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Stability and existence of critical Kaehler metrics on ruled manifolds

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Abstract. In this article we discuss how the existence of Kaehler metrics with constant scalar curvature on the projectivization of a holomorphic vector bundle over a Kaehler manifold M is related to a moment map condition for the action of the automorphism group of M on the moduli of vector bundles.

In this note we will prove existence results (Theorem III.A and Corollary III.A) for Kaehler metrics with constant scalar curvature on ruled manifolds when the base manifold admits nontrivial holomorphic vector fields, as mentioned in page 28 of [**14**], based on the solvability result of [**9**]. It is expected that this work will be helpful to clarify the relation between the various stability conditions [**14**] and the solvability of constant scalar curvature equation in the future. This work originated from discussions with Professor Simon Donaldson who suggested the author to characterize the (symplectic) stability condition presented in this article from the moment map point of view. The (symplectic) stability condition for the Hermitian-Yang-Mills case from the moment map point of view has been explained in Chapter 6 of [**6**].

When the base manifold does not admit nontrivial holomorphic vector fields the slope stability of the holomorphic vector bundle over the base manifold suffices to ensure the existence of Kaehler metrics with constant scalar curvature on ruled manifolds [**8**]. When the base manifold admits nontrivial holomorphic vector fields, as considered in this article, we need an extra (symplectic) stability condition, originating from the action of the Lie algebra of nontrivial holomorphic vector fields (on the base manifold) on the moduli of holomorphic structures on a vector bundle endowed with a fixed Hermitian metric over the base manifold, to ensure the existence of Kaehler metrics with constant scalar curvature on ruled manifolds. Our result Corollary III.A is derived from Theorem III.A following the

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argument explained in page 210, lines 15–25, of [**6**]. In Section III we will need the fact that the asymptotic expansion (page 409, line 4,) constructed in [**9**] is uniform, similar to (3) of Proposition 6 in [**4**], which can be obtained by tracing through the same arguments in [**9**].

We will prove the existence results in the differential-geometric setting based on the solvability of Gauge-Fixing Constant Scalar Curvature Equations on Ruled Manifolds introduced in [**9**] through the following idea: At each point of an orbit the corresponding Gauge-Fixing Constant Scalar Curvature Equation can be solved (proved in [**9**]), but there is at most a non-degenerate critical point (modulo discrete subgroup) where the solvability of the Constant Scalar Curvature Equation is equivalent *to* the solvability of the Gauge-Fixing Constant Scalar Curvature Equation. An orbit with a non-degenerate critical point is exactly a stable orbit in the moment map sense. In Section I we will introduce a moment map μ which turns out to be the correct one needed in this work.

In this article we will adopt the approach similar to the one explained in page 210, lines 15–25, of [**6**]. Hence in Section I we will introduce the action of the Lie algebra of nontrivial holomorphic vector fields (on the base manifold) on the moduli of holomorphic structures on a vector bundle endowed with a fixed Hermitian metric over the base manifold. In Section III we will switch our approach to the other but equivalent one (varying the Hermitian metric but fixing the holomorphic structure on a vector bundle) to derive Corollary III.A from Theorem III.A. Section II is devoted to the introduction of notation and results of [**9**]. However Corollary II.A has not been derived in [**9**]. It should be remarked that the method presented in this article can also be used to produce "extremal Kaehler metrics" (critical Kaehler metrics) on ruled manifolds for some "semi-stable" case.

Assume that $(M : \omega_M)$ is an m-dimensional compact Kaehler manifold with constant scalar curvature. We have the following facts [**12**] about the structure of the Lie algebra of holomorphic vector fields on M:

THEOREM 0. *Assume that* $(M : \omega_M)$ *is an* m*-dimensional compact Kaehler manifold with constant scalar curvature. Here* ω_M *is the Kaehler form of* M. *Let* $\mathfrak{h}(M)$ *denote the complex Lie algebra of holomorphic vector fields on* M. *Let* $\mathfrak{h}_o(M) = \{Z \in \mathfrak{h}(M) : i_Z\omega_M$ *is* $\bar{\partial}$*-exact*$\}$. *Then we have the following direct sum decomposition* (*in the Lie algebra sense*) *of the Lie algebra* $\mathfrak{h}(M)$:

$$\mathfrak{h}(M) = \mathfrak{h}_o(M) \oplus \mathfrak{c}(M)$$

in which $\mathfrak{c}(M) = \{Z \in \mathfrak{h}(M) : i_Z\omega_M \in H^{(0:1)}(M : \boldsymbol{C})\}$. *Note that the complex Lie algebra* $\mathfrak{c}(M)$ *is commutative and is a Lie sub-algebra of the Lie algebra of the* *isometry group of* $(M : \omega_M)$. *Besides* $\mathfrak{h}_o(M)$ *is the complexification of the intersection* $\mathfrak{k}_{(M:\omega_M)}$ *of* $\mathfrak{h}_o(M)$ *with the Lie algebra of the isometry group of* $(M : \omega_M)$.

Let $\mathbf{E}$ be a smooth complex vector bundle of rank n endowed with a Hermitian metric $H_{\mathbf{E}}$ over $(M : \omega_M)$. Suppose that $\mathbf{A}$ is an Einstein-Hermitian connection on $\mathbf{E}$, compatible with $H_{\mathbf{E}}$, defining a simple holomorphic structure on $\mathbf{E}$. Let $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$ denote the projectivization of $\mathbf{E}$ over M endowed with the holomorphic structure defined by $\mathbf{A}$. Let $\check{\pi}_{\mathbf{A}} : \boldsymbol{P}_{\mathbf{A}}(\mathbf{E}) \longrightarrow M$ denote the natural surjective holomorphic map. Let $L_{\mathbf{A}}$ denote the universal line bundle on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$. Let $e(L_{\mathbf{A}})$ denote the Euler class of $L_{\mathbf{A}}$ on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$. In [**9**] we introduce the Gauge-Fixing Constant Scalar Curvature Equation on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$, depending on sufficiently large $k \in \boldsymbol{N}$, which can be expressed concisely as follows:

$$\mathscr{S}^{\mathbf{A}}_{G\text{-}F}\bullet = \mathscr{S}^{\mathbf{A}}\,\bullet + \frac{k^m}{k \cdot k} \cdot \mathscr{K}^{\mathbf{A}}\bullet = 0$$

with $\bullet$ being a Kaehler form on $\mathscr{P}_{\mathbf{A}}(\mathbf{E})$ lying in the Kaehler class $-e(L_{\mathbf{A}}) + k \cdot [\check{\pi}^*_{\mathbf{A}}\omega_M]$. Note that

$$\mathscr{S}^{\mathbf{A}}\bullet = 0$$

is exactly the usual Constant Scalar Curvature Equation. Besides $\mathscr{K}^{\mathbf{A}}$ (the *gauge-fixing* operator) is a natural projection operator, depending on sufficiently large $k \in \boldsymbol{N}$, which, by identifying $i \cdot \mathfrak{k}_{(M:\omega_M)}$ with the space of infinitesimal deformations of ω_M with *constant* scalar curvature in the Kaehler class $[\omega_M]$, essentially takes value in $i \cdot \mathfrak{k}_{(M:\omega_M)}$.

In [**9**] the *solvability* of the Gauge-Fixing Constant Scalar Curvature Equation $\mathscr{S}^{\mathbf{A}}_{G\text{-}F}\bullet = 0$ on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$ has been established for sufficiently large $k > 0$. Besides, in [**9**], asymptotic expansion for the solutions of the Gauge-Fixing Constant Scalar Curvature Equation on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$, as $k \to +\infty$, has been shown to exist. By tracing through the same arguments in [**9**] it can be shown that the solutions of the Gauge-Fixing Constant Scalar Curvature Equation, depending on sufficiently large $k > 0$, on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$ constructed in [**9**] actually depend smoothly on the Einstein-Hermitian connection $\mathbf{A}$ which defines simple holomorphic structure on $\mathbf{E}$. Let $\hat{\mathscr{K}}^{\mathbf{A}}$ denote the $i \cdot \mathfrak{k}_{(M:\omega_M)}$-value of $\mathscr{K}^{\mathbf{A}}$ substituted by the corresponding solution constructed in [**9**] for the Gauge-Fixing Constant Scalar Curvature Equation, depending on sufficiently large $k > 0$, on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$. It is obvious that to solve the Constant Scalar Curvature Equation, depending on sufficiently large $k > 0$, on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$ it suffices to find natural conditions which lead to the vanishing of $\hat{\mathscr{K}}^{\mathbf{A}}$.

Let $\hat{\mathscr{K}}_o^{\mathbf{A}}$ denote the nontrivial leading term of the asymptotic expansion of $\hat{\mathscr{K}}^{\mathbf{A}}$ as $k \to +\infty$. By considering $\hat{\mathscr{K}}^{\mathbf{A}}$, depending on sufficiently large $k > 0$, as a vector-valued function on the moduli space of simple Einstein-Hermitian connections on $\mathbf{E}$ it will be shown in Section III that we can solve the Constant Scalar Curvature Equation, with $\frac{1}{k}$ *near* 0, on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$ through the Implicit Function Theorem by finding non-degenerate zero points of $\hat{\mathscr{K}}_o^{\mathbf{A}}$ on the moduli space of simple Einstein-Hermitian connections on $\mathbf{E}$. Besides in Section III it will be inferred from Corollary II.A that the "non-degenerate zero point" condition is equivalent to the stability condition (Definition I.A) introduced in Section I. Hence the moment map μ introduced in Section I is crucial to the solvability of constant scalar curvature equations on ruled Kaehler manifolds. However the relation between the stability condition (Definition I.A) introduced in Section I and the various stability conditions discussed in [**14**] is so far not clear to the author.

I. Moment maps and stability.

In this article we will use the Real version of Theorem 0. Thus we define

$$\mathfrak{h}_o^R(M) = \{Z + \bar{Z} \in \Gamma(M : T(M)) : Z \in \mathfrak{h}_o(M)\}.$$

Note that $\mathfrak{h}_o^R(M)$ is naturally isomorphic to $\mathfrak{h}_o(M)$ in the Lie algebra sense. Besides $\mathfrak{h}_o^R(M)$ is the complexification of

$$\mathfrak{k}_{(M:\omega_M)} = \{X \in \Gamma(M : T(M)) : \mathscr{L}_X \mathrm{I}_M = 0 \text{ and } i_X\omega_M \text{ is } d\text{-exact}\}.$$

Here I_M is the complex structure of M.

Assume that $\mathbf{E}$ is a smooth complex vector bundle of rank n endowed with a Hermitian metric $H_{\mathbf{E}}$ over $(M : \omega_M)$. Let $\mathbf{A}_{\wedge^n\mathbf{E}}$ be a fixed Einstein-Hermitian connection on $\wedge^n\mathbf{E}$, compatible with the Hermitian metric $H_{\mathbf{E}}$ on $\mathbf{E}$, defining holomorphic structure on $\wedge^n\mathbf{E}$. Let $\mathscr{A}$ denote the affine space of smooth connections on $\mathbf{E}$ compatible with the Hermitian metric $H_{\mathbf{E}}$ on $\mathbf{E}$. Then there is a symplectic form $\omega_{\mathscr{A}}$ on $\mathscr{A}$ defined as follows:

$$\omega_{\mathscr{A}}(\alpha : \beta) = \int_M \frac{\operatorname{trace}(\alpha \wedge \beta) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2}.$$

Note that this symplectic form is compatible with the natural complex structure on $\mathscr{A}$ and so $(\mathscr{A} : \omega_{\mathscr{A}})$ becomes formally a flat Kaehler manifold.

