$S_\tau=\frac{-q^2+\dfrac{A_E-A_H}{A_E+A_H}}{(1-q^2)\sqrt{\dfrac{A_E+A_H}{A_E+A_H}}}=\frac{-\tanh^2\dfrac{r_H}{2}+\tanh^2\dfrac{\rho_H}{2}}{\left(1-\tanh^2\dfrac{r_H}{2}\right)\cdot\tanh\dfrac{\rho_H}{2}}$$

$$=\frac{\cosh^2\dfrac{\rho_H}{2}-\cosh^2\dfrac{r_H}{2}}{\sinh\dfrac{\rho_H}{2}\cosh\dfrac{\rho_H}{2}}=\frac{\cosh\rho_H-\cosh r_H}{\sinh\rho_H}.$$

Thus we have proved the theorem. Π

The corollary now follows at once. Indeed, the maximum of S in the whole class $C(R, I)$ is obviously attained for $r_H=0$, that is, at the hyperbolic center τ_H^*, and the maximum is equal to

$$\frac{\cosh^2\dfrac{\rho_H}{2}-1}{\sinh\dfrac{\rho_H}{2}\cosh\dfrac{\rho_H}{2}}=\tanh\frac{\rho_H}{2},$$

which proves the corollary. D

9. The spherical span and an extremal problem.

We now discuss some properties of the spherical span. For this purpose, let G and ζ be as before and set

$$\Delta[f]=\frac{\delta[f]}{1+|\kappa_f|^2},$$

where $f(z)=1/(z-\zeta)+\kappa_f(z-\zeta)+\cdots\in F(G,\zeta)$

THEOREM D_0. (I) *For any fixed* $\kappa\in\mathfrak{K}(G,\zeta)$ *there exists a unique element in* $F_\kappa(G,\zeta)$ *which maximizes* $\Delta[f]$ *in this class.*

(II) *The maximum is constant on each spherical concentric circle* $\{\kappa\in\boldsymbol{C}|\,d_S(\kappa,\kappa_S^*)=r_S\}$ *and the constant is equal to*

$$\pi\frac{\tan^2\dfrac{\rho_S}{2}-\tan^2\dfrac{r_S}{2}}{\tan\dfrac{\rho_S}{2}\left(1+\tan^2\dfrac{r_S}{2}\right)}.$$

Assertion (I) is almost trivial. To prove assertion (II), we first note the following lemma, whose proof is straightforward and hence omitted.

LEMMA 7. *Let* $c\in \boldsymbol{C}$ *and* r>0. *Then the eudidean center and the eudidean radius of the spherical circle* $\{z\in \boldsymbol{C}\,|\,d_S(z, c)=r\}$ *are given by*

$$\frac{\left(1+\tan^2\frac{r}{2}\right)c}{1-|c|^2\tan^2\frac{r}{2}} \quad \textit{and} \quad \frac{(1+|c|^2)\tan\frac{r}{2}}{1-|c|^2\tan^2\frac{r}{2}}$$

respectively.

Now, let $\kappa\in\mathfrak{K}(G, \zeta)$. We recall that κ_S^* and ρ_S denote the spherical center and the spherical radius of the disk $\mathfrak{K}(G, \zeta)$ respectively. For simplicity we set

$$q_0=\tan\frac{\rho_S}{2}$$

and

$$q=\tan\frac{d_S(\kappa, \kappa_S^*)}{2}.$$

Then, we have $|\kappa-\kappa_S^*|/|1+\bar{\kappa}\kappa_S^*|=q$. This identity yields

$$2\,\mathrm{Re}[e^{-i\theta}\kappa]=\frac{(1-q^2|\kappa_S^*|^2)|\kappa|^2+|\kappa_S^*|^2-q^2}{(1+q^2)|\kappa_S^*|},$$

where $\theta=\arg\kappa_E^*$. Hence we have

$$\begin{aligned}\delta_\kappa&=\pi\frac{\rho_E^2-|\kappa-\kappa_E^*|^2}{PE}\\&=\pi\frac{\rho_E^2-|\kappa|^2-|\kappa_E^*|^2+2|\kappa_E^*|\cdot\mathrm{Re}[e^{-i\theta}\kappa]}{PE}\\&=\pi\frac{(\rho_E^2-|\kappa|^2-|\kappa_E^*|^2)(1+q^2)|\kappa_S^*|+|\kappa_E^*|\{(1-q^2|\kappa_S^*|^2)|\kappa|^2+|\kappa_S^*|^2-q^2\}}{\rho_E(1+q^2)|\kappa_S^*|}\end{aligned}$$

Applying Lemma 7 to the circle $\partial\mathfrak{K}(G, \zeta)$, we now have

$$\frac{\delta_\kappa}{1+|\kappa|^2}=\pi\frac{q_0^2-q^2}{q_0(1+q^2)},$$

from which assertion (II) follows immediately. ∏

As a counterpart to the classical theorem of Grunsky and its refinement (see Corollary B_0) we have now the following

COROLLARY D_0. (I) *There exists a unique element* $[\hat{\boldsymbol{C}}, \infty, f_S]$ *which max-*

imizes $\Delta[f]$ *in* $F(G,\zeta)$.

