J. Math. Soc. Japan
Vol. 60, No. 1 (2008) pp. 51–64
doi: 10.2969/jmsj/06010051

Common maxima of distance functions on orientable Alexandrov surfaces

By Costin Vîlcu

(Received Nov. 30, 2005)
(Revised Feb. 7, 2007)

Abstract. We find properties of the sets M_y^{-1} of all points on a compact orientable Alexandrov surface S, the distance functions of which have a common maximum at $y \in S$. For example, the components of M_y^{-1} are arcwise connected and their number is at most $\max\{1, 10g-5\}$, where g is the genus of S. A special attention receives the case of local tree components of M_y^{-1}, providing a relationship to the unit tangent cone at y.

1. Introduction.

In this paper, by surface we always mean a compact 2-dimensional Alexandrov space with curvature bounded below (without boundary), as defined by Burago, Gromov and Perelman in [**3**]. It is well-known that our surfaces are topological manifolds. We refer the reader to [**3**], [**10**], [**11**] for basic facts on surfaces, such as convergence theorems on shortest paths or on angles, the generalized Toponogov theorem, and a description of the structure of the cut loci. Let $\mathscr{A}$ be the space of all surfaces.

For any two points x, y on the surface S, denote by $\rho(x,y)$ the geodesic distance between them, and by ρ_x the distance function from x, $\rho_x(y) = \rho(x,y)$. Let M_x denote the set of all relative maxima of ρ_x, and M the naturally induced multivalued mapping, associating to any point $x \in S$ the set M_x. Similarly, F_x is the set of all farthest points from x (absolute maxima of ρ_x), Q_x the set of all critical points with respect to ρ_x, and F, Q, are the corresponding multivalued mappings.

As usual, the point $y \in S$ is called *critical* with respect to ρ_x if for any vector v tangent to S at y there exists a segment from y to x whose direction at y makes an angle not larger than $\pi/2$ with v. For an interesting presentation of the principles, as well as the applications, of the critical point theory for distance functions, see the survey [**5**] by K. Grove.

2000 *Mathematics Subject Classification.* Primary 53C45.
Key Words and Phrases. Alexandrov surface, distance function.

Properties of the mappings M, Q and F have recently been obtained in [**1**], [**6**] and [**14**]. Various results concerning the mapping F on convex surfaces can be found in the survey [**13**], which also announces some results of this paper, in the particular framework of convex surfaces.

For any surface S, the space T_y of all unit tangent directions at $y \in S$ is a closed Jordan curve of length λT_y at most 2π [**3**]. Call the point y *conical* if $\lambda T_y < 2\pi$, and smooth otherwise.

If f is a multivalued mapping defined on S with the set f_x as image of x, put $f_y^{-1} = \{x \in S : y \in f_x\}$.

This study focuses on the sets M_y^{-1} and, since $M_y^{-1} \subset Q_y^{-1}$ (see Lemma 3), it complements properties of the sets Q_y^{-1} obtained in [**1**]; for example [**1**], if S is orientable of genus g and y is a smooth point in S then $1 \leq \mathrm{card} Q_y^{-1} \leq \max\{1, 8g-4\}$, and an easy example shows the existence of conical points y with infinitely many "inverse critical points".

Our Theorem 1 is valid for orientable surfaces. Roughly speaking, it states that the components of the sets M_y^{-1} are arcwise connected, and further describes these components. A main consequence of these properties is Theorem 2: for every orientable surface S of genus g and every point y in S, M_y^{-1} has at most $\max\{1, 10g-5\}$ components. If moreover $\lambda T_y > \pi$ then M_y^{-1} is a local tree (a tree if $g = 0$), with at most $2g$ generating cycles and less than $\frac{\lambda T_y}{\lambda T_y - \pi}$ extremities outside the cycles of the cut locus of y (Theorem 5). Theorem 4 prepares a part of Theorem 5 by proving it, slightly more generally, for the sets Q_y^{-1}, while Theorem 3 shows that every finite tree can be realized as the set F_y^{-1}, for some point y on some surface $S \in \mathscr{A}$. Finally, Theorem 6 expresses the dependence of λT_y on M_y^{-1} and has a nice consequence in Corollary 3: a convex surface contains at most 7 points y such that M_y^{-1} is a tree with at least 3 extremities. Several examples are given to complete the presentation.

