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NONLINEAR EIGENVALUE PROBLEM AND SINGULAR VARIATION OF DOMAINS

Dedicated to Professor Hiroki Tanabe on his 60 th birthday

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1. Introduction.

Recently we have huge amount of research papers concerning semi-linear elliptic boundary value problems. See, for example Berestychi-Lions-Peletier [3], Dancer [4], Lin [5], Ni-Serrin [6], Rabinowitz [14], Wang [15] and the literatures cited there.

In this paper we want to discuss the following quantitative result for non-linear eigenvalue problem with the Robin condition.

Let $\Omega\subset R^2$ be a bounded domain with smooth boundary $\partial\Omega$. Let w be a fixed point in Ω. Let $B(\varepsilon;w)$ denote the ball of center w with radius ε. We remove $B(\varepsilon;w)$ from Ω and we get $\Omega_\varepsilon=\Omega\setminus\overline{B(\varepsilon;w)}$. We write $B(\varepsilon;w)$ as B_ε.

Fix $p\in(1,\infty)$. We fix $k>0$. We put

$$\lambda(\varepsilon)=\inf_{u\in X}\left(\int_{\Omega_\varepsilon}|\nabla u|^2dx+k\int_{\partial B_\varepsilon}u^2d\sigma_x\right), \tag{1.1$_\varepsilon$}$$

where $X=\{u\in H^1(\Omega_\varepsilon),\ u=0$ on $\partial\Omega$ and $u\geqq 0$ in $\Omega_\varepsilon,\ \|u\|_{L^{p+1}(\Omega_\varepsilon)}=1\}$. We see that there exists at least one solution v_ε of the above problem which attains $(1.1)_\varepsilon$.

We see that v_ε satisfies

$$\begin{aligned}
&-\Delta v_\varepsilon(x)=\lambda(\varepsilon)v_\varepsilon(x)^p && x\in\Omega_\varepsilon\\
&v_\varepsilon(x)=0 && x\in\partial\Omega\\
&kv_\varepsilon(x)+\frac{\partial}{\partial\nu_x}v_\varepsilon(x)=0 && x\in\partial B(\varepsilon;w).
\end{aligned}$$

Here $\partial/\partial\nu_x$ denotes the derivative along the exterior normal vector with respect to Ω_ε.

We write

$$\lambda=\inf_{u\in Y}\int_\Omega|\nabla u|^2dx, \tag{1.2}$$

where $Y=\{u;\ u\in H_0^1(\Omega),\ u\geqq 0$ in $\Omega,\ \|u\|_{L^{p+1}(\Omega)}=1\}$. There exists at least one

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positive solution v which attains (1.2). It satisfies $-\Delta v=\lambda v^p$ in Ω, $v=0$ on $\partial\Omega$.

Main result of this paper is the following.

THEOREM. *Assume that the positive function* $\boldsymbol{u}_\varepsilon$ *which attains* $(1.1)_\varepsilon$ *is unique for* $0<\varepsilon\ll 1$. *Assume also that the positive function* $\boldsymbol{u}$ *which attains* (1.2) *is unique. Assume that* $Ker(\Delta+p\lambda(\varepsilon)\boldsymbol{u}_\varepsilon^{p-1})=\{0\}$ *for* $0<\varepsilon\ll 1$. *Assume that* $\|\boldsymbol{u}_\varepsilon^p-\boldsymbol{u}^p\|_{L^{\tilde{q}}(\Omega_\varepsilon)}\to 0$ *for some* $\tilde{q}>1$ *and* $\sup_\varepsilon \varepsilon^2\|\boldsymbol{u}_\varepsilon^p\|_{L^q(\Omega_\varepsilon)}$ *is finite for fixed large* q. *Under these assumptions, we have*

$$\lambda(\varepsilon)-\lambda=2\pi k\varepsilon\boldsymbol{u}(w)^2+o(\varepsilon)\,. \tag{1.3}$$