Let $\mathscr{A}(\mathbf{E} : \wedge^n\mathbf{E})$ denote the space of smooth connections on $\mathbf{E}$, compatible with the Hermitian metric $H_{\mathbf{E}}$ on $\mathbf{E}$, which, modulo the group of unitary transformations of the holomorphic line bundle $\wedge^n\mathbf{E}$ over M, induce the connection $\mathbf{A}_{\wedge^n\mathbf{E}}$ on $\wedge^n\mathbf{E}$. Let $\mathscr{A}^{\text{E-H}}_{\text{simple}}$ denote the subspace of $\mathscr{A}(\mathbf{E} : \wedge^n\mathbf{E})$ consisting of Einstein-Hermitian connections defining simple holomorphic structures on $\mathbf{E}$.

We will now introduce a family of smooth vector fields on $\mathscr{A}^{\text{E-H}}_{\text{simple}}$ associated with $\mathfrak{h}^R_o(M)$. Let $\mathfrak{u}(\mathbf{E})$ denote the sub-bundle of $\mathrm{Hom}_{\boldsymbol{C}}(\mathbf{E} : \mathbf{E})$ over M defined as follows:

$$\mathfrak{u}(\mathbf{E}) = \{u \in \mathrm{Hom}_{\boldsymbol{C}}(\mathbf{E} : \mathbf{E}) : u + u^* = 0\}$$

in which u^* is the adjoint of u with respect to the Hermitian metric $H_{\mathbf{E}}$ on $\mathbf{E}$. Let Λ_M denote the adjoint of the $\boldsymbol{C}$-linear map

$$\bullet \longmapsto \omega_M \wedge \bullet$$

on M with respect to the Kaehler form ω_M. Given an element $X = Z_X + \bar{Z}_X$ of $\mathfrak{h}^R_o(M)$ with Z_X being holomorphic we consider at each $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ the following equation for $g_X \in \Gamma(M : \mathrm{Hom}_{\boldsymbol{C}}(\mathbf{E} : \mathbf{E}))$:

$$\Lambda_M \circ \partial_{\mathbf{A}}(\bar{\partial}_{\mathbf{A}} g_X + i_{Z_X} F_{\mathbf{A}}) = 0 \iff \partial_{\mathbf{A}}(\bar{\partial}_{\mathbf{A}} g_X + i_{Z_X} F_{\mathbf{A}}) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!} = 0$$

in which $F_{\mathbf{A}}$ is the curvature form of $\mathbf{E}$ defined by $\mathbf{A}$.

PROPOSITION I.A. *Let* id. *denote the identity transformation of* $\mathbf{E}$ *over* M. *Given* $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ *there exists, modulo* $\{c \cdot \text{id.} : c \in \boldsymbol{C}\}$, *a unique solution* $g_X \in \Gamma(M : \mathrm{Hom}_{\boldsymbol{C}}(\mathbf{E} : \mathbf{E}))$, *depending on* $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$, *for the equation*

$$\Lambda_M \circ \partial_{\mathbf{A}}(\bar{\partial}_{\mathbf{A}} g_X + i_{Z_X} F_{\mathbf{A}}) = 0.$$

PROOF. Note that, according to a result of Andre Lichnerowicz [**12**], there exists for each $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ a corresponding $f \in \Gamma(M : \boldsymbol{C})$ such that

$$\mathrm{trace}(i_{Z_X} F_{\mathbf{A}}) = i_{Z_X}(\mathrm{trace}\ F_{\mathbf{A}}) = \bar{\partial} f.$$

In particular we have

$$\int_M (\text{trace } \Lambda_M \circ \partial_{\mathbf{A}} \circ i_{Z_X} F_{\mathbf{A}}) \wedge \frac{\omega_M^m}{m!} = i \cdot \int_M \text{trace}(\bar{\partial}_A^* \circ i_{Z_X} F_{\mathbf{A}}) \wedge \frac{\omega_M^m}{m!}$$
$$= i \cdot \int_M \bar{\partial}^* \circ \bar{\partial} f \wedge \frac{\omega_M^m}{m!} = 0.$$

Thus the solvability of $\Lambda_M \circ \partial_{\mathbf{A}}(\bar{\partial}_{\mathbf{A}} g_X + i_{Z_X} F_{\mathbf{A}}) = 0$ follows immediately from the simplicity of the holomorphic structure on $\mathbf{E}$ defined by $\mathbf{A}$. □

Note that for each solution g_X of $\Lambda_M \circ \partial_{\mathbf{A}}(\bar{\partial}_{\mathbf{A}} g_X + i_{Z_X} F_{\mathbf{A}}) = 0$ found in Proposition I.A we have $\bar{\partial}_{\mathbf{A}} g_X + i_{Z_X} F_{\mathbf{A}}$ being traceless. This fact can be observed readily from the proof of Proposition I.A. Besides we note that $(-\partial_{\mathbf{A}} g_X^* + \bar{\partial}_{\mathbf{A}} g_X + i_X F_{\mathbf{A}})$ satisfies $\Lambda_M \circ d_{\mathbf{A}}(-\partial_{\mathbf{A}} g_X^* + \bar{\partial}_{\mathbf{A}} g_X + i_X F_{\mathbf{A}}) = 0$ and is traceless because $F_{\mathbf{A}}^* = -F_{\mathbf{A}}$. We can now define the smooth vector field θ_X on $\mathscr{A}^{\text{E-H}}_{\text{simple}}$ associated with $X \in \mathfrak{h}_o^R(M)$ as follows: Given $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ we define the tangent vector $\theta_X|_{\mathbf{A}}$ assigned by θ_X at $\mathbf{A}$ as

$$\theta_X|_{\mathbf{A}} = -(-\partial_{\mathbf{A}} g_X^* + \bar{\partial}_{\mathbf{A}} g_X + i_X F_{\mathbf{A}}).$$

LEMMA I.A. *Let $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$. Suppose that $w \in \Gamma(M : \mathrm{Hom}_{\boldsymbol{C}}(\mathbf{E} : \mathbf{E}))$ satisfies*

$$\Lambda_M \circ d_{\mathbf{A}}(-\partial_{\mathbf{A}} w^* + \bar{\partial}_{\mathbf{A}} w) = 0$$

with $w = u + i \cdot v$ in which $u \in \Gamma(M : \mathfrak{u}(\mathbf{E}))$ and $v \in \Gamma(M : \mathfrak{u}(\mathbf{E}))$. Then we have $\bar{\partial}_{\mathbf{A}} v = 0 = \partial_{\mathbf{A}} v$ and $-\partial_{\mathbf{A}} w^* + \bar{\partial}_{\mathbf{A}} w = d_{\mathbf{A}} u$.

PROOF. Note that

$$-\partial_{\mathbf{A}} w^* + \bar{\partial}_{\mathbf{A}} w = d_{\mathbf{A}} u + i \cdot (-\partial_{\mathbf{A}} v + \bar{\partial}_{\mathbf{A}} v)$$

and so Lemma I.A follows readily from the simplicity of the holomorphic structure on $\mathbf{E}$ defined by $\mathbf{A}$. (See page 476 of [**8**].) □

Let $\mathscr{G}$ denote the gauge group of gauge transformations of $\mathbf{E}$ generated by $\Gamma(M : \mathfrak{u}(\mathbf{E}))$. Let $\mathscr{G}_o$ denote the quotient group

$$\mathscr{G}_o = \frac{\mathscr{G}}{\{c \cdot id. \in \mathscr{G} : c \in \boldsymbol{C} \text{ with } |c| = 1\}}.$$

Here id. $\in \mathrm{Hom}_{\boldsymbol{C}}(\mathbf{E} : \mathbf{E})$ is the identity map of $\mathbf{E}$ over M. Our next result (Proposition I.B) implies that the action of $\mathfrak{h}_o^R(M)$ on the complex moduli space

$$\mathscr{M}^{\text{E-H}}_{\text{simple}} = \frac{\mathscr{A}^{\text{E-H}}_{\text{simple}}}{\mathscr{G}_o}$$

is Lie-algebraic. Hence $\mathscr{M}^{\text{E-H}}_{\text{simple}}$ is divided into orbits generated by $\mathfrak{h}_o^R(M)$.

PROPOSITION I.B. *Given elements X and Y of $\mathfrak{h}_o^R(M)$ we have, at each $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$, the following result*

$$-\theta_{[X:Y]} + [\theta_X : \theta_Y] = -\theta_{[X:Y]} + (-\nabla_{\theta_Y}\theta_X + \nabla_{\theta_X}\theta_Y) \ \in \ d_{\mathbf{A}}\, \Gamma(M : \mathfrak{u}(\mathbf{E})).$$

PROOF. Direct computation shows that $[\theta_X : \theta_Y] = -\nabla_{\theta_Y}\theta_X + \nabla_{\theta_X}\theta_Y$ can be expressed as

$$(-\partial_{\mathbf{A}} g^* + \bar{\partial}_{\mathbf{A}} g) + (-i_X \circ d_{\mathbf{A}} \circ i_Y F_{\mathbf{A}} + i_Y \circ d_{\mathbf{A}} \circ i_X F_{\mathbf{A}})$$

in which g is a smooth section of $\mathrm{Hom}_{\boldsymbol{C}}(\mathbf{E} : \mathbf{E})$ over M. Since $d_{\mathbf{A}} F_{\mathbf{A}} = 0$ we have

$$d_{\mathbf{A}}\ F_{\mathbf{A}}(X : Y) + i_{[X:Y]} F_{\mathbf{A}} = (-i_Y \circ d_{\mathbf{A}} \circ i_X F_{\mathbf{A}} + i_X \circ d_{\mathbf{A}} \circ i_Y F_{\mathbf{A}})$$

and so

$$[\theta_X : \theta_Y] = -\nabla_{\theta_Y}\theta_X + \nabla_{\theta_X}\theta_Y = -(d_{\mathbf{A}}\ F_{\mathbf{A}}(X : Y) + i_{[X:Y]} F_{\mathbf{A}}) + (-\partial_{\mathbf{A}} g^* + \bar{\partial}_{\mathbf{A}} g).$$

It follows from the Einstein-Hermitian condition that there is a constant λ such that $\Lambda_M F_\bullet = \lambda \cdot id. \iff F_\bullet \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!} = \lambda \cdot id. \frac{\omega_M^m}{m!}$ over M for each $\bullet \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ in which $F_\bullet = d\bullet + \bullet \wedge \bullet$ is the curvature 2-form associated with the connection $\bullet$. When the infinitesimal deformation $\delta\bullet$ of $\bullet$ is θ_Y it can be inferred from the Einstein-Hermitian condition $\Lambda_M F_\bullet = \lambda \cdot id.$ on $\mathscr{A}^{\text{E-H}}_{\text{simple}}$ that we have

$$0 = \delta(\lambda \cdot id.) = \delta(\Lambda_M F_\bullet) = \Lambda_M(\delta F_\bullet) = \Lambda_M(d\theta_Y + \bullet \wedge \theta_Y + \theta_Y \wedge \bullet) = \Lambda_M \circ d_\bullet \theta_Y.$$