The properties of a set M_y^{-1} are not necessarily inherited by its subsets. Nevertheless, one can easily see, following the proofs, that F_y^{-1} enjoys as well these properties. (This is why Theorem 3 deals with the mapping F^{-1}, instead of M^{-1} as all other results.) Therefore, our work can also be regarded as treating global maxima, and thus it contributes to a description of the farthest points H. Steinhaus had asked for (see Section A35 in [**4**]).

Throughout this paper, by segment we mean a shortest path between its extremities. The *cut locus* $C(x)$ of a point x in S is the set of all endpoints different from x, called cut points, of maximal (with respect to inclusion) segments starting at x. It is known that $C(x)$ is a *local tree* (that is, each of its points z has a neighbourhood V in S where the component $K_z(V)$ of z is a tree), even a tree if $g = 0$.

There are points x on surfaces, the cut locus of which is dense in the surface.

(A large class of examples is provided, for example, by Theorem 4 in [**16**].) It will turn out that, for our study, very important is the *cyclic part* $C^{cp}(x)$ of $C(x)$. It is the minimal (with respect to inclusion) subset of $C(x)$, the exclusion of which from S provides a topological disk. For every point x in every surface $S \in \mathscr{A}$, $C^{cp}(x)$ is a local tree with finitely many vertices [**9**], and each component of $C \setminus C^{cp}(x)$ is a tree.

Recall that a *tree* is a set T any two points of which can be joined by a unique Jordan arc included in T. The *degree* of a point y of a local tree is the number of components of $K_y(V) \setminus \{y\}$, if V is chosen such that $K_y(V)$ is a tree. A point $y \in T$ is called an *extremity* of T if it has degree 1, and a *ramification point* of T if it has degree at least 3. An *internal edge* of T is a Jordan arc which connects ramification points of T.

For a set $M \subset S$, $\mathrm{cl}M$, $\mathrm{int}M$ and $\mathrm{card}M$ stand – as usually – for the closure, the interior and the cardinality of M, respectively. We denote by λG the length of the curve G, by $B(x,r)$ the open intrinsic ball of radius r centered at $x \in S$ and by $[xv]$ the line-segment determined by the points $x, v \subset \mathbf{R}^2$.

2. General properties of M_y^{-1}.

The goal of this section is to characterize the components of M_y^{-1}, via Theorem 1 and its consequences. The proof of Theorem 1 makes use of several lemmas, with which we start.

Lemma 1. *Let $S \in \mathscr{A}$ and $y \in S$. Suppose the points $v, z \in C(y)$ are each joined to y by two (possibly coinciding) segments γ^1_{vy}, γ^2_{vy} and respectively γ^1_{zy}, γ^2_{zy}, the union of which cuts off from S a closed set Δ contractible to a topological circle. Then there exists a Jordan arc $J_{vz} \subset C(y)$ joining v to z, every interior point of which belongs to Δ and can be joined to y by two segments, the union of which separates v from z in Δ.*

Proof. The existence of the Jordan arc $J_{vz} \subset C(y)$ joining v to z follows from the properties of $C(y)$ (see, for example, [**11**]). The separability was established, for convex surfaces, by Lemma 1 in [**15**]. The arguments therein also hold under our more general assumptions, and will not be repeated here. □

Next result was implicitly established for convex surfaces within the proof of Theorem 5 in [**12**], but the same arguments are valid in a more general framework.

Lemma 2. *Let (A, ρ) be an Alexandrov space with curvature bounded below, and γ_{ac}, γ_{bd} be segments joining the points $a, c \in A$ and respectively $b, d \in A$. If $\gamma_{ac} \cap \gamma_{bd} = \{e\}$ and $\rho(a,b) + \rho(c,d) \geq \rho(a,c) + \rho(b,d)$, then $a = d$, or $b = c$, or*

$a = c$, *or* $b = d$.

The following statement is easily proven using Proposition 2.4 in [**11**].

LEMMA 3. *On* $S \in \mathscr{A}$, *let* γ, γ' *be (possibly coinciding) segments from* x *to* y *and* D *a component of the complement of* $\gamma \cup \gamma'$ *in an open disc around* y. *If the angle of* γ, γ' *at* y *toward* D *is smaller than* π *then there exists* $\varepsilon > 0$ *such that* y *is a strict maximum for the restriction of the distance function* ρ_x *to* $B(y, \varepsilon\rho(x, y)) \cap D$. *Conversely, if* $y \in M_x$ *then* $y \in Q_x$; *in particular, if* $\lambda T_y > \pi$ *then there are at least two segments from* x *to* y.