Remark. Here the operator $\Delta+p\lambda(\varepsilon)\boldsymbol{u}_\varepsilon^{p-1}$ means the operator associated with the boundary condition with respect to $(1.1)_\varepsilon$. The inequality $\lambda(\varepsilon)\leqq\lambda+O(\varepsilon)$ is easy to show. Let $\chi_\varepsilon(x)$ be the characteristic function of Ω_ε. Then we put $F_\varepsilon(x)=\chi_\varepsilon(x)\boldsymbol{u}(x)$. Then,

$$\begin{aligned}\lambda(\varepsilon)&\leqq\Big(\int_{\Omega_\varepsilon}|\nabla F_\varepsilon|^2dx+\int_{\partial\Omega_\varepsilon}kF_\varepsilon(x)^2d\sigma_x\Big)\Big/\Big(\int_{\Omega_\varepsilon}F_\varepsilon(x)^{p+1}dx\Big)^{2/(p+1)}\\&=\lambda+O(\varepsilon)\,.\end{aligned}$$

There are many papers concerning eigenvalues of the Laplacian under singular variation of domains. See Ozawa [8], [9], [10], [11], Besson [2] and the literature cited there.

Our proof of Theorem is given by a systematic use of the Hadamard variational formula developed by Osawa [7] and the techniques in Ozawa [10]. The authors think that the techniques developed in this paper have wide class applications to investigation of semi-linear boundary value problems.

We quote the following theorem from Osawa [7]. It should be remarked that more general theorem is treated in [7].

THEOREM ([7]). *Fix* ε. *Assume that positive minimizer* u_ε *associated with* $(1.1)_\varepsilon$ *is unique and* $Ker(\Delta+p\lambda(\varepsilon)u_\varepsilon^{p-1})=0$. *Then,*

$$\frac{\partial}{\partial\varepsilon}\lambda(\varepsilon)=-\int_{\partial B_\varepsilon}(|\tilde{\nabla}u_\varepsilon|^2-(2\lambda(\varepsilon)/(p+1))u_\varepsilon^{p+1}-(k^2-kH_1)u_\varepsilon^2)d\sigma_x\,, \tag{1.4}$$

where $H_1\equiv-\varepsilon^{-1}$ *is the first mean curvature of the boundary point with respect to the interior normal vector at* x. *Here* $\tilde{\nabla}$ *denotes the gradient operator on the tangent line.*

Thus,

$$\lambda(\varepsilon)-\lambda=\int_0^\varepsilon(I_1(t)+I_2(t)+I_3(t)+I_4(t))dt\,,$$

where

$$I_1(t)=-\int_{\partial B_t}|\tilde{\nabla}u_t|^2d\sigma_x\,,\quad I_2(t)=2\int_{\partial B_t}(\lambda(t)/(p+1))u_t^{p+1}d\sigma_x\,,$$

$$I_3(t)=k^2\int_{\partial B_t} u_t^2 d\sigma_x\,, \quad I_4(t)=+k\int_{\partial B_t} t^{-1}u_t^2 d\sigma_x\,.$$

2. Preliminary Lemma.

LEMMA 2.1. *Fix $L\in C^\infty(\partial B_\varepsilon)$. Let u be the solution of*

$$\begin{aligned}
&\Delta u(x)=0 && x\in\Omega\setminus\overline{B(\varepsilon\,;w)}\\
&u(x)=0 && x\in\partial\Omega\\
&ku(x)+\frac{\partial}{\partial\nu_x}u(x)=L(\theta) && x=w+\varepsilon(\cos\theta,\ \sin\theta).
\end{aligned}\tag{2.1}$$

Then, $u(x)$ satisfies

$$\begin{aligned}
&|u(x)|\leqq C\varepsilon\,\mathrm{Max}_{\theta}|L(\theta)|\\
&|\mathrm{grad}\,u(x)|\leqq C(\|L\|_{L^2(S^1)}+\|L\|_{C^{(3/4)+\sigma}})^{2(1+\alpha)/3}\|L\|_{L^\infty(S^1)}^{(1-2\alpha)/3}+C\|L\|_{L^\infty(S^1)}
\end{aligned}\tag{2.2}$$

for any $\alpha\in(0,\ 1/2)$, $\sigma>0$.