(II) *The Laurent expansion of the extremal function* f_S *about* ζ *is of the form*

$$\frac{1}{z-\zeta}+\kappa_S^*(z-\zeta)+\cdots .$$

(III) $\Delta[f_S]=\pi\tan(\sigma_S(G,\zeta)/4)$.

(III′) $\delta[f_S]=\pi(1+|\kappa_S^*|^2)\tan(\sigma_S(G,\zeta)/4)$.

(IV) $\sigma_S(G,\zeta)\geqq 4\tan^{-1}(\Delta[f]/\pi)$ *for all* $f\in F(G,\zeta)$.

It is now apparent that similar results for $\alpha[R',\chi',i']/\{1+|\tau[R',\chi',i']|^2\}$, $[R',\chi',i']\in\mathfrak{M}(R,$ 30, can be obtained by the same argument.

10. Applications of the Area Theorems.

We will give three theorems which readily follow from Theorems B_0 and B_1. The first application is the following classical theorem (see £6], pp. 83-84; [17], pp. 760-762). Jenkins used the method of extremal length to prove the theorem, while Tsuji used the Rengel inequality for the same purpose.

THEOREM E_0. *Let* G_∞ *be an extremal horizontal slit domain and suppose that* f *maps* G_∞ *conformally onto a domain* G' *containing* oo *whose* (*euclidean*) *complementary area is* δ. *If* $f(z)=z+\kappa/z+\cdots$ *about* oo, *then*

$$\operatorname{Re}\kappa\leqq-\frac{\delta}{2\pi}.$$

More generally we have

THEOREM E_0'. *For any* $f\in F_\kappa(G,\zeta)$

$$\operatorname{Re}\kappa(1)+\frac{1}{2\pi}\delta[f]\leqq\operatorname{Re}\kappa\leqq\operatorname{Re}\kappa(0)-\frac{1}{2\pi}\delta[f].$$

Proof. It follows immediately from Theorem B_0 that

$$\delta[f]\leqq\pi\frac{(\rho_E-|\kappa-\kappa_E^*|)(\rho_E+|\kappa-\kappa_E^*|)}{PE}$$
$$\leqq 2\pi(\rho_E-|\kappa-\kappa_E^*|).$$

Since $\rho_E=\operatorname{Re}(\kappa(0)-\kappa_E^*)$ and $|\kappa-\kappa_E^*|\geqq\operatorname{Re}(\kappa-\kappa_E^*)$, we have

$$\delta[f]\leqq 2\pi\operatorname{Re}[\kappa(0)-\kappa].$$

The other inequality is similarly proved. D

As a counterpart of Theorem E_0 we have

THEOREM E_1. *Let* $\tau\in\mathfrak{M}(R, I)$ *and* $[R', \chi', i'] \in C_\tau(R, \chi)$. *Then the complementary area* $\alpha[R'', \chi', i']$ *satisfies the inequality*

$$\operatorname{Im}\tau(0)+\alpha[R',\chi', i']\leqq\operatorname{Im}\tau\leqq\operatorname{Im}\tau(1)-\alpha[R',\chi', i'].$$

Theorem E_0 implies the estimate of the Schiffer span which has been already proved. See assertion (IV) in Corollary B_0. Similarly assertion (IV) of Corollary B_1 immediately follows from Theorem E_1 above.

We finally remark an almost direct but curious consequence of the Area Theorem (Theorem BO. Note that there is no counterpart of this theorem for plane domains.