The next result will help to reduce the study of M_y^{-1} to that of $M_y^{-1} \cap C(y)$.

LEMMA 4. *Assume* $S \in \mathscr{A}$ *and* $y \in S$. *If* $x \in M_y^{-1} \setminus C(y)$ *then* M_y^{-1} *contains the whole segment from* x *to the cut point of* y *in the direction of* x.

PROOF. We show that M_y^{-1} contains, together with x, the cut point z of y along the segment γ_{yx}, as well as the arc γ_{xz} of the segment $\gamma_{yz} \supset \gamma_{yx}$. To see this, consider a neighbourhood V of y such that $\rho(x, v) \leq \rho(x, y)$ for all $v \in V$. If $u \in \gamma_{xz} \setminus \{x\}$ then we have, for all $w \in V$,

$$\rho(u, y) = \rho(u, x) + \rho(x, y) \geq \rho(u, x) + \rho(x, w) \geq \rho(u, w),$$

and the proof is complete. □

COROLLARY 1. *For every point* y *on every orientable surface* S, $S \setminus M_y^{-1}$ *is connected.*

PROOF. Suppose $S \setminus M_y^{-1}$ is disconnected. Denote by S' the component of $S \setminus M_y^{-1}$ containing y, and take a point u in a component $S'' \neq S'$ of $S \setminus M_y^{-1}$. Then each segment γ_{yu} from y to u meets $\mathrm{bd}S' \subset M_y^{-1}$. Take a point x in $M_y^{-1} \cap \gamma_{yu}$, so $y \in M_x$ and, by Lemma 4, all points of γ_{yu} from x to u also belong to M_y^{-1}. In particular $u \in M_y^{-1}$ and a contradiction is obtained. □

THEOREM 1. *Let* $S \in \mathscr{A}$ *be an orientable surface and* y *a point in* S.

a) If two points of M_y^{-1} *lie in the same edge of* $C^{cp}(y)$, *or on the same component of* $C(y) \setminus C^{cp}(y)$, *then they belong to the same arcwise connected component of* M_y^{-1}.

b) If there exists a point v *in* $M_y^{-1} \cap C(y) \setminus C^{cp}(y)$ *then* M_y^{-1} *is connected or the component of* v *in* M_y^{-1} *intersects* $C^{cp}(y)$.

c) For any two points in the same component of M_y^{-1} *there exists a Jordan arc* $J \subset M_y^{-1}$ *joining them such that* $J \setminus C(y)$ *is the union of at most two segments. In particular, each component of* M_y^{-1} *is arcwise connected.*

PROOF. *a*) If the points $v, z \in M_y^{-1}$ are interior to the same edge of $C^{cp}(y)$, or to the same component of $C(y) \setminus C^{cp}(y)$, then there exists a set Δ as in Lemma 1 such that, moreover, Δ contains all segments from v and z to y. Let γ_{vy}^1, γ_{vy}^2 and respectively γ_{zy}^1, γ_{zy}^2, denote the segments bounding Δ.

We claim that the Jordan arc $J_{vz} \subset C(y)$ joining v to z in Δ is included in M_y^{-1}.

To prove the claim, consider a neighbourhood V of y such that $\rho(v,w) \leq \rho(v,y)$ and $\rho(z,w) \leq \rho(z,y)$ hold for all points $w \in V$. By possibly passing to an open subset of V, we may assume that $\mathrm{cl}(\Delta \cup V)$ is a topological cylinder, because S is orientable. Choose $u \in J_{vz} \setminus \{v, z\}$, and assume y is not a local maximum for ρ_u. Then there exist points $y' \to y$ such that $\rho(u,y') \geq \rho(u,y)$.

Let $\gamma_{uy}^1, \gamma_{uy}^2$ be two segments from y to u, the union of which separates v from z in Δ. Then $O = \gamma_{uy}^1 \cup \gamma_{uy}^2$ also separates y' in $\mathrm{cl}(\Delta \cup V)$ either from v or from z. Assume the former be true and choose a segment $\gamma_{vy'}$ from v to y' (see Figure 1).