Proof. We put

$$f(x)=a_0\log r+\sum_{j=1}^{\infty}(b_j\cos j\theta+c_j\sin j\theta)(-j)^{-1}r^{-j}\,.$$

Then, it satisfies $\Delta f(x)=0$ $x\in R^2\setminus\bar{B}_\varepsilon$. We expand $L(\theta)$ in a Fourier series

$$L(\theta)=s_0+\sum_{j=1}^{\infty}(s_j\cos j\theta+t_j\sin j\theta)\,.$$

Therefore,

$$\begin{aligned}
&a_0=k^{-1}s_0(\log\varepsilon-k\varepsilon^{-1})^{-1}\\
&b_j=k^{-1}s_j((-j)^{-1}-k^{-1}\varepsilon^{-1})^{-1}\varepsilon^j\\
&c_j=k^{-1}t_j((-j)^{-1}-k^{-1}\varepsilon^{-1})^{-1}\varepsilon^j\,.
\end{aligned}$$

We see that

$$|f(x)_{|\partial\Omega}|\leqq C\varepsilon$$

observing

$$\sum_j(s_j^2+t_j^2)\leqq C\max_\theta L(\theta)^2$$

Then, we solve $\Delta v(x)=0$, $x\in\Omega$ and $v(x)=f(x)$ for $x\in\partial\Omega$. And we put

$$L^{(2)}(\theta)=v(x)_{|x=w+\varepsilon(\cos\theta,\sin\theta)}\,.$$

We solve

$$\begin{aligned}
&\Delta f^{(2)}(x)=0 \qquad x\in R^2\setminus\bar{B}_\varepsilon\\
&kf^{(2)}(x)+\frac{\partial}{\partial\nu_x}f^{(2)}(x)=L^{(2)}(\theta)\,.
\end{aligned}$$

We continue this procedure, then we get $u(x)=f(x)-v(x)+f^{(2)}(x)\cdots$ satisfies (2.1). Observing this step, we get

$$
\begin{aligned}
(2.3)\qquad |u(x)| &\leqq C\Big(|s_0|\varepsilon+\sum_{j=1}^{\infty}(|s_j|+|t_j|)(-j)^{-1}\varepsilon^{j+1}r^{-j}\Big) \\
&\leqq C\varepsilon\Big(|s_0|+\Big(\sum_{j=1}^{\infty}(s_j^2+t_j^2)\Big)^{1/2}\Big(\sum_{j=1}^{\infty}j^{-2}\varepsilon^{2j}r^{-2j}\Big)^{1/2}\Big).
\end{aligned}
$$

We use

$$2\pi s_0^2+\pi\sum_{j=1}^{\infty}(s_j^2+t_j^2)=\int_0^{2\pi}L(\theta)^2d\theta\leqq 2\pi\max_{\theta}L(\theta)^2.$$

Therefore, we get the first part of (2.2). By the above construction of u we see that

$$
\begin{aligned}
|\mathrm{grad}\,u(x)| &\leqq C\Big(\sum_{j=1}^{\infty}|s_j|+|t_j|)(r^{-(j+1)}\varepsilon^{j+1})+C|s_0|\varepsilon r^{-1}\Big) \\
&\leqq C\Big(\Big(\sum_{j=1}^{\infty}(j^{1+\alpha}(s_j^2+t_j^2)\Big)^{1/2}\Big(\sum_{j=1}^{\infty}j^{-(1+\alpha)}(\varepsilon/r)^{2(j+1)}\Big)^{1/2}
\end{aligned}
$$

for $\alpha>0$.

We have the inequality

$$\Big(\sum_{j=1}^{\infty}j^{1+\alpha}(s_j^2+t_j^2)\Big)^{1/2}\leqq\Big(\sum_{j=1}^{\infty}j^{3/2}(s_j^2+t_j^2)\Big)^{(1+\alpha)/3}\Big(\sum_{j=1}^{\infty}(s_j^2+t_j^2)\Big)^{(1-2\alpha)/6}$$

for $\alpha\in(0, 1/2)$.