By deforming this equality $0 = \Lambda_M(d\theta_Y + \bullet \wedge \theta_Y + \theta_Y \wedge \bullet)$ on $\mathscr{A}^{\text{E-H}}_{\text{simple}}$ along θ_X so that $\delta\bullet = \theta_X$ we have

$$
\begin{aligned}
0 &= \Lambda_M \circ \delta(d\theta_Y + \bullet \wedge \theta_Y + \theta_Y \wedge \bullet) \\
&= \Lambda_M(d\nabla_{\theta_X}\theta_Y + \theta_X \wedge \theta_Y + \bullet \wedge \nabla_{\theta_X}\theta_Y + \nabla_{\theta_X}\theta_Y \wedge \bullet + \theta_Y \wedge \theta_X) \\
&= \Lambda_M(d_\bullet \nabla_{\theta_X}\theta_Y + \theta_X \wedge \theta_Y + \theta_Y \wedge \theta_X)
\end{aligned}
$$

in which $d_\bullet \nabla_{\theta_X}\theta_Y = d\nabla_{\theta_X}\theta_Y + \bullet \wedge \nabla_{\theta_X}\theta_Y + \nabla_{\theta_X}\theta_Y \wedge \bullet$. Hence we have

$$\Lambda_M \circ ([\theta_X : \theta_Y]_{\mathrm{s}} + d_{\mathbf{A}}\ \nabla_{\theta_X}\theta_Y) = 0$$

at $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ in which $[\ :\ ]_{\mathrm{s}}$ is the super-symmetric bracket operation so that

$$[\theta_X : \theta_Y]_{\mathrm{s}} = \theta_X \wedge \theta_Y + \theta_Y \wedge \theta_X.$$

Similarly, by deforming the equality $\Lambda_M \circ d_\bullet \theta_X = 0$ on $\mathscr{A}^{\text{E-H}}_{\text{simple}}$ along θ_Y, we obtain $\Lambda_M \circ ([\theta_Y : \theta_X]_{\mathrm{s}} + d_{\mathbf{A}}\ \nabla_{\theta_Y}\theta_X) = 0$. In particular we infer from these equalities that

$$\Lambda_M \circ d_{\mathbf{A}}([\theta_X : \theta_Y]) = \Lambda_M \circ d_{\mathbf{A}}(-\nabla_{\theta_Y}\theta_X + \nabla_{\theta_X}\theta_Y) = 0$$

and so

$$\Lambda_M \circ d_{\mathbf{A}}[-(d_{\mathbf{A}}\ F_{\mathbf{A}}(X:Y) + i_{[X:Y]}F_{\mathbf{A}}) + (-\partial_{\mathbf{A}} g^* + \bar{\partial}_{\mathbf{A}} g)] = 0.$$

Now by comparing this equation with

$$-\Lambda_M \circ d_{\mathbf{A}}\theta_{[X:Y]} = \Lambda \circ d_{\mathbf{A}}(-\partial_{\mathbf{A}} g^*_{[X:Y]} + \bar{\partial}_{\mathbf{A}} g_{[X:Y]} + i_{[X:Y]}F_{\mathbf{A}}) = 0$$

we infer immediately that

$$\Lambda_M \circ d_{\mathbf{A}}(-d_{\mathbf{A}}\ F_{\mathbf{A}}(X:Y) + (-\partial_{\mathbf{A}} g^* + \bar{\partial}_{\mathbf{A}} g) + (-\partial_{\mathbf{A}} g^*_{[X:Y]} + \bar{\partial}_{\mathbf{A}} g_{[X:Y]})) = 0.$$

Hence according to Lemma I.A we must have

$$-d_{\mathbf{A}}\ F_{\mathbf{A}}(X:Y) + (-\partial_{\mathbf{A}} g^* + \bar{\partial}_{\mathbf{A}} g) = -\bar{\partial}_{\mathbf{A}} g_{[X:Y]} + \partial_{\mathbf{A}} g^*_{[X:Y]} + d_{\mathbf{A}} u$$

for some $u \in \Gamma(M : \mathfrak{u}(\mathbf{E}))$. In particular we conclude that

$$\begin{aligned}[\theta_X : \theta_Y] &= -\nabla_{\theta_Y}\theta_X + \nabla_{\theta_X}\theta_Y = -(-\partial_{\mathbf{A}} g^*_{[X:Y]} + \bar{\partial}_{\mathbf{A}} g_{[X:Y]} + i_{[X:Y]}F_{\mathbf{A}}) + d_{\mathbf{A}}u \\ &= \theta_{[X:Y]} + d_{\mathbf{A}}u\end{aligned}$$

and so Proposition I.B is true. □

Note that $\omega_{\mathscr{A}}$ induces a symplectic form $\omega_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}$ on $\mathscr{M}^{\text{E-H}}_{\text{simple}}$. Actually, for each tangent vector θ at $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$, we must have $\Lambda_M \circ d_{\mathbf{A}}\theta = 0$ and so

$$\begin{aligned}\omega_{\mathscr{A}}(d_{\mathbf{A}}u : \theta) &= \int_M \frac{\operatorname{trace}(d_{\mathbf{A}}u \wedge \theta) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2} = -\int_M \frac{\operatorname{trace}(u \wedge d_{\mathbf{A}}\theta) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2} \\ &= -\int_M \frac{\operatorname{trace}(u \wedge \Lambda_M \circ d_{\mathbf{A}}\theta) \wedge \frac{\omega_M^m}{m!}}{2\pi^2} = 0\end{aligned}$$

for any $u \in \Gamma(M : \mathfrak{u}(\mathbf{E}))$. This result implies the existence of $\omega_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}$ on $\mathscr{M}^{\text{E-H}}_{\text{simple}}$.

We can now introduce a moment map

$$\mu^e : \mathscr{M}^{\text{E-H}}_{\text{simple}} \longrightarrow \operatorname{Hom}_{\boldsymbol{R}}(\mathfrak{k}_{(M:\omega_M)} : \boldsymbol{R}).$$

Given $X \in \mathfrak{k}_{(M:\omega_M)}$ we denote by $f_X \in \Gamma(M : \boldsymbol{R})$ the unique smooth $\boldsymbol{R}$-valued function on M satisfying $i_X\omega_M = d\,f_X$ and $\int_M f_X \cdot \frac{\omega_M^m}{m!} = 0$. We define the value of μ^e_X at $[\mathbf{A}] \in \mathscr{M}^{\text{E-H}}_{\text{simple}}$ as

$$\mu^e_X([\mathbf{A}]) = \int_M f_X \cdot \operatorname{trace}\left(\frac{i \cdot F_{\mathbf{A}}}{2\pi} \wedge \frac{i \cdot F_{\mathbf{A}}}{2\pi}\right) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}.$$

It can be checked readily that this definition of $\mu^e_X([\mathbf{A}])$ is independent of the representative $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ used for $[\mathbf{A}] \in \mathscr{M}^{\text{E-H}}_{\text{simple}}$.

Lemma I.B. *Given $X \in \mathfrak{k}_{(M:\omega_M)}$ we have $\Lambda_M \circ d_{\mathbf{A}} \circ i_X F_{\mathbf{A}} = 0$ at any $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$. Besides the tangent vector assigned by θ_X at $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ can be expressed in the following form*

$$\theta_X|_{\mathbf{A}} = -(d_{\mathbf{A}}u + i_X F_{\mathbf{A}})$$

for some $u \in \Gamma(M : \mathfrak{u}(\mathbf{E}))$.

Proof. Since $d \circ i_X\omega_M = \mathscr{L}_X\omega_M = 0$ it can be inferred from the Einstein-Hermitian condition

$$\Lambda_M F_{\mathbf{A}} = \lambda \cdot \mathrm{id.} \iff F_{\mathbf{A}} \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!} = \lambda \cdot \mathrm{id.} \frac{\omega_M^m}{m!}$$

satisfied by $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ that

$$i_X F_{\mathbf{A}} \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!} + F_{\mathbf{A}} \wedge i_X \frac{\omega_M^{(-1+m)}}{(-1+m)!} = \lambda \cdot \mathrm{id.} \cdot i_X \frac{\omega_M^m}{m!}$$

and so

$$(\Lambda_M \circ d_{\mathbf{A}} \circ i_X F_{\mathbf{A}}) \cdot \frac{\omega_M^m}{m!} = d_{\mathbf{A}} \circ i_X F_{\mathbf{A}} \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!} = 0.$$

Now using this result and Lemma I.A we conclude that the remaining assertion of Lemma I.B is true. $\square$

PROPOSITION I.C. *$\mu^e : \mathscr{M}^{\text{E-H}}_{\text{simple}} \longrightarrow \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{k}_{(M:\omega_M)} : \boldsymbol{R})$ is an equivariant moment map.*

PROOF. Let θ be a tangent vector at $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$. Then θ is traceless and satisfies $\Lambda_M \circ d_{\mathbf{A}} \theta = 0$. Given $X \in \mathfrak{k}_{(M:\omega_M)}$ it follows from the Stokes Theorem and Lemma I.B that

$$\begin{aligned}
(d\,\mu_X^e)[\theta] &= \int_M 2 \cdot \mathrm{trace}\left(\frac{i \cdot d_{\mathbf{A}}\theta}{2\pi} \wedge \frac{i \cdot F_{\mathbf{A}}}{2\pi}\right) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!} \wedge f_X \\
&= -\int_M \frac{\mathrm{trace}(\theta \wedge F_{\mathbf{A}}) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!} \wedge d\,f_X}{2\pi^2} \\
&= -\int_M \frac{\mathrm{trace}(\theta \wedge F_{\mathbf{A}}) \wedge i_X \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2} \\
&= -\int_M \frac{\mathrm{trace}(i_X\theta \wedge F_{\mathbf{A}}) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2} + \int_M \frac{\mathrm{trace}(\theta \wedge i_X F_{\mathbf{A}}) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2} \\
&= -0 + \int_M \frac{\mathrm{trace}(\theta \wedge i_X F_{\mathbf{A}}) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2} = -\int_M \frac{\mathrm{trace}(\theta \wedge \theta_X) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2}.
\end{aligned}$$

Hence we have $(d\,\mu_X^e)[\theta] = \omega_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\theta_X] : [\theta])$ and so $\mu^e : \mathscr{M}^{\text{E-H}}_{\text{simple}} \longrightarrow$

$\mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{k}_{(M:\omega_M)} : \boldsymbol{R})$ is a moment map.