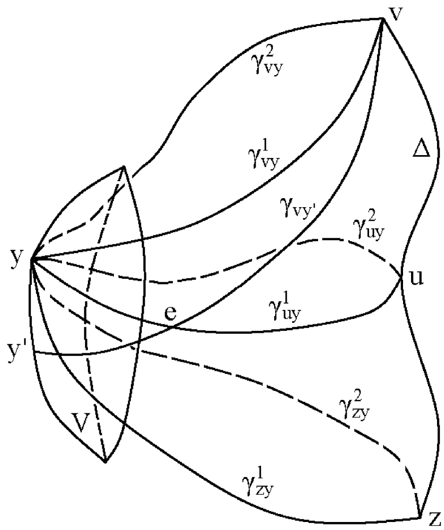

Figure 1.

Then, for y' close to y, $\gamma_{vy'}$ is close to a segment from v to y, and therefore it cuts O, say at $e(\neq y)$. Assume $e \in \gamma_{uy}^1$. Summing up the inequalities $\rho(u,y') \geq \rho(u,y)$ and $\rho(v,y) \geq \rho(v,y')$, we obtain

$$\rho(u,y') + \rho(v,y) \geq \rho(v,y') + \rho(u,y).$$

Then, the other equality cases in Lemma 2 being easily excluded, $y' = y$ and the claim is proven: $J_{vz} \subset M_y^{-1}$.

b) Assume there exist points $v, z \in M_y^{-1} \cap C(y)$ such that $v \notin C^{cp}(y)$ and z belongs either to $C^{cp}(y)$ or to another component of $C(y) \setminus C^{cp}(y)$ than v. Figure 2 a) presents the case $v, z \notin C^{cp}(y)$ (the arrows are indicating segments to y), while Figure 2 b) illustrates images of the points in Figure 2 a) onto T_y.

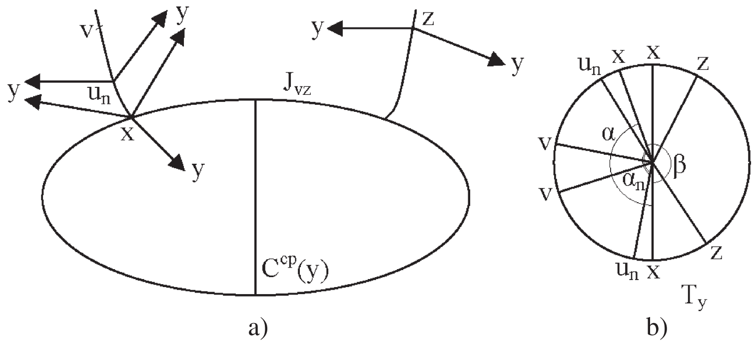

Figure 2.

Denote by J_{vz} a (minimal with respect to inclusion) Jordan arc of $C(y)$ joining v to z, and by x the point of $J_{vz} \cap C^{cp}(y)$ closest to v along J_{vz}. Let J_{vx} be the subarc of J_{vz} from v to x. Eventhought J_{vz} needs not to be unique, x and J_{vx} are uniquely determined by the assumption $v \notin C^{cp}(y)$.

Then, for each point u interior to J_{vx}, the union of segments from u to y separates v from z in S, hence the arguments proving *a*) completely apply to show $\mathrm{int} J_{vx} \subset M_y^{-1}$.

We claim that x belongs to M_y^{-1}, too. This is not directly implied by the previous considerations and passing to the limit, because the set M_y^{-1} is not necessarily closed. Assume $x \neq z$, since otherwise there is nothing to justify.

To prove the claim, choose a sequence of points $u_n \in J_{vx}$ converging to x, each point of which is joint to y by precisely two segments, say $\gamma^1_{u_n y}$ and $\gamma^2_{u_n y}$. This choice is possible because $C(y)$ has at most countably many ramification points.

Denote by S^v_n, respectively S^z_n, the component of $S \setminus (\gamma^1_{u_n y} \cup \gamma^2_{u_n y})$ containing v, respectively z. Also denote by α_n, β_n the angles at y of $\gamma^1_{u_n y}$ and $\gamma^2_{u_n y}$ towards S^v_n, respectively S^z_n. Then, by the last part of Lemma 3, $\alpha_n \leq \pi$ and $\beta_n \leq \pi$. Passing to the limit we get segments $\gamma^1_{xy} = \lim_{n\to\infty} \gamma^1_{u_n y}$ and $\gamma^2_{xy} = \lim_{n\to\infty} \gamma^2_{u_n y}$ from x to y such that their angles α, respectively β at y towards v, respectively z, verify

$$\alpha \leq \liminf_{n\to\infty} \alpha_n \leq \pi$$

and

$$\beta \le \liminf_{n\to\infty}\beta_n \le \pi.$$

Observe now that $\alpha < \pi$, because $z \in M_y^{-1}$. Indeed, since $x \neq z$, there exist two segments from y to z, the angle η of which at y strictly contains α. But $\eta \le \pi$, by the last part of Lemma 3, hence $\alpha < \pi$.