We know that $H^{3/4}(S^1)$-norm of h is equivalent to the following norm. See Adams [1].

$$\|h\|_{L^2(S^1)}+\Big(\int_{S^1}\int_{S^1}|h(x)-h(y)|^2|x-y|^{-5/2}dxdy\Big)^{1/2}$$

Thus, we have

$$\|h\|_{H^{3/4}(S^1)}\leqq C(\|h\|_{L^2(S^1)}+\|h\|_{C^{(3/4)+\sigma}(S^1)})$$

for any $\sigma>0$. Summing up these facts we get the second part of (2.2).

3. Approximation of the Geen function.

This section is heavily depend on the previous paper of one of the authors [10]. We introduce the following kernel $p_\varepsilon(x, y)$.

$$(3.1)\qquad p_\varepsilon(x, y)=G(x, y)+g(\varepsilon)G(x, w)G(w, y)+h(\varepsilon)\langle\nabla_w G(x, w), \nabla_w G(w, y)\rangle,$$

where

$$\langle\nabla_w a, \nabla_w b\rangle=\sum_{i=1}^{2}\frac{\partial}{\partial\tilde{w}_i}a(\tilde{w})\frac{\partial}{\partial\tilde{w}_i}b(\tilde{w})|_{\tilde{w}=w}$$

for orthnomal frame (w_1, w_2) of $\boldsymbol{R}^2$ and where

$$g(\varepsilon)=-(\gamma-(2\pi)^{-1}\log\varepsilon+(k\,2\pi)^{-1}\varepsilon^{-1})^{-1} \tag{3.2}$$

and

$$h(\varepsilon)((2\pi\varepsilon)^{-1}+(2\pi)^{-1}k^{-1}\varepsilon^{-2})=k^{-1}. \tag{3.3}$$

Here

$$\gamma=\lim_{x\to w}(G(x, w)+(2\pi)^{-1}\log|x-w|).$$

Let $G_\varepsilon(x, y)$ be the Green function of the Laplacian in Ω_ε associated with the boundary conditions

$$G_\varepsilon(x, y)=0 \qquad x\in\partial\Omega$$

$$kG_\varepsilon(x, y)+\frac{\partial}{\partial\nu_x}G_\varepsilon(x, y)=0 \qquad x\in\partial B_\varepsilon.$$

We put

$$\boldsymbol{G}_\varepsilon f(x)=\int_{\Omega_\varepsilon}G_\varepsilon(x, y)f(y)dy$$

$$\boldsymbol{P}_\varepsilon f(x)=\int_{\Omega_\varepsilon}p_\varepsilon(x, y)f(y)dy.$$

We want to prove the following. We put $\boldsymbol{Q}_\varepsilon f(x)=\boldsymbol{P}_\varepsilon f(x)-\boldsymbol{G}_\varepsilon f(x)$.

There exists a constant C independent of ε such that

$$\max_{x\in\partial B_\varepsilon}\left|k\boldsymbol{Q}_\varepsilon f(x)+\frac{\partial}{\partial\nu_x}\boldsymbol{Q}_\varepsilon f(x)\right|\leqq C\varepsilon\|f\|_{L^q(\Omega_\varepsilon)} \tag{3.4}$$

$$\max_{x\in\partial B_\varepsilon}|\nabla\boldsymbol{Q}_\varepsilon f(x)|\leqq C\varepsilon^{(1-2\alpha)/3}\|f\|_{L^q(\Omega_\varepsilon)} \tag{3.5}$$

for any $\alpha\in(0, 1/2)$, $q>8$.

Proof of (3.4), (3.5). Since $\boldsymbol{G}_\varepsilon f(x)$ satisfies the third boundary condition, then we have only to calculate

$$k\boldsymbol{P}_\varepsilon f(x)+\frac{\partial}{\partial\nu_x}\boldsymbol{P}_\varepsilon f(x) \tag{3.6}$$

on ∂B_ε. First we get

$$\begin{aligned}\boldsymbol{P}_\varepsilon f(x)=&\boldsymbol{G}f(x)+g(\varepsilon)(-(2\pi)^{-1}\log\varepsilon+\gamma+O(\varepsilon))\boldsymbol{G}f(w)\\&+h(\varepsilon)\Big((2\pi\varepsilon)^{-1}\frac{\partial}{\partial w_1}G(w, y)\Big)+h(\varepsilon)\langle\nabla_w S(x, w), \nabla_w G(w, y)\rangle\end{aligned} \tag{3.7}$$