Now we check the equivariance of $\mu^e : \mathscr{M}^{\text{E-H}}_{\text{simple}} \longrightarrow \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{k}_{(M:\omega_M)} : \boldsymbol{R})$. Given $X \in \mathfrak{k}_{(M:\omega_M)}$ and $Y \in \mathfrak{k}_{(M:\omega_M)}$ it can be inferred from the Einstein-Hermitian condition

$$\Lambda_M F_{\mathbf{A}} = \lambda \cdot id. \Longleftrightarrow F_{\mathbf{A}} \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!} = \lambda \cdot id. \frac{\omega_M^m}{m!}$$

satisfied by $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ that

$$\begin{aligned} &\mathrm{trace}(i_Y F_{\mathbf{A}} \wedge i_X F_{\mathbf{A}}) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!} + \mathrm{trace}(i_Y F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \wedge i_X \frac{\omega_M^{(-1+m)}}{(-1+m)!} \\ &\quad = \mathrm{trace}(i_Y F_{\mathbf{A}} \wedge \lambda \cdot id.) \wedge i_X \frac{\omega_M^m}{m!}. \end{aligned}$$

Since $i_X\omega_M$ is d-exact and $\Lambda_M \circ d_{\mathbf{A}} \circ i_Y F_{\mathbf{A}} = 0$ (Lemma I.B) it can be checked readily, using the Stokes Theorem, that $\int_M \mathrm{trace}(i_Y F_{\mathbf{A}} \wedge \lambda \cdot id.) \wedge i_X \frac{\omega_M^m}{m!} = 0$. Hence, using the Stokes Theorem, we have

$$\begin{aligned} \int_M \frac{\mathrm{trace}(i_Y F_{\mathbf{A}} \wedge i_X F_{\mathbf{A}}) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2} &= -\int_M \frac{\mathrm{trace}(i_Y F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \wedge i_X \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2} \\ &= -\int_M \frac{i_Y \circ \mathrm{trace}(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \wedge i_X \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{4\pi^2} \\ &= \int_M \frac{\mathrm{trace}(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \wedge i_Y \circ i_X \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{4\pi^2} \\ &= \int_M \frac{\mathrm{trace}(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!} \wedge \omega_M(X : Y)}{4\pi^2} \end{aligned}$$

because $i_X\omega_M$ and $i_Y\omega_M$ are d-exact. Note that

$$\omega_M(X : Y) = -f_{[X:Y]}$$

in which $f_{[X:Y]}$ is the unique smooth $\boldsymbol{R}$-valued function on M satisfying $i_{[X:Y]}\omega_M = d\, f_{[X:Y]}$ and $\int_M f_{[X:Y]} \cdot \frac{\omega_M^m}{m!} = 0$. With this result we conclude, using Lemma I.B and the Stokes Theorem, that

$$(d\mu^e_Y)[\theta_X] = \omega_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\theta_Y] : [\theta_X]) = \int_M \frac{\operatorname{trace}(i_Y F_{\mathbf{A}} \wedge i_X F_{\mathbf{A}}) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{2\pi^2}$$
$$= -\int_M f_{[X:Y]} \cdot \frac{\operatorname{trace}(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{4\pi^2} = \mu^e_{[X:Y]}$$

and so the moment map $\mu^e : \mathscr{M}^{\text{E-H}}_{\text{simple}} \longrightarrow \operatorname{Hom}_{\boldsymbol{R}}(\mathfrak{k}_{(M:\omega_M)} : \boldsymbol{R})$ is equivariant. $\square$

Let F_{ω_M} denote the curvature form of the holomorphic tangent bundle of M induced by ω_M. Given $\mathbf{A} \in \mathscr{A}(\mathbf{E} : \wedge^n\mathbf{E})$ we define a smooth m-form $\eta_{\wedge^n\mathbf{E}}$ on M as follows:

$$\eta_{\wedge^n\mathbf{E}} = \frac{\left[\operatorname{trace}\left(\frac{i\cdot F_{\mathbf{A}}}{2\pi}\right) \wedge \operatorname{trace}\left(\frac{i\cdot F_{\mathbf{A}}}{2\pi}\right) + (n+1)\cdot \operatorname{trace}\left(\frac{i\cdot F_{\mathbf{A}}}{2\pi}\right) \wedge \operatorname{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right] \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{n}.$$

Since for any $\mathbf{A} \in \mathscr{A}(\mathbf{E} : \wedge^n\mathbf{E})$ we have $\operatorname{trace}(\frac{i\cdot F_{\mathbf{A}}}{2\pi}) = \frac{i\cdot F_{\mathbf{A}_{\wedge^n\mathbf{E}}}}{2\pi}$ it follows that $\eta_{\wedge^n\mathbf{E}}$ does not depend on the connection $\mathbf{A} \in \mathscr{A}(\mathbf{E} : \wedge^n\mathbf{E})$ used in the above definition. Note that $\eta_{\wedge^n\mathbf{E}}$ defines a linear functional $\mathbf{L}_{\eta_{\wedge^n\mathbf{E}}}$ on $\mathfrak{k}_{(M:\omega_M)}$ as follows:

$$\mathbf{L}_{\eta_{\wedge^n\mathbf{E}}}(X) = \int_M f_X \cdot \eta_{\wedge^n\mathbf{E}}$$

$\forall X \in \mathfrak{k}_{(M:\omega_M)}$. Let $\mu : \mathscr{M}^{\text{E-H}}_{\text{simple}} \longrightarrow \operatorname{Hom}_{\boldsymbol{R}}(\mathfrak{k}_{(M:\omega_M)} : \boldsymbol{R})$ be defined as follows:

$$\mu = -\mathbf{L}_{\eta_{\wedge^n\mathbf{E}}} + \mu^e.$$

Then $\mu : \mathscr{M}^{\text{E-H}}_{\text{simple}} \longrightarrow \operatorname{Hom}_{\boldsymbol{R}}(\mathfrak{k}_{(M:\omega_M)} : \boldsymbol{R})$ is a moment map. Motivated by the Kempf-Ness principle we introduce

DEFINITION I.A. Given $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ we denote by $\mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\mathbf{A}])$ the orbit of $[\mathbf{A}]$ generated by $\mathfrak{h}^R_o(M)$ in $\mathscr{M}^{\text{E-H}}_{\text{simple}}$. We say that the orbit $\mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\mathbf{A}])$ is stable if and only if there exists a non-degenerate zero point $[\mathbf{A}_\infty] \in \mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\mathbf{A}])$ for μ: $\mu|_{[\mathbf{A}_\infty]} = 0$ with $\{X \in \mathfrak{k}_{(M:\omega_M)} : [\theta_X]|_{[\mathbf{A}_\infty]} = 0\} = \langle 0\rangle$.

II. Some fundamental results.

In this section we introduce some notation and certain facts taken from [**9**] as background material for the arguments presented in Section III. We will use

Corollary II.A, which has not been indicated in [**9**], in Section III.

Let $\pi : E \longrightarrow M$ be a simple holomorphic vector bundle of rank n endowed with Einstein-Hermitian metric H_E over the Kaehler manifold $(M : \omega_M)$ with constant scalar curvature. Let A denote the Einstein-Hermitian connection on E induced by H_E. Let $\boldsymbol{P}(E)$ denote the projectivization of E over M. Then $\boldsymbol{P}(E)$ is a compact complex manifold with $(-1+m+n)$ dimensions. Let L be the universal holomorphic line bundle over $\boldsymbol{P}(E)$. Then the Einstein-Hermitian metric H_E induces a Hermitian metric H_{L^*} on the dual L^* of L over $\boldsymbol{P}(E)$. Let A_{L^*} denote the Hermitian connection on L^* induced by H_{L^*}. Thus there is a representative

$$\frac{i \cdot F_{A_{L^*}}}{2\pi} = \frac{i}{2\pi} \cdot \bar{\partial}\partial \log H_{L^*} = -\frac{i}{2\pi} \cdot \bar{\partial}\partial \log H_L$$

of the Euler class $e(L^*)$ of L^* on $\boldsymbol{P}(E)$ induced by the Hermitian connection A_{L^*}. Here H_L is the Hermitian metric on L over $\boldsymbol{P}(E)$ induced by the Einstein-Hermitian metric H_E on E over M. Note that the representative $\frac{i \cdot F_{A_{L^*}}}{2\pi}$ of $e(L^*)$ on $\boldsymbol{P}(E)$ induces the Fubini-Study metric on each fiber $\boldsymbol{P}(\boldsymbol{C}^n)$ of $\check{\pi} : \boldsymbol{P}(E) \longrightarrow M$. Thus, for each large $k > 0$, $\frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \check{\pi}^* \omega_M$ is a Kaehler form on $\boldsymbol{P}(E)$.

Since the restriction of $\frac{i \cdot F_{A_{L^*}}}{2\pi}$ on each fiber $\boldsymbol{P}(\boldsymbol{C}^n)$ of $\check{\pi} : \boldsymbol{P}(E) \longrightarrow M$ is simply the Fubini-Study Kaehler form there is a well-defined smooth vector bundle W over M whose fiber (vector space over $\boldsymbol{R}$) W_z over $z \in M$ is the eigen-space of the lowest non-zero eigen-value of the (Fubini-Study) Laplacian on the fiber $\boldsymbol{P}(\boldsymbol{C}^n)$ of $\boldsymbol{P}(E)$ over M. On the other hand integration along the fibers of $\check{\pi} : \boldsymbol{P}(E) \longrightarrow M$ maps a smooth function on $\boldsymbol{P}(E)$ onto a smooth function on M. Let $\Gamma(M : W)$ denote the space of smooth sections of W over M. Then for each smooth $\boldsymbol{R}$-valued function $f \in \Gamma(\boldsymbol{P}(E) : \boldsymbol{R})$ on $\boldsymbol{P}(E)$ we have the following corresponding decomposition:

$$f = \hat{\sigma}(f) \oplus \sigma(f) \oplus \tilde{\sigma}(f)$$

in which $(\hat{\sigma}(f) : \sigma(f)) \in \Gamma(M : \boldsymbol{R}) \oplus \Gamma(M : W)$ while the restriction of $\tilde{\sigma}(f)$ on each fiber $\boldsymbol{P}(\boldsymbol{C}^n)$ of $\check{\pi} : \boldsymbol{P}(E) \longrightarrow M$ over $z \in M$ is *orthogonal to* both the space W_z and the space of constant functions on that fiber (over $z \in M$).

Note that the Einstein-Hermitian connection A on E over M defines a smooth distribution $\mathscr{H}$ of horizontal spaces on $\boldsymbol{P}(E)$:

$$T(\boldsymbol{P}(E)) = V \oplus \mathscr{H}.$$

Here V is the sub-bundle of $T(\boldsymbol{P}(E))$ over $\boldsymbol{P}(E)$ consisting of tangent vectors which are tangential to the fibers of $\breve{\pi} : \boldsymbol{P}(E) \longrightarrow M$. Let $V^{[*]}$ denote the maximal sub-bundle of $T^*(\boldsymbol{P}(E))$ over $\boldsymbol{P}(E)$ whose action on $\mathscr{H}$ is identically zero. Then the decomposition $T(\boldsymbol{P}(E)) = V \oplus \mathscr{H}$ of $T(\boldsymbol{P}(E))$ over $\boldsymbol{P}(E)$ induces the following corresponding decomposition $T^*(\boldsymbol{P}(E)) = V^{[*]} \oplus \breve{\pi}^*(T^*(M))$ of $T^*(\boldsymbol{P}(E))$ over $\boldsymbol{P}(E)$. Thus we have the following decomposition

$$\wedge^* T^*(\boldsymbol{P}(E)) = \mathscr{C}_V \oplus \mathscr{C}_m \oplus \mathscr{C}_M$$

of $\wedge^* T^*(\boldsymbol{P}(E))$ over $\boldsymbol{P}(E)$. Here $\mathscr{C}_V = \wedge^* V^{[*]}$ and $\mathscr{C}_M = \wedge^* \breve{\pi}^* T^*(M)$ while $\mathscr{C}_m$ is the sub-bundle of $\wedge^* T^*(\boldsymbol{P}(E))$ over $\boldsymbol{P}(E)$ consisting of the mixed components of $\wedge^* T^*(\boldsymbol{P}(E))$. Thus we have the following diagram

$$\begin{tikzcd}
\mathscr{C}_V & \wedge^* T^* \left(\boldsymbol{P}(E)\right) \arrow[l, "\Pi_{\mathscr{C}_V}"'] \arrow[r, "\Pi_{\mathscr{C}_M}"] \arrow[d, "\Pi_{\mathscr{C}_m}"] & \mathscr{C}_M \\
& \mathscr{C}_m &
\end{tikzcd}$$

of projection maps over $\boldsymbol{P}(E)$ such that $id. = \Pi_{\mathscr{C}_V} \oplus \Pi_{\mathscr{C}_m} \oplus \Pi_{\mathscr{C}_M}$ on $\wedge^* T^*(\boldsymbol{P}(E))$. Since the decomposition $T^*(\boldsymbol{P}(E)) = V^{[*]} \oplus \breve{\pi}^*(T^*(M))$ of $T^*(\boldsymbol{P}(E))$ is defined by the Einstein-Hermitian connection A on E over M we note that the representative $\frac{i \cdot F_{A_{L^*}}}{2\pi}$ of the Euler class $e(L^*)$ of L^* on $\boldsymbol{P}(E)$ has no nontrivial mixed components of $\wedge^* T^*(\boldsymbol{P}(E))$:

$$\frac{i \cdot F_{A_{L^*}}}{2\pi} = \Pi_{\mathscr{C}_V}\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right) \oplus \Pi_{\mathscr{C}_M}\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right).$$