Since x is a ramification point of $C(y)$, there exists a segment γ_{xy}^3 from y to x whose direction at y divides β into angles strictly less than π.

Therefore, by the first part of Lemma 3, $x \in M_y^{-1}$ and the proof of *b*) is complete.

c) Choose points x, x^* in a component C of M_y^{-1} in S.

For any real number $\delta > 0$ there exists a finite covering of C (since it exists for S) with closed intrinsic balls $B_1, \dots, B_n$ in S of diameter $\max_{j=1}^n \mathrm{diam} B_j < \delta$, where the integer $n \ge 1$ depends on δ. Assume $x \in B_1$ and $x^* \in B_n$. By the connectedness of C, $B_1 \cap \bigcup_{j=2}^n B_j \neq \emptyset$, say $B_1 \cap B_2 \neq \emptyset$. Then $(B_1 \cup B_2) \cap \bigcup_{j=3}^n B_j \neq \emptyset$ as well. Iterating, we can find a finite sequence of balls $B_1 = B_{i_1}, \dots, B_{i_m} = B_n$ such that $B_{i_\alpha} \cap B_{i_{\alpha+1}} \neq \emptyset$, for $\alpha = 1, \dots, m-1$ and $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. Therefore, it suffices to verify that points in C close enough to each other, say in the same closed intrinsic ball of diameter δ, belong to the same arcwise connected component of M_y^{-1}.

Assume $x, x^* \in C \cap B_1$ and let z, z^* be the cut points of y in the directions of x, x^*, respectively. Our choice directly implies that z is close to z^*, by the convergence of segments.

If $\{z, z^*\} \subset C(y) \setminus C^{cp}(y)$, or z, z^* belong to the same edge E of $C^{cp}(y)$, then the conclusion follows by Lemma 4 and *a*). Indeed, the Jordan arc J joining z to z^* in $C(y) \setminus C^{cp}(y)$, respectively in E, is included in M_y^{-1}, as well as the whole segment from x to z, respectively from x^* to z^*.

Assume now z belongs to a tree component T of $C(y) \setminus C^{cp}(y)$ and z^* is in $C^{cp}(y)$, or z, z^* belong to different edges E, E^* of $C^{cp}(y)$. By letting $\delta \to 0$, we get – say – sequences $x_n \to x^*$ and respectively $z_n \to z^*$, where $x = x_1$, $z = z_1$, z_n is the cut point of y in the direction of x_n and moreover, according to the case, all points z_n are either in T or in E. Therefore, either T connects to $C^{cp}(y)$ at z^*, or $z^* \in E \cap E^*$.

To end the proof, observe that the whole Jordan arc J joining z to z^* either in $T \cup \{z^*\}$, or in E, is included in M_y^{-1}. Indeed, for any point $u \in J$ there exists a point $x_{n_u} \in J$ closer to z^* than u, hence u lies in J between x and x_{n_u} and therefore the arguments at *a*) completely apply. Moreover, Lemma 4 shows that the whole segment from x to z, respectively from x^* to z^*, is also included in M_y^{-1}. $\square$

Theorem 2. *For every orientable surface S of genus g and every point y in S, M_y^{-1} has at most* $\max\{1, 10g-5\}$ *components.*

Proof. If $g=0$ then M_y^{-1} is connected, as follows easily from the proof of *a*) in Theorem 1, so we may assume from now on that $g>0$.