on $x=w+(\varepsilon, 0)$. Here we notice the formulas

$$
\begin{aligned}
(3.8)\qquad & \langle \nabla_w G(x, w), \nabla_w G(w, y)\rangle_{|x=w+(\varepsilon, 0)} \\
&= (2\pi\varepsilon)^{-1}\frac{\partial}{\partial w_1} G(w, y) + \langle \nabla_w S(x, w), \nabla_w G(w, y)\rangle
\end{aligned}
$$

$$
\begin{aligned}
(3.9)\qquad & \frac{\partial}{\partial x_1}\langle \nabla_w G(x, w), \nabla_w G(w, y)\rangle_{|x=w+(\varepsilon, 0)} \\
&= -(2\pi)^{-1}\varepsilon^{-2}\frac{\partial}{\partial w_1} G(w, y) + \frac{\partial}{\partial x_1}\langle \nabla_w S(x, w), \nabla_w G(w, y)\rangle .
\end{aligned}
$$

By using relations (3.2), (3.3), (3.8), (3.9) we get the equation

$$
\begin{aligned}
(3.10)\qquad (3.6) &= k(\boldsymbol{G}f(x) - \boldsymbol{G}f(w) + g(\varepsilon)O(1)\boldsymbol{G}f(w) \\
&\quad - k^{-1}\frac{\partial}{\partial x_1}\boldsymbol{G}f(x) + k^{-1}\frac{\partial}{\partial w_1}\boldsymbol{G}f(w) + h(\varepsilon)\langle \nabla_w S(x, w), \nabla_w \boldsymbol{G}f(w)\rangle \\
&\quad - k^{-1}h(\varepsilon)\frac{\partial}{\partial x_1}\langle \nabla_w S(x, w), \nabla_w \boldsymbol{G}f(w)\rangle .
\end{aligned}
$$

We know that

$$
\begin{aligned}
g(\varepsilon) &= -(2\pi k)\varepsilon + O(\varepsilon^2 |\log \varepsilon|) \\
h(\varepsilon) &= 2\pi\varepsilon^2 + O(\varepsilon^3) .
\end{aligned}
$$

Therefore, we have (3.4).

Next we want to estimate

$$
(3.11)\qquad \|L\|_{L^2(S^1)} + \|L\|_{C^{(3/4)+\sigma}(S^1)} .
$$

We have

$$
\begin{aligned}
(3.12)\qquad (3.11) &\leqq C(\|\boldsymbol{G}f\|_{C^{(7/4)+\sigma}(S^1)} + O(\varepsilon^2)\|\nabla \boldsymbol{G}f\|_{L^\infty(\Omega)}) \\
&\leqq C'\|f\|_{L^q(\Omega_\varepsilon)}
\end{aligned}
$$

for $q>8$. It should be remarked that $\|\boldsymbol{G}f\|_{C^2(S^1)}$ can not be estimated by $C\|f\|_{L^q(\Omega_\varepsilon)}$ for any q. Thus, we used delicate technique of considering $H^{3/4}(S^1)$-norm. Summing up (2.2), (3.4), (3.12) we get (3.5).

4. Proof of Theorem.

First we consider the term $I_l(t)$. We have $u_t = v(t) + \lambda(t)\boldsymbol{P}_t u_t^p$, where

$$
v(t) = \lambda(t)(\boldsymbol{G}_t u_t^p - \boldsymbol{P}_t u_t^p) .
$$

Therefore,

$$
I_1(t) = I_{1,1}(t) + I_{1,2}(t) + I_{1,3}(t) ,
$$

where

$$
I_{1,1}(t) = -\int_{\partial B_t} |\tilde{\nabla} v(t)|^2 d\sigma_x
$$

$$I_{1,2}(t)=-2\lambda(t)\int_{\partial B_t}\tilde{\nabla}v(t)\cdot\tilde{\nabla}\boldsymbol{P}_t u_t^p d\sigma_x$$

$$I_{1,3}(t)=-\lambda(t)^2\int_{\partial B_t}|\tilde{\nabla}\boldsymbol{P}_t u_t^p|^2 d\sigma_x .$$