Now we introduce a Hermitian form (metric) $\breve{\omega}$ on $\boldsymbol{P}(E)$ by setting

$$\breve{\omega} = \Pi_{\mathscr{C}_V}\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right) + \breve{\pi}^* \omega_M.$$

Note that the derivation operator $d : \Gamma(\boldsymbol{P}(E) : \boldsymbol{R}) \longrightarrow \Gamma(\boldsymbol{P}(E) : T^*(\boldsymbol{P}(E)) \otimes \boldsymbol{R})$ can be expressed as

$$d = d_V + d_M$$

in which $d_V : \Gamma(\boldsymbol{P}(E) : \boldsymbol{R}) \longrightarrow \Gamma(\boldsymbol{P}(E) : \boldsymbol{R} \otimes V^{[*]})$ and $d_M : \Gamma(\boldsymbol{P}(E) : \boldsymbol{R}) \longrightarrow \Gamma(\boldsymbol{P}(E) : \boldsymbol{R} \otimes \breve{\pi}^*(T^*(M)))$. Let d_V^* and d_M^* be respectively the adjoint operators of d_V and d_M with respect to the Hermitian form (metric) $\check{\omega}$ on $\boldsymbol{P}(E)$. Then we have

$$\Delta = d^* \circ d = \Delta_V + \Delta_M$$

in which $\Delta_V \equiv d_V^* \circ d_V$ and $\Delta_M \equiv d_M^* \circ d_M$. Similarly we have $\bar{\partial} = \bar{\partial}_V + \bar{\partial}_M$ and $\partial = \partial_V + \partial_M$. Let Λ_V and Λ_M be respectively the adjoint operators of

$$\bullet \longmapsto \left(\Pi_{\mathscr{C}_V} \frac{i \cdot F_{A_{L^*}}}{2\pi}\right) \wedge \bullet$$

and

$$\bullet \longmapsto \check{\pi}^* \omega_M \wedge \bullet$$

on $\boldsymbol{P}(E)$ with respect to the Hermitian form (metric) $\check{\omega}$. Then we have the following results (proved in the Appendix of [**8**]).

PROPOSITION II.A. *Given $f \in \Gamma(\boldsymbol{P}(E) : \boldsymbol{R})$ we have the following equalities $i \cdot \Lambda_V \circ \bar{\partial} \circ \partial f = \frac{\Delta_V f}{2}$ and $i \cdot \Lambda_M \circ \bar{\partial} \circ \partial f = \frac{\Delta_M f}{2}$.*

PROPOSITION II.B. $\Delta_M \circ \Delta_V = \Delta_V \circ \Delta_M$.

In particular we have $\Delta_M \circ (-4\pi n \cdot id. + \Delta_V) = (-4\pi n \cdot id. + \Delta_V) \circ \Delta_M$ and so Δ_M preserves $\Gamma(M : W)$. It has been shown in [**8**] that the invertibility of the linear partial differential operator Δ_M acting on $\Gamma(M : W)$ is equivalent to the simplicity of the holomorphic vector bundle E over M.

Let $\mathscr{V}_M$ denote the deformation operator for the constant scalar curvature equation on $(M : \omega_M)$:

$$\begin{aligned}\mathscr{V}_M \bullet &= i\bar{\partial}\partial\left(\tfrac{\Delta_M \bullet}{4\pi}\right) \wedge \tfrac{\omega_M^{(-1+m)}}{(-1+m)!} + \left[-\Lambda_M \mathrm{trace}\left(\tfrac{i \cdot F_{\omega_M}}{2\pi}\right)\right] \cdot i\bar{\partial}\partial \bullet \wedge \tfrac{\omega_M^{(-1+m)}}{(-1+m)!} + \\ &\quad i \cdot \bar{\partial}\partial \bullet \wedge \mathrm{trace}\left(\tfrac{i \cdot F_{\omega_M}}{2\pi}\right) \wedge \tfrac{\omega_M^{(-2+m)}}{(-2+m)!} \\ &= \tfrac{\Delta_M \circ \Delta_M \bullet}{8\pi} \cdot \tfrac{\omega_M^m}{m!} + \left[-\Lambda_M \mathrm{trace}\left(\tfrac{i \cdot F_{\omega_M}}{2\pi}\right)\right] \cdot \tfrac{\Delta_M \bullet}{2} \cdot \tfrac{\omega_M^m}{m!} + \\ &\quad i \cdot \bar{\partial}\partial \bullet \wedge \mathrm{trace}\left(\tfrac{i \cdot F_{\omega_M}}{2\pi}\right) \wedge \tfrac{\omega_M^{(-2+m)}}{(-2+m)!}.\end{aligned}$$

Here F_{ω_M} is the curvature form of the holomorphic tangent bundle of M induced by the Kaehler form ω_M on M. Hence $\Lambda_M \mathrm{trace}(\frac{i \cdot F_{\omega_M}}{2\pi})$ is the scalar curvature of

$(M:\omega_M)$. Let $\Gamma_o(M:\boldsymbol{R})$ denote the space of smooth $\boldsymbol{R}$-valued functions f on M satisfying $\int_M f\cdot\Omega_M=0$ in which $\Omega_M\equiv\frac{\omega_M^m}{m!}$. Then the linear partial differential operator

$$\frac{\mathscr{V}_M}{\Omega_M}=\frac{\mathscr{V}_M}{\frac{\omega_M^m}{m!}}$$

acting on $\Gamma_o(M:\boldsymbol{R})$ is both non-negative and symmetric (with respect to the Kaehler form ω_M on M). Note that the kernel of the linear partial differential operator $\mathscr{V}_M$ acting on $\Gamma_o(M:\boldsymbol{R})$ is isomorphic to the vector space

$$\frac{\mathfrak{h}_o(M)}{\mathfrak{k}_{(M:\omega_M)}}$$

over $\boldsymbol{R}$. Let $\mathbf{N}_{\mathscr{V}_M}$ denote the kernel of $\mathscr{V}_M$ acting on $\Gamma_o(M:\boldsymbol{R})$. We can then decompose the function space $\Gamma_o(M:\boldsymbol{R})$ into the direct sum of $\mathbf{N}_{\mathscr{V}_M}$ and the orthogonal complement of $\mathbf{N}_{\mathscr{V}_M}$ in $\Gamma_o(M:\boldsymbol{R})$. Thus for $f\in\Gamma_o(M:\boldsymbol{R})$ we have

$$f=\tau^{+}_{\mathbf{N}_{\mathscr{V}_M}}(f)\oplus\tau_{\mathbf{N}_{\mathscr{V}_M}}(f)$$

in which $\tau^{+}_{\mathbf{N}_{\mathscr{V}_M}}(f)$ is orthogonal to $\mathbf{N}_{\mathscr{V}_M}$ while $\tau_{\mathbf{N}_{\mathscr{V}_M}}(f)$ is the $\mathbf{N}_{\mathscr{V}_M}$-component of f: $\mathscr{V}_M\circ\tau_{\mathbf{N}_{\mathscr{V}_M}}(f)=0$.

Let ${}_oH_{\#k}$ denote the Kaehler metric on $\boldsymbol{P}(E)$ induced by the Kaehler form

$${}_o\omega_{\#k}\equiv\frac{i\cdot F_{A_{L^*}}}{2\pi}+k\cdot\check{\pi}^*\omega_M.$$

Suppose that, for each sufficiently large $k>0$, $\omega_{\#k}$ is a Kaehler form on $\boldsymbol{P}(E)$ lying in the Kaehler class $[{}_o\omega_{\#k}]$ so that

$$\omega_{\#k}={}_o\omega_{\#k}+i\cdot\bar{\partial}\partial\psi_k$$

with $\psi_k\in\Gamma(\boldsymbol{P}(E):\boldsymbol{R})$ satisfying $\int_{\boldsymbol{P}(E)}\psi_k\cdot\Omega_{\boldsymbol{P}(E)}=0\Longleftrightarrow\int_M\hat{\sigma}(\psi_k)\cdot\Omega_M=0$ in which $\Omega_M=\frac{\omega_M^m}{m!}$ and $\Omega_{\boldsymbol{P}(E)}\equiv\frac{\breve{\omega}^{(-1+m+n)}}{(-1+m+n)!}=\frac{(\frac{i\cdot F_{A_{L^*}}}{2\pi})^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^m}{m!}$. Let $\check{c}_k\in\boldsymbol{R}$, depending on the parameter $k>0$, denote the topological constant satisfying the following equality

$$\check{c}_k\cdot\int_{\boldsymbol{P}(E)}\frac{{}_o\omega_{\#k}^{(-1+m+n)}}{(-1+m+n)!}=\int_{\boldsymbol{P}(E)}\frac{i\cdot\bar{\partial}\partial\log\det{}_oH_{\#k}}{2\pi}\wedge\frac{{}_o\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}.$$

Let $H_{\#k}$ be the Kaehler metric on $\boldsymbol{P}(E)$ induced by the Kaehler form $\omega_{\#k}$. Then the Constant Scalar Curvature Equation for $\omega_{\#k}$ is

$$\mathscr{S}^A(\omega_{\#k}) = 0$$

in which

$$\mathscr{S}^A(\omega_{\#k}) \equiv -\check{c}_k \cdot \tfrac{\omega_{\#k}^{(-1+m+n)}}{(-1+m+n)!} + \tfrac{i \cdot \bar{\partial}\partial \log \det H_{\#k}}{2\pi} \wedge \tfrac{\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}.$$

Since the constant scalar curvature equation is invariant under the action of the group $\mathrm{Aut}(\boldsymbol{P}(E))$ of holomorphic automorphisms of $\boldsymbol{P}(E)$ we introduced in [**9**] the Gauge-Fixing Constant Scalar Curvature Equation as follows: Let

$$\begin{aligned}\mathscr{S}^A_{G\text{-}F}(\omega_{\#k}) &\equiv \mathscr{S}^A(\omega_{\#k}) + \tfrac{\tau_{\mathbf{N}_{\mathscr{V}_M}} \circ \hat{\sigma}(\psi_k)}{k \cdot k} \cdot k^m \cdot \Omega_{\boldsymbol{P}(E)} \\ &= -\check{c}_k \cdot \tfrac{\omega_{\#k}^{(-1+m+n)}}{(-1+m+n)!} + \tfrac{i \cdot \bar{\partial}\partial \log \det H_{\#k}}{2\pi} \wedge \tfrac{\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!} + \\ &\quad \tfrac{\tau_{\mathbf{N}_{\mathscr{V}_M}} \circ \hat{\sigma}(\psi_k)}{k \cdot k} \cdot k^m \cdot \Omega_{\boldsymbol{P}(E)}.\end{aligned}$$

We define the Gauge-Fixing Constant Scalar Curvature Equation, depending on large $k > 0$, as

$$\mathscr{S}^A_{G\text{-}F}(\omega_{\#k}) = 0.$$

This equation is not invariant under the action of the group $\mathrm{Aut}(\boldsymbol{P}(E))$ of holomorphic automorphisms of $\boldsymbol{P}(E)$ because the gauge-fixing term

$$\frac{\tau_{\mathbf{N}_{\mathscr{V}_M}} \circ \hat{\sigma}(\psi_k)}{k \cdot k} \cdot k^m \cdot \Omega_{\boldsymbol{P}(E)}$$

has been added to $\mathscr{S}^A(\omega_{\#k})$.