THEOREM F_1. *Let* $[R_E, \chi_E, i_E](=[R_{\tau_E^*}, \chi_{\tau_E^*}, i_{\tau_E^*}])$ *be the compact continuation of* (R, χ) *that maximizes* $\alpha[R', \chi', i']$ *in the class* $C(R, \chi)$. *Then the complementary area is less than the image area*

$$S[R_E, \chi_E, i_E]<\frac{1}{2}.$$

In other words· Let T be a torus, X a closed subset of T such that $T\backslash X$ *is connected and of genus one. Suppose furthermore that the area of* $T\backslash X$ *does not exceed the area of X. Then there exist another torus T' and a closed subset X' of T' such that* $T\backslash X$ *is conformally equivalent to* $T'\backslash X'$ *and that the area of X' is greater than that of X.*

Proof. Since $A[R_E, \chi_E, i_E]=\operatorname{Im}\tau_E^*>\rho_E=2\alpha[R_E, \chi_E, i_E]$ by Corollary B_1, we have the assertion. D

11. The Strebel continuation.

Let now R be the interior of a compact bordered Riemann surface of genus one. We also refer to R, extending the conventional usage, as a *finite open torus.* In [16] Strebel proved that there exists a conformal mapping i_S of R into a torus R_S such that each component of $R_S\backslash i_S(R)$ is either a disk or a point. The torus R_S is essentially unique. Althogh his main interest was to give a normal form of a finite open torus, we can regard his result as a theorem in the framework of continuation problems: If we attach suitable canonical homology bases χ and χ_S to R and R_S respectively, we have a compact continuation $[R_S, \chi_S, is]$ of (R, χ). We call $[R_S, \chi_S, i_S]$ the *Strebel continuation* of *(R, T).* It is interesting to specify the Strebel continuation in the set $C(R, I)$. For example, we ask where the modulus of (R_S, χ_S) is in the moduli disk $\mathfrak{M}(R, X)$. Although we suspect that $\tau[R_S, \chi_S, i_S]=\tau_H^*$, we content ourselves at present by the following claim:

The modulus of the Strebel continuation of a finite open torus cannot be characterized as the euclidean center of the moduli disk.

To see this we have only to observe a torus with a large disk removed and to apply the last theorem.

REFERENCES

[1] L. V. AHLFORS AND L. SARIO, *Riemann surface,* Princeton Univ. Press, Princeton, Princeton, 1960, 382 pp.

[2] A. F. BEARDON, *The geometry of discrete groups,* Springer-Verlag, New York-Heidelberg-Berlin, 1983, 337 pp.

[3] G. M. GOLUZIN, *Geometric theory of functions of a complex variable* (translated from Russian), Amer. Math. Soc., Providence, 1969, 676 pp.

[4] H. GRÖTZSCH, *Die Werte des Doppelverhältnisses bei schlichter konformer Abbildung,* Sitzungsber. Preuss. Akad. Wiss. Berlin (1933), 501-515.

[5] H. GRUNSKY, *Lectures on theory of functions in multiply connected domains,* Vandenhoeck & Ruprecht, Göttingen, 1978, 253 pp.

[6] J. A. JENKINS, *Univalent functions and conformal mapping,* Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958, 169 pp.

[7] J. A. JENKINS, *On some span theorems,* Ill. J. Math. *1* (1963), 104-117.

[8] Z. NEHARI, *Conformal mapping,* McGraw-Hill, New York-Toronto-London, 1952, 396 pp. (Reprint: Dover, 1975.)

[9] CHR. POMMERENKE, *Univalentfunctions,* Vandenhoeck & Ruprecht, Göttingen-Zürich, 1975, 376 pp.

[10] B. RODIN AND L. SARIO, *Principal functions* (with an Appendix by M. NAKAI), Van Nostrand, Princeton, 1968, 347 pp.

[11] M. SCHIFFER, *The span of multiply connected domains,* Duke Math. J. 10 (1943), 209-216.

[12] M. SHIBA, *The Riemann-Hurwitz relation, parallel slit covering map, and continuation of an open Riemann surface of finite genus,* Hiroshima Math. J. 14 (1984), 371-399.

[13] M. SHIBA, *The moduli of compact continuations of an open Riemann surface of genus one,* Trans. Amer. Math. Soc. 301 (1987), 299-311.

[14] M. SHIBA, *The period matrices of compact continuations of an open Riemann surface of finite genus,* in "Holomorphic Functions and Moduli", Vol. 1, edited by D. DRASIN et al , Springer-Verlag, New York-Berlin-Heidelberg-London-Paris-Tokyo, 1988, pp. 237-246.

[15] M. SHIBA, *Conformal embeddings of an open Riemann surface into closed surfaces of the same genus,* in "Analytic Function Theory of One Complex Variable", edited by Y. KOMATU et al., Longman Scientific & Technical, Essex, 1989, pp. 287-298.

[16] K. STREBEL, *Ein Klassifizierungsproblem für Riemannsche Flächen vom Geschlecht 1,* Arch. Math. 48 (1987), 77-81.

[17] M. TSUJI, *Theory of conformal mapping of a multiply connected domain I,* Jap. J. Math. 18 (1943), 759-775.

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