We want to estimate $I_{1,1}(t)$. We know that

$$\begin{aligned}\tilde{V}(t)&\equiv kv(t)(x)+\frac{\partial}{\partial\nu_x}v(t)(x)_{|x\in\partial B_t}\\&=\lambda(t)\Big(-k\boldsymbol{P}_t u_t^p-\frac{\partial}{\partial\nu_x}\boldsymbol{P}_t u_t^p\Big)_{|x\in\partial B_t}\end{aligned}$$

satisfies

$$\max_\theta|\tilde{V}(t)(\theta)|\leqq Ct\|u_t^p\|_{L^q(\Omega_t)}$$

$$\max_{x\in\partial B_t}|\nabla\tilde{V}(t)(x)|\leqq Ct^{(1-2\alpha)/3}\|u_t^p\|_{L^q(\Omega_t)}$$

by (3.4), (3.5) for large fixed q.

On the other hand $u_t=\lambda(t)(\boldsymbol{G}_t-\boldsymbol{P}_t)u_t^p+\lambda(t)\boldsymbol{P}_t u_t^p$. We see that

$$\begin{aligned}|(\boldsymbol{G}_t-\boldsymbol{P}_t)u_t^p|&\leqq Ct\max_\theta|L(\theta)|\\&\leqq C't^2\|u_t^p\|_{L^q(\Omega_t)}\end{aligned}$$

for large q and we see that

$$|\boldsymbol{P}_t u_t^p|\leqq|\boldsymbol{G}\tilde{u}_t^p|+|g(\varepsilon)||G(x,w)||\boldsymbol{G}\tilde{u}_t^p(w)|+|h(\varepsilon)||\nabla G(x,w)||\nabla\boldsymbol{G}\tilde{u}_t^p(w)| .$$

Here $\tilde{u}_t$ is the extension of u_t which is zero outside Ω_t. Since we have $\|u_t\|_{L^{p+1}(\Omega_t)}=1$, we get $\|\boldsymbol{G}\tilde{u}_t^p\|<C'$. Therefore, $|\boldsymbol{P}_t u_t^p|\leqq C''$ by observing

$$\begin{aligned}|\nabla\boldsymbol{G}\tilde{u}_t^p(w)|&\leqq C\Big(\int_{\Omega_t}|w-y|^{-(p+1)}dy\Big)^{1/(p+1)}\\&\leqq C't^{-(p-1)/(p+1)}\end{aligned}$$

and

$$|h(t)||G(x,w)||\boldsymbol{G}\tilde{u}_t^p(w)|\leqq Ct^{1-(p-1)(p+1)^{-1}}$$

Summing up these fact we get

$$|u_t|\leqq C+Ct^2\|u_t^p\|_{L^q(\Omega_t)} .$$

By the assumption of Theorem we get

$$\sup_t\sup_{x\in\Omega_t}|u_t(x)|<C'<\infty . \tag{4.1}$$

Then,

$$\max_\theta|\tilde{V}(t)(\theta)|\leqq C''t \tag{4.2}$$

(4.3) $$\max_{x\in\partial B_t}|\nabla\tilde{V}(t)(x)|\leqq C''t^{(1-2\alpha)/3}.$$

Therefore, $I_{1,1}(t)=O(t^{1+2(1-2\alpha)3-1})$

$$I_{1,2}(t)=O(t^{(1/2)+(1-2\alpha)/3})\Big(\int_{\partial B_t}|\tilde{\nabla}\boldsymbol{P}_t u_t^p|^2 d\sigma_x\Big)^{1/2}.$$