Let $\Gamma_o(\boldsymbol{P}(E) : \boldsymbol{R})$ denote the space of smooth $\boldsymbol{R}$-valued functions f on $\boldsymbol{P}(E)$ satisfying $\int_{\boldsymbol{P}(E)} f \cdot \Omega_{\boldsymbol{P}(E)} = 0 \iff \int_M \hat{\sigma}(f) \cdot \Omega_M = 0$. It has been shown in [**9**] that the gauge-fixing constant scalar curvature equation, depending on large $k > 0$, can be solved by considering $\psi_k \in \Gamma_o(\boldsymbol{P}(E) : \boldsymbol{R})$ admitting asymptotic expansion of the following form

$$\psi_k \sim \phi_0 + \sum_{p \in \boldsymbol{N}} \frac{\phi_p}{k^p}$$

as $k \to +\infty$. Here each $\phi_\bullet \in \Gamma_o(\boldsymbol{P}(E) : \boldsymbol{R})$ is a smooth $\boldsymbol{R}$-valued function, independent of the parameter k, on $\boldsymbol{P}(E)$. Besides the following Induction Condition

$$\sigma(\phi_0) = \tilde{\sigma}(\phi_0) = 0 = \tilde{\sigma}(\phi_1) \Longleftrightarrow \phi_0 \in \Gamma_o(M : \boldsymbol{R}) \text{ and } \phi_1 \in \Gamma_o(M : \boldsymbol{R}) \oplus \Gamma(M : W)$$

is imposed on the leading terms ϕ_0 and ϕ_1. Actually in [**9**] we suppose that $\mathscr{S}^A_{G\text{-}F}(\omega_{\#k})$ admits asymptotic expansion as $k \to \infty$:

$$\mathscr{S}^A_{G\text{-}F}(\omega_{\#k}) \sim k^m \cdot \left(\mathbf{B}^A_0 + \sum_{p \in \boldsymbol{N}} \frac{\mathbf{B}^A_p}{k^p} \right) \quad \text{as } k \to +\infty$$

in which each $\mathbf{B}^A_\bullet$ is independent of the parameter k. By substituting

$$\omega_{\#k} = {}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k \sim {}_o\omega_{\#k} + i\bar{\partial}\partial\phi_0 + \sum_{p \in \boldsymbol{N}} \frac{i\bar{\partial}\partial\phi_p}{k^p} \qquad \text{as } k \to +\infty$$

into the asymptotic expansion of $\mathscr{S}^A_{G\text{-}F}(\omega_{\#k})$ it can be shown that $\mathbf{B}^A_0 = 0 = \mathbf{B}^A_1$ provided the Induction Condition is satisfied by ϕ_0 and ϕ_1. Moreover in [**9**] it can be shown that all $\phi_p \in \Gamma_o(\boldsymbol{P}(E) : \boldsymbol{R})$ are uniquely determined through solving the family of equations $\mathbf{B}^A_{p+2} = 0$ by induction on p.

PROPOSITION II.C. *By choosing the Induction Condition*

$$\phi_0 \in \Gamma_o(M : \boldsymbol{R}) \text{ and } \phi_1 \in \Gamma_o(M : \boldsymbol{R}) \oplus \Gamma(M : W)$$

there exists a unique family of smooth $\boldsymbol{R}$-valued functions $\phi_p \in \Gamma_o(\boldsymbol{P}(E) : \boldsymbol{R})$ on $\boldsymbol{P}(E)$, depending on integers $p \geq 0$, such that $\mathbf{B}^A_p = 0$ for any integer $p \geq 0$.

Now for each large $N \in \boldsymbol{N}$ we define a Kaehler form ${}_N\omega_{\#k}$ on $\boldsymbol{P}(E)$, depending on large $k > 0$, as follows:

$$\begin{aligned} {}_N\omega_{\#k} &\equiv {}_o\omega_{\#k} + i\bar{\partial}\partial\phi_0 + \sum_{p \in \boldsymbol{N} \text{ with } p \leq N} \frac{i\bar{\partial}\partial\phi_p}{k^p} \\ &= \frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \omega_M + i\bar{\partial}\partial\phi_0 + \sum_{p \in \boldsymbol{N} \text{ with } p \leq N} \frac{i\bar{\partial}\partial\phi_p}{k^p}. \end{aligned}$$

Here each $\phi_\bullet$ is taken from the unique family of smooth $\boldsymbol{R}$-valued functions on

$\boldsymbol{P}(E)$ stated in Proposition II.C.

Given integer $\gamma \geq 0$ we define the Sobolev norm $\| \bullet \|_{H^{[2\gamma]}(\boldsymbol{P}(E):\breve{\omega})}$ of $\bullet$ as follows:

$$\| \bullet \|_{H^{[2\gamma]}(\boldsymbol{P}(E):\breve{\omega})} \equiv \| \bullet \|_{L^2(\boldsymbol{P}(E):\breve{\omega})} + \|(\Delta_V + \Delta_M) \bullet \|_{L^2(\boldsymbol{P}(E):\breve{\omega})} + \\ \cdots + \|(\Delta_V + \Delta_M)^{\gamma} \bullet \|_{L^2(\boldsymbol{P}(E):\breve{\omega})}.$$

Besides we denote by $H_o^{[2\gamma]}(\boldsymbol{P}(E) : \breve{\omega})$ the Sobolev space consisting of $\boldsymbol{R}$-valued functions $f \in H^{[2\gamma]}(\boldsymbol{P}(E) : \breve{\omega})$ on $\boldsymbol{P}(E)$ satisfying $\int_{\boldsymbol{P}(E)} f \cdot \Omega_{\boldsymbol{P}(E)} = 0$. Then we have the following result proved in [**9**].

THEOREM II.A. *When the parameter $k > 0$ is sufficiently large the corresponding gauge-fixing constant scalar curvature equation $\mathscr{S}^A_{G\text{-}F}({}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k) = 0$ can be solved by some smooth $\boldsymbol{R}$-valued function $\psi_k \in \Gamma_o(\boldsymbol{P}(E) : \boldsymbol{R})$ on $\boldsymbol{P}(E)$. Besides this family of smooth $\boldsymbol{R}$-valued functions $\psi_k \in \Gamma_o(\boldsymbol{P}(E) : \boldsymbol{R})$ on $\boldsymbol{P}(E)$ admits asymptotic expansion of the following form*

$$\psi_k \sim \phi_0 + \sum_{p \in \boldsymbol{N}} \frac{\phi_p}{k^p}$$

as $k \to +\infty$. Here each $\phi_\bullet$ is taken from the unique family of smooth $\boldsymbol{R}$-valued functions on $\boldsymbol{P}(E)$ stated in Proposition II.C. Actually, for each pair $(\gamma : q) \in \boldsymbol{N} \times \boldsymbol{N}$ of large enough integers, we may even require, when $N \in \boldsymbol{N}$ is chosen sufficiently large, that

$${}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k = {}_N\omega_{\#k} + i\bar{\partial}\partial\psi_{(k:N)}$$

with $\psi_{(k:N)} \in \Gamma_o(\boldsymbol{P}(E) : \boldsymbol{R})$ satisfying $\|\psi_{(k:N)}\|_{H^{[2\gamma+4]}(\boldsymbol{P}(E):\breve{\omega})} \leq \frac{1}{k^q}$ whenever $k > 0$ is large enough. In this case the choice of the solution

$${}_N\omega_{\#k} + i\bar{\partial}\partial\psi_{(k:N)}$$

for the gauge-fixing constant scalar curvature equation $\mathscr{S}^A_{G\text{-}F}({}_N\omega_{\#k} + i\bar{\partial}\partial\psi_{(k:N)}) = 0$ with $\psi_{(k:N)} \in \Gamma_o(\boldsymbol{P}(E) : \boldsymbol{R})$ satisfying $\|\psi_{(k:N)}\|_{H^{[2\gamma+4]}(\boldsymbol{P}(E):\breve{\omega})} \leq \frac{1}{k^q}$ is, for each sufficiently large k, unique.

Actually (in Section V of [**9**]) the proof of this theorem is based on the Contraction Mapping Theorem and the invertibility of the linearization of $\mathscr{S}^A_{G\text{-}F}$ at approximate solutions ${}_N\omega_{\#k}$.

We will now compute the detailed expression of $\hat{\sigma}(\frac{\mathbf{B}_2^A}{\Omega_{\boldsymbol{P}(E)}})$ to find out the nontrivial leading term in the asymptotic expansion of $\frac{\mathscr{S}^A({}_o\omega_{\#k}+i\bar{\partial}\partial\psi_k)}{\Omega_{\boldsymbol{P}(E)}}$ as $k \to +\infty$. Let ξ_A denote the smooth $(-1+m+n)$-form on $\boldsymbol{P}(E)$ defined by A as follows:

$$\xi_A \equiv (n+n) \cdot \frac{\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right)^{(n+1)}}{(n+1)!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!} +$$
$$\check{\pi}^* \mathrm{trace}\left(\frac{i \cdot F_A}{2\pi}\right) \wedge \frac{\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right)^{n}}{n!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!} +$$
$$\check{\pi}^* \mathrm{trace}\left(\frac{i \cdot F_{\omega_M}}{2\pi}\right) \wedge \frac{\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right)^{n}}{n!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}.$$

Let $\frac{\xi_A}{\Omega_{\boldsymbol{P}(E)}}$ denote the smooth $\boldsymbol{R}$-valued function on $\boldsymbol{P}(E)$ satisfying $\xi_A = \frac{\xi_A}{\Omega_{\boldsymbol{P}(E)}} \cdot \Omega_{\boldsymbol{P}(E)}$. Then, using the constancy of the scalar curvature of $(M : \omega_M)$ and the Einstein-Hermitian condition satisfied by A, it can be shown that [**9**, p. 428]

$$\hat{\sigma}\left(\frac{\mathbf{B}_2^A}{\Omega_{\boldsymbol{P}(E)}}\right) = \left(\frac{\mathscr{V}_M}{\Omega_M} + \tau_{\mathbf{N}_{\mathscr{V}_M}}\right)\phi_0 + \hat{\sigma}\left(-c_{\xi_A} + \frac{\xi_A}{\Omega_{\boldsymbol{P}(E)}}\right)$$

in which $\phi_0 \in \Gamma_o(M : \boldsymbol{R})$ is the smooth $\boldsymbol{R}$-valued function on M described in Proposition II.C while $c_{\xi_A} \in \boldsymbol{R}$ is the constant satisfying $c_{\xi_A} \cdot \int_{\boldsymbol{P}(E)} \Omega_{\boldsymbol{P}(E)} = \int_{\boldsymbol{P}(E)} \xi_A$.