We know that

(4.4) $$\int_{\partial B_t}|\tilde{\nabla}\boldsymbol{P}_t u_t^p|^2 d\sigma_x\leqq C\Big(\int_{\partial B_t}|\tilde{\nabla}\boldsymbol{G}u_t^p|^2 d\sigma_x+g(t)^2\int_{\partial B_t}|G(x, w)\boldsymbol{G}u_t^p(w)|^2 d\sigma_x$$
$$+h(t)^2\int_{\partial B_t}|\langle\tilde{\nabla}\nabla_w G(x, w), \nabla_w \boldsymbol{G}u_t^p(w)\rangle|^2 d\sigma_x\Big).$$

The first term in the right hand side of (4.4) is $O(t)$. The second term in the right hand side of (4.4) is $O(g(t)^2)tt^{-2}=O(t)$. The third term in the right hand side of (4.4) is $O(h(t)^2)t^{-4}t=O(t)$. Here we used the fact that $|\tilde{\nabla}\nabla_w G(x, w)|=O(t^{-2})$. Therefore, $I_l(t)=O(t)$. Thus,

(4.5) $$\int_0^\varepsilon I_1(t)dt=O(\varepsilon^2).$$

Second we consider the term $I_2(t)$. By (4.1) we have $I_2(t)=O(t)$. Thus,

(4.6) $$\int_0^\varepsilon I_2(t)dt=O(\varepsilon^2).$$

Similarly we have

(4.7) $$\int_0^\varepsilon I_3(t)dt=O(\varepsilon^2).$$

We would like to consider the integral of $I_4(t)$ from 0 to ε which is a main term of our analysis. We have $u_t=\lambda(t)\boldsymbol{G}_t u_t^p$ we get $I_4(t)=O(1)$. Thus, we have

(4.8) $$\int_0^\varepsilon I_4(t)dt=O(\varepsilon).$$

Summing up these estimates we have

(4.9) $$\lambda(\varepsilon)-\lambda=O(\varepsilon).$$

We need more delicate analysis to get Theorem. We have

(4.10) $$\Big(\int_{\partial B_t} u_t^2 d\sigma_x\Big)=\tilde{I}_5(t)+\tilde{I}_6(t)+\tilde{I}_7(t),$$

where

$$\tilde{I}_5(t)=\lambda(t)^2\int_{\partial B_t}(\boldsymbol{P}_t u_t^p)^2 d\sigma_x$$

$$\tilde{I}_6(t)=2\lambda(t)^2\int_{\partial B_t}(\boldsymbol{P}_t u_t^p)(\boldsymbol{G}_t-\boldsymbol{P}_t)u_t^p d\sigma_x$$

$$\tilde{I}_7(t)=\lambda(t)^2\int_{\partial B_t}((\boldsymbol{G}_t-\boldsymbol{P}_t)u_t^p)^2 d\sigma_x .$$

Since we have (3.4), Lemma 2.1 we get

$$|(\boldsymbol{G}_t-\boldsymbol{P}_t)u_t^p|\leqq C''t^2\|u_t^p\|_{L^q(\Omega_t)}\leqq Ct^2 .$$

Therefore,

$$\tilde{I}_6(t)\leqq C\Big(\int_{\partial B_t}(\boldsymbol{P}_t u_t^p)^2 d\sigma_x\Big)^{1/2} t^{5/2}\leqq \tilde{C}\tilde{I}_5(t)^{1/2}t^{5/2} \tag{4.11}$$

by the Schwartz inequality. We have

$$\tilde{I}_7(t)=O(t^5) . \tag{4.12}$$

We want to calculate $\tilde{I}_5(t)$.