LEMMA II.A. *Let $H_{\boldsymbol{C}^n} = \delta_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta$ denote the standard Hermitian metric on $\boldsymbol{C}^n$. Assume that C and D are $n \times n$ matrices defining smooth functions*

$$Q_C = \frac{C_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta}{H_{\boldsymbol{C}^n}} \quad \text{and} \quad Q_D = \frac{D_{\alpha\beta} \cdot w_\alpha \cdot \bar{w}_\beta}{H_{\boldsymbol{C}^n}}$$

on the projective space $\boldsymbol{P}(\boldsymbol{C}^n)$ endowed with the Fubini-Study Kaehler form $\omega_{F\text{-}S}$. Then we have the following results:

$$\int_{\boldsymbol{P}(\boldsymbol{C}^n)} Q_C = \int_{\boldsymbol{P}(\boldsymbol{C}^n)} \frac{\mathrm{trace}\, C}{n}$$

and

$$\int_{\boldsymbol{P}(\boldsymbol{C}^n)} Q_C \cdot Q_D = \int_{\boldsymbol{P}(\boldsymbol{C}^n)} \frac{\operatorname{trace}(CD) + (\operatorname{trace} C) \cdot (\operatorname{trace} D)}{n + n^2}.$$

PROOF. These results can be proved through direct computation following the method used in the Appendix of [**8**]. □

Let Ξ_A denote the smooth m-form on M defined by A as follows:

$$\begin{aligned}\Xi_A = &- \frac{\operatorname{trace}\left(\frac{i \cdot F_A}{2\pi}\right) \wedge \operatorname{trace}\left(\frac{i \cdot F_{\omega_M}}{2\pi}\right) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{n} + \\ &- \frac{\operatorname{trace}\left(\frac{i \cdot F_A}{2\pi}\right) \wedge \operatorname{trace}\left(\frac{i \cdot F_A}{2\pi}\right) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{n \cdot (n+1)} + \\ &\frac{\operatorname{trace}\left(\frac{i \cdot F_A}{2\pi} \wedge \frac{i \cdot F_A}{2\pi}\right) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{(n+1)}.\end{aligned}$$

Now using Lemma II.A and the following decomposition of $\frac{i \cdot F_{A_{L^*}}}{2\pi}$: $\frac{i \cdot F_{A_{L^*}}}{2\pi} = \Pi_{\mathscr{C}_V}(\frac{i \cdot F_{A_{L^*}}}{2\pi}) \oplus \Pi_{\mathscr{C}_M}(\frac{i \cdot F_{A_{L^*}}}{2\pi})$ it can be shown that $\hat{\sigma}(\frac{\xi_A}{\Omega_{\boldsymbol{P}(E)}}) \cdot \Omega_M = \Xi_A$. Hence

$$\hat{\sigma}\left(\frac{\mathbf{B}_2^A}{\Omega_{\boldsymbol{P}(E)}}\right) \cdot \Omega_M = \mathscr{V}_M \phi_0 + (\tau_{\mathbf{N}_{\mathscr{V}_M}} \phi_0) \cdot \Omega_M + (-c_{\xi_A} \cdot \Omega_M + \Xi_A).$$

COROLLARY II.A. *Let $g_{\langle \Xi_A \rangle}$ denote the smooth $\boldsymbol{R}$-valued function on M satisfying*

$$\Xi_A = g_{\langle \Xi_A \rangle} \cdot \Omega_M.$$

Assume that, for each large k, ${}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k$ is the solution, described in Theorem II.A, *for the gauge-fixing constant scalar curvature equation $\mathscr{S}^A_{G\text{-}F}({}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k) = 0$. Then $\frac{\mathscr{S}^A({}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k)}{\Omega_{\boldsymbol{P}(E)}}$ stays in $\mathbf{N}_{\mathscr{V}_M}$ and admits asymptotic expansion*:

$$\frac{\mathscr{S}^A({}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k)}{\Omega_{\boldsymbol{P}(E)}} \sim k^m \cdot \left(\frac{\tau_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\Xi_A} + g_{\langle \Xi_A \rangle})}{k \cdot k} + \textit{higher order terms}\right)$$

as $k \to +\infty$. Here $c_{\Xi_A} \in \boldsymbol{R}$ is the constant satisfying $c_{\Xi_A} \cdot \int_M \Omega_M = \int_M \Xi_A$.

PROOF. Note that $c_{\Xi_A} = c_{\xi_A}$ and

$$\hat{\sigma}\Big(\frac{\mathbf{B}_2^A}{\Omega_{\boldsymbol{P}(E)}}\Big) \cdot \Omega_M = \mathscr{V}_M \phi_0 + \tau^+_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\xi_A} + g_{\langle \Xi_A \rangle}) \cdot \Omega_M + \\ \tau_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\xi_A} + g_{\langle \Xi_A \rangle}) \cdot \Omega_M + (\tau_{\mathbf{N}_{\mathscr{V}_M}} \phi_0) \cdot \Omega_M$$

in which $\phi_0 \in \Gamma_o(M : \boldsymbol{R})$ is the smooth $\boldsymbol{R}$-valued function on M described in Proposition II.C. Since $\mathbf{B}_2^A = 0$ it can be inferred from the above equality that $\phi_0 \in \Gamma_o(M : \boldsymbol{R})$ must satisfy

$$\mathscr{V}_M \phi_0 + \tau^+_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\xi_A} + g_{\langle \Xi_A \rangle}) \cdot \Omega_M = 0$$

and so $\hat{\sigma}(\frac{\mathbf{B}_2^A}{\Omega_{\boldsymbol{P}(E)}})$ can be expressed as follows:

$$\hat{\sigma}\Big(\frac{\mathbf{B}_2^A}{\Omega_{\boldsymbol{P}(E)}}\Big) = \tau_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\Xi_A} + g_{\langle \Xi_A \rangle}) + \tau_{\mathbf{N}_{\mathscr{V}_M}} \phi_0.$$

Since the family of smooth $\boldsymbol{R}$-valued functions $\psi_k \in \Gamma_o(\boldsymbol{P}(E) : \boldsymbol{R})$ on $\boldsymbol{P}(E)$ admits, as $k \to +\infty$, asymptotic expansion of the following form $\psi_k \sim \phi_0 + \sum_{p \in \boldsymbol{N}} \frac{\phi_p}{k^p}$ and

$$\mathscr{S}^A_{G\text{-}F}({}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k) = \mathscr{S}^A({}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k) + \frac{\tau_{\mathbf{N}_{\mathscr{V}_M}} \circ \hat{\sigma}(\psi_k)}{k \cdot k} \cdot k^m \cdot \Omega_{\boldsymbol{P}(E)}$$

it follows that the leading term of the asymptotic expansion of $\frac{\mathscr{S}^A({}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k)}{\Omega_{\boldsymbol{P}(E)}}$, as $k \to +\infty$, is $k^m \cdot \frac{\tau_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\Xi_A} + g_{\langle \Xi_A \rangle})}{k \cdot k}$ and so Corollary II.A is true. $\square$

III. Existence of critical Kaehler metrics on ruled manifolds.

Given $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ we consider the Gauge-Fixing Constant Scalar Curvature Equation $\mathscr{S}^{\mathbf{A}}_{G\text{-}F} \bullet = 0$ on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$, depending on sufficiently large $k > 0$, with $\bullet$ being a Kaehler form on $\boldsymbol{P}_{\mathbf{A}}(\mathbf{E})$ lying in the Kaehler class $-e(L_{\mathbf{A}}) + k \cdot [\breve{\pi}^*_{\mathbf{A}} \omega_M]$. Let ${}_o\omega^{\mathbf{A}}_{\#k} + i \cdot \bar{\partial} \circ \partial \psi^{\mathbf{A}}_k$ denote the solution for $\mathscr{S}^{\mathbf{A}}_{G\text{-}F} \bullet = 0$ found in Theorem II.A for the Gauge-Fixing Constant Scalar Curvature Equation (depending on sufficiently large $k > 0$): $\mathscr{S}^{\mathbf{A}}_{G\text{-}F}({}_o\omega^{\mathbf{A}}_{\#k} + i \cdot \bar{\partial} \circ \partial \psi^{\mathbf{A}}_k) = 0$. Let $t = \frac{1}{k}$. Since the asymptotic expansion, described in Theorem II.A, is uniform, as mentioned in the Introduction of this article, it follow that for each small open subset U of $\mathscr{A}^{\text{E-H}}_{\text{simple}}$ around $\mathbf{A} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ there exists a corresponding $\epsilon_U > 0$ such that

$$t \cdot \left({}_o\omega^{\mathbf{B}}_{\#\frac{1}{t}} + i \cdot \overline{\partial} \circ \partial \psi^{\mathbf{B}}_{\frac{1}{t}} \right)$$

depends smoothly on $(\mathbf{B} : t) \in \overline{U} \times [0 : \epsilon_U)$. In particular, according to Corollary II.A, we have

$$\frac{\mathscr{S}^{\mathbf{B}}\left({}_o\omega^{\mathbf{B}}_{\#\frac{1}{t}} + i \cdot \bar{\partial} \circ \partial\psi^{\mathbf{B}}_{\frac{1}{t}}\right)}{\Omega_{\boldsymbol{P}_{\mathbf{B}}(\mathbf{E})}} \sim \frac{t^2 \cdot \tau_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\Xi_{\mathbf{B}}} + g_{\langle\Xi_{\mathbf{B}}\rangle}) + \text{higher order terms}}{t^m}$$

uniformly for $\mathbf{B} \in \overline{U}$ as $t \to 0$. Let $\mathbf{v}_{\boldsymbol{P}(\boldsymbol{C}^n)} > 0$ denote the volume of $\boldsymbol{P}(\boldsymbol{C}^n)$ endowed with the Fubini-Study Kaehler form. Since $\bullet = \tau^{+}_{\mathbf{N}_{\mathscr{V}_M}}(\bullet) \oplus \tau_{\mathbf{N}_{\mathscr{V}_M}}(\bullet)$ for each $\bullet \in \Gamma_o(M : \boldsymbol{R})$ it follows that at each $\mathbf{B} \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ we have

$$\begin{aligned}\int_{\boldsymbol{P}_{\mathbf{B}}(\mathbf{E})} \tau_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\Xi_{\mathbf{B}}} + g_{\langle\Xi_{\mathbf{B}}\rangle}) \cdot f_X \cdot \Omega_{\boldsymbol{P}_{\mathbf{B}}(\mathbf{E})} &= \mathbf{v}_{\boldsymbol{P}(\boldsymbol{C}^n)} \cdot \int_M \tau_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\Xi_{\mathbf{B}}} + g_{\langle\Xi_{\mathbf{B}}\rangle}) \cdot f_X \cdot \Omega_M \\ &= \mathbf{v}_{\boldsymbol{P}(\boldsymbol{C}^n)} \cdot \int_M (-c_{\Xi_{\mathbf{B}}} \cdot \Omega_M + \Xi_{\mathbf{B}}) \cdot f_X \\ &= \frac{\mathbf{v}_{\boldsymbol{P}(\boldsymbol{C}^n)}}{n+1} \cdot \mu_X|_{\mathbf{B}} \quad \forall X \in \mathfrak{k}_{(M:\omega_M)}\end{aligned}$$

in which $f_X \in \Gamma(M : \boldsymbol{R})$ the unique smooth $\boldsymbol{R}$-valued function on M satisfying $i_X\omega_M = d\,f_X$ and $\int_M f_X \cdot \frac{\omega_M^m}{m!} = 0$.