$$\begin{aligned}\tilde{I}_5(t)=&\lambda(t)^2\int_{\partial B_t}(\boldsymbol{G}\tilde{u}_t^p)(x)^2 d\sigma_x\\ &+2\lambda(t)^2\int_{\partial B_t}\boldsymbol{G}\tilde{u}_t^p(x)g(t)G(x, w)\boldsymbol{G}\tilde{u}_t^p(w)d\sigma_x\\ &+2\lambda(t)^2\int_{\partial B_t}\boldsymbol{G}\tilde{u}_t^p(x)h(t)\langle\nabla_w G(x, w), \nabla_w\boldsymbol{G}\tilde{u}_t^p(w)\rangle d\sigma_x\\ &+\lambda(t)^2 g(t)^2\int_{\partial B_t}G(x, w)^2(\boldsymbol{G}\tilde{u}_t^p)(w)^2 d\sigma_x\\ &+2\lambda(t)g(t)h(t)\int_{\partial B_t}G(x, w)\boldsymbol{G}\tilde{u}_t^p(w)\langle\nabla_w G(x, w), \nabla_w\boldsymbol{G}u_t^p(w)\rangle d\sigma_x\\ &+\lambda(t)^2h(t)^2\int_{\partial B_t}\langle\nabla_w G(x, w), \nabla_w\boldsymbol{G}u_t^p(w)\rangle^2 d\sigma_x\\ =&\tilde{I}_8(t)+\tilde{I}_9(t)+\cdots+\tilde{I}_{13}(t) .\end{aligned}$$

We have $\tilde{I}_9(t)=O(t^2|\log t|)$, $\tilde{I}_{10}(t)=O(t^2)$, $\tilde{I}_{11}(t)=O(t^3(\log t)^2)$, $\tilde{I}_{12}(t)=O(t^3|\log t|)$, $\tilde{I}_{13}(t)=O(t^3)$. Since we have (4.9) we get

$$\tilde{I}_8(t)=\lambda^2\int_{\partial B_t}(\boldsymbol{G}\tilde{u}_t^p)(x)^2 d\sigma_x+O(t^2) .$$

Thus,

$$\tilde{I}_5(t)=\lambda^2\int_{\partial B_t}(\boldsymbol{G}\tilde{u}_t^p)(x)^2 d\sigma_x+O(t^2)=O(t) .$$

Thus, $\tilde{I}_6(t)=O(t^3)$. Therefore,

$$I_4(t)=k\lambda^2 t^{-1}\int_{\partial B_t}\boldsymbol{G}\tilde{u}_t^p(x)^2 d\sigma_x+O(t) .$$

Summing up these facts we get

$$\lambda(\varepsilon)-\lambda=k\lambda^2\int_0^\varepsilon\left(t^{-1}\int_{\partial B_t} G\tilde{u}_t^p(x)^2 d\sigma_x\right)dt+O(\varepsilon^2).$$

By the assumption of Theorem we have $G\tilde{u}_t^p(x)-Gu^p(x)=o(1)$ uniformly for x. Therefore we get Theorem.

5. Comments.

We know that the condition

$$\sup_\varepsilon \varepsilon^2\|u_\varepsilon^p\|_{L^q(\Omega_\varepsilon)}<C'<+\infty$$

in Theorem can be replaced by

$$\sup_\varepsilon \sup_{x\in\Omega_\varepsilon} |u_\varepsilon(x)|<C<+\infty. \tag{5.1}$$

The author conjectures that (5.1) follows from other conditions in Theorem.

Our proof of Theorem of this paper is quite different from the proof of Theorem 1 (with $\sigma=0$) in Ozawa [11]. Our proof of this paper used Hadamard's variational formula for non-linear eigenvalue in [7].

The authors want to get the asymptotic estimate of eigenvalues of q-Laplacian under singular variation of domains. Here this problem is related to minimizing problem of

$$\inf_{u\in X}\int_\Omega |\nabla u|^q dx,$$

where $X=\{\|u\|_{L^{p+1}(\Omega)}=1,\ u\in W^{1,q}(\Omega),\ u=0 \text{ on } \partial\Omega\}$. However the Euler equation is complicated compared with the case $q=2$. Can one get any result?

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Addendum: Right hand side of formula (3.4) should be corrected as $C\varepsilon^h\|f\|_{L^p(\Omega_\varepsilon)}$ for $h<1$ and large q. And it suffices to get our Theorem, if an assumption of Theorem, which is

$$\sup_\varepsilon \varepsilon^2\|u_\varepsilon^p\|_{L^p(\Omega_\varepsilon)}<C<+\infty$$

for large q is replaced by

$$\sup_\varepsilon \varepsilon^{1+h}\|u_\varepsilon^p\|_{L^p(\Omega_\varepsilon)}<C'_{q,h}<+\infty$$

for any $h<1$ and large q.