Let $\mathrm{I}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}$ denote the natural complex structure on $\mathscr{M}^{\text{E-H}}_{\text{simple}}$. When the orbit $\mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\mathbf{A}])$ in $\mathscr{M}^{\text{E-H}}_{\text{simple}}$ is stable (in the sense of Definition I.A) with $[\mathbf{A}_\infty] \in \mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\mathbf{A}])$ being a non-degenerate zero point for μ we have

$$\frac{\mathbf{v}_{\boldsymbol{P}(\boldsymbol{C}^n)}}{n+1} \cdot \mu_X|_{\mathbf{A}_\infty} = \int_{\boldsymbol{P}_{\mathbf{A}_\infty}(\mathbf{E})} \tau_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\Xi_{\mathbf{A}_\infty}} + g_{\langle\Xi_{\mathbf{A}_\infty}\rangle}) \cdot f_X \cdot \Omega_{\boldsymbol{P}_{\mathbf{A}_\infty}(\mathbf{E})} = 0 \quad \forall X \in \mathfrak{k}_{(M:\omega_M)}.$$

Since $\{X \in \mathfrak{k}_{(M:\omega_M)} : [\theta_X]|_{[\mathbf{A}_\infty]} = 0\} = \langle 0 \rangle$ it follows that the pairing

$$(d\mu_X)\mathrm{I}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}[\theta_Y] = \omega_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\theta_X] : \mathrm{I}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}[\theta_Y]) \quad \forall (X : Y) \in \mathfrak{k}_{(M:\omega_M)} \times \mathfrak{k}_{(M:\omega_M)}$$

at $[\mathbf{A}_\infty]$ is non-degenerate. Hence it can be inferred readily from the Implicit Function Theorem, through Corollary II.A,

$$\int_{\boldsymbol{P}_{\mathbf{B}}(\mathbf{E})} \tau_{\mathbf{N}_{\mathscr{V}_M}}(-c_{\Xi_{\mathbf{B}}} + g_{\langle\Xi_{\mathbf{B}}\rangle}) \cdot f_X \cdot \Omega_{\boldsymbol{P}_{\mathbf{B}}(\mathbf{E})} = \frac{\mathbf{v}_{\boldsymbol{P}(\boldsymbol{C}^n)}}{n+1} \cdot \mu_X|_{\mathbf{B}} \quad \forall X \in \mathfrak{k}_{(M:\omega_M)},$$

and the non-degeneracy of μ at $[\mathbf{A}_\infty]$ that, there exists a family of Einstein-Hermitian connections

$$\{\mathbf{A}_k \in \mathscr{A}^{\text{E-H}}_{\text{simple}} : [\mathbf{A}_k] \in \mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\mathbf{A}_\infty]) \text{ with } k > 0 \text{ being sufficiently large}\}$$

on $\mathbf{E}$ with $\lim_{k\to+\infty} \mathbf{A}_k = \mathbf{A}_\infty$ such that $\frac{\mathscr{S}^{\mathbf{A}_k}\left({}_o\omega^{\mathbf{A}_k}_{\#\frac{1}{t}} + i\cdot\bar{\partial}\circ\partial\psi^{\mathbf{A}_k}_{\frac{1}{t}}\right)}{\Omega_{\boldsymbol{P}_{\mathbf{A}_k}(\mathbf{E})}} = 0$ in which $t = \frac{1}{k}$.

THEOREM III.A. *Assume that $\mathbf{E}$ is a smooth complex vector bundle of rank n endowed with a Hermitian metric $H_{\mathbf{E}}$ over the compact Kaehler manifold $(M : \omega_M)$ with constant scalar curvature. Let $\mathbf{A}_{\wedge^n\mathbf{E}}$ be a fixed Einstein-Hermitian connection on $\wedge^n\mathbf{E}$, compatible with the Hermitian metric $H_{\mathbf{E}}$ on $\mathbf{E}$, defining holomorphic structure on $\wedge^n\mathbf{E}$. Suppose that $\mathbf{A}_\infty \in \mathscr{A}^{\text{E-H}}_{\text{simple}}$ and the orbit $\mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\mathbf{A}_\infty])$ in $\mathscr{M}^{\text{E-H}}_{\text{simple}}$ is stable (in the sense of Definition* I.A*) with $[\mathbf{A}_\infty] \in \mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\mathbf{A}_\infty])$ being a non-degenerate zero point for μ. Let $t = \frac{1}{k}$. Then there exists a family of Einstein-Hermitian connections*

$$\{\mathbf{A}_k \in \mathscr{A}^{\text{E-H}}_{\text{simple}} : [\mathbf{A}_k] \in \mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\mathbf{A}_\infty]) \textit{ with } k > 0 \textit{ being sufficiently large}\}$$

on $\mathbf{E}$ with $\lim_{k\to+\infty} \mathbf{A}_k = \mathbf{A}_\infty$ such that each ${}_o\omega^{\mathbf{A}_k}_{\#k} + i\cdot\bar{\partial}\circ\partial\psi^{\mathbf{A}_k}_k = {}_o\omega^{\mathbf{A}_k}_{\#\frac{1}{t}} + i\cdot\bar{\partial}\circ\partial\psi^{\mathbf{A}_k}_{\frac{1}{t}}$ defines a Kaehler form on $\boldsymbol{P}_{\mathbf{A}_k}(\mathbf{E})$ with constant scalar curvature.

By switching our approach to the other but equivalent one (varying the Hermitian metric but fixing the holomorphic structure on a vector bundle) as explained in page 210, lines 15–25, of [**6**] we obtain the following result:

COROLLARY III.A. *Let $\pi : E \longrightarrow M$ be a simple holomorphic vector bundle of rank n endowed with Einstein-Hermitian metric H^∞_E over the compact Kaehler manifold $(M : \omega_M)$ with constant scalar curvature. Let $\mathbf{E}$ denote the smooth complex vector bundle E over M endowed with the Hermitian metric $H_{\mathbf{E}} = H^\infty_E$ forgetting the holomorphic structure on E. Let A_∞ denote the Einstein-Hermitian connection on E induced by H^∞_E. Let $\boldsymbol{P}_{A_\infty}(\mathbf{E})$ denote the projectivization of $\mathbf{E}$ over M endowed with the holomorphic structure defined by A_∞. Suppose that the orbit $\mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([A_\infty])$ in $\mathscr{M}^{\text{E-H}}_{\text{simple}}$ is stable (in the sense of Definition* I.A*) with $[A_\infty] \in \mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([A_\infty])$ being a non-degenerate zero point for μ. Let $t = \frac{1}{k}$. Let*

$$\{\mathbf{A}_k \in \mathscr{A}^{\text{E-H}}_{\text{simple}} : [\mathbf{A}_k] \in \mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([A_\infty]) \textit{ with } k > 0 \textit{ being sufficiently large}\}$$

denote the family of Einstein-Hermitian connections on $\mathbf{E}$, stated in Theorem III.A*, with $\lim_{k\to+\infty} \mathbf{A}_k = A_\infty$ such that each ${}_o\omega^{\mathbf{A}_k}_{\#k} + i\cdot\bar{\partial}\circ\partial\psi^{\mathbf{A}_k}_k = {}_o\omega^{\mathbf{A}_k}_{\#\frac{1}{t}} + i\cdot\bar{\partial}\circ\partial\psi^{\mathbf{A}_k}_{\frac{1}{t}}$ defines a Kaehler form on $\boldsymbol{P}_{\mathbf{A}_k}(\mathbf{E})$ with constant scalar curvature. Then there exists a corresponding family of holomorphic diffeomorphism maps*

$$\Phi_{\frac{1}{t}} : \boldsymbol{P}_{A_\infty}(\mathbf{E}) \longrightarrow \boldsymbol{P}_{\mathbf{A}_{\frac{1}{t}}(\mathbf{E})}$$

preserving the complex vector bundle structure of $\mathbf{E}$ *over* M*, depending smoothly on* $t \in [0 : \epsilon)$ *for some* $\epsilon > 0$*, with* $\Phi_\infty : \boldsymbol{P}_{A_\infty}(\mathbf{E}) \longrightarrow \boldsymbol{P}_{A_\infty}(\mathbf{E})$ *being the identity map such that, for each* $t \in [0 : \epsilon)$*, the holomorphic diffeomorphism map* $\hat{\Phi}_{\frac{1}{t}} : M \longrightarrow M$ *induced by* $\Phi_{\frac{1}{t}}$*, which makes the following diagram*

$$\begin{array}{ccc} \boldsymbol{P}_{A_\infty}(\mathbf{E}) & \xrightarrow{\Phi_{\frac{1}{t}}} & \boldsymbol{P}_{\mathbf{A}_{\frac{1}{t}}}(\mathbf{E}) \\ \Big\downarrow & & \Big\downarrow \\ M & \xrightarrow{\hat{\Phi}_{\frac{1}{t}}} & M \end{array}$$

commutative, is actually an element of the Lie group of holomorphic transformations of M *generated by* $\mathfrak{h}_o^R(M)$*. In particular*

$$\left\{ \Phi^*_{\frac{1}{t}} \left({}_o\omega^{\mathbf{A}_k}_{\#\frac{1}{t}} + i \cdot \bar{\partial} \circ \partial \psi^{\mathbf{A}_k}_{\frac{1}{t}} \right) : t \in (0 : \epsilon) \right\}$$

is a smooth family of Kaehler forms on $\boldsymbol{P}_{A_\infty}(\mathbf{E})$ *carrying constant scalar curvature with each* $\Phi^*_{\frac{1}{t}} \left({}_o\omega^{\mathbf{A}_k}_{\#\frac{1}{t}} + i \cdot \bar{\partial} \circ \partial \psi^{\mathbf{A}_k}_{\frac{1}{t}} \right)$ *lying in the Kaehler class* $-e(L_{A_\infty}) + k \cdot [\breve{\pi}^*_{A_\infty} \omega_M]$*.*

Proof. Corollary III.A follows from the fact that the orbit $\mathbf{O}_{\mathscr{M}^{\text{E-H}}_{\text{simple}}}([\mathbf{A}_\infty])$ is generated in $\mathscr{M}^{\text{E-H}}_{\text{simple}}$ by the action of $\mathfrak{h}_o^R(M)$ on $\mathbf{E}$. $\square$

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