Notice that, by *b*) in Theorem 1, if M_y^{-1} has a component disjoint to $C^{cp}(y)$ then M_y^{-1} consists of precisely that component. Assume this is not the case. Then the interior of each edge of $C^{cp}(y)$ may intersect at most one component of M_y^{-1}, by *a*) in Theorem 1, and moreover each vertex of $C^{cp}(y)$ may belong to a component of M_y^{-1}. Since $C^{cp}(y)$ is a graph with $2g$ generating cycles, its maximal number of edges is $6g-3$, while for counting together edges and vertices yields at most $10g-5$ (a proof of this fact is given in [**1**]). $\square$

3. Local tree components of M_y^{-1}.

The main purpose of the last part of this paper is to highlight a strong relationship between λT_y and the structure of M_y^{-1}, with Theorems 4 to 6. Before this we show, by Theorem 3, that the objects we shall talk about do exist.

The following result slightly strengthens Theorem 9 in [**8**]. The proof follows the same argument, with a simple modification, and will not be repeated here.

Lemma 5. *Every combinatorial type of finite tree can be realized as the cut locus $C(y)$ of some point y on some doubly covered convex polygon, such that the internal edges of $C(y)$ are arbitrarily small compared to the external ones.*

We shall employ the following hinge variant of the Toponogov's comparaison theorem. For, let ρ_H denote the distance on the simply connected 2-dimensional space M_H of constant curvature H.

Lemma 6. *Let Δ be a domain in the surface $S \in \mathscr{A}$, bounded by the segments $\gamma_i : [0, l_i] \to S$, $i = 1, 2, 3$. Assume the curvature exists on Δ and verifies $K \leq H$. Assume moreover that the segments γ_1, γ_2 make an angle of α at the point $\gamma_1(0) = \gamma_2(0)$. Consider segments $\bar{\gamma}_i : [0, l_i] \to M_H$, $i = 1, 2$, making an angle of α at the point $\bar{\gamma}_1(0) = \bar{\gamma}_2(0)$. Then $\rho(\gamma_1(l_1), \gamma_2(l_2)) \geq \rho_H(\bar{\gamma}_1(l_1), \bar{\gamma}_2(l_2))$.*

We shall write $T \sim T'$ if the trees T and T' have the same combinatorial structure.

Theorem 3. *Every finite tree can be realized as the set F_y^{-1}, for some point y on some surface $S \in \mathscr{A}$.*

Proof. For every tree T there exists, by Lemma 5, a convex surface P and a point z in P such that $C(z) \sim T$. Remark that F_z contains at least one smooth point o of P. Therefore, since P is piecewise Euclidean, there are at least three segments from z to o, hence o is a ramification point of $C(z)$. Moreover, since z is not a vertex of P, there exists a circle $C \subset P$ centered at z of radius smaller than the injectivity radius at z.

Choose a circle C^* parallel to C and of smaller radius. Cut along C^* and smoothly connect to $P \setminus C^*$ a right circular cone of apex y whose total angle is $\lambda T_y = \pi$, with y on the line orthogonal to the centre of C^*. The above connection can be done such that the curvature of the glued piece V is nonpositive everywhere except at y. The resulting surface S is Alexandrov, $S \in \mathscr{A}$, and the distance function from y on S clearly coincides, on $P \setminus C^* = S \setminus V$, to the distance function from z on P. Next considerations will all refer to S.

Notice that C is a distance circle from y, so $C(y) \sim T$. Moreover, $o \in F_y^{-1}$ by the construction.

Observe now that each point in F_y^{-1} is necessarily joined to y by at least 2 segments. For, choose $v \in S$ with a unique shortest path γ_{vy} to y, and a segment γ_{yw} starting at y orthogonally to γ_{vy}. If w is close enough to y then the triangle wvy contains no vertex of P, so its curvature exists and is nonpositive. Construct a planar triangle $\bar{w}\bar{v}\bar{y}$ such that $\|\bar{w} - \bar{y}\| = \rho(w, y)$, $\|\bar{y} - \bar{v}\| = \rho(y, v)$ and $\angle\bar{v}\bar{y}\bar{w} = \pi/2$. By Lemma 6,

$$\rho(v, w) \geq \|\bar{v} - \bar{w}\| > \|\bar{v} - \bar{y}\| = \rho(v, y).$$

Then, since F_y^{-1} is closed, it consists of points interior to $C(y)$ with respect to the relative topology.

We claim that every point $x \in C(y) \setminus \{o\}$ close enough to o also belongs to F_y^{-1}. Indeed, such x is joined to y by precisely two segments, which make at y an angle $\alpha_x < \pi$. By Lemma 3, $\rho(x, y) > \rho(x, u)$ for all points u in some small ball $B(y, \varepsilon\rho(x, y)) \setminus \{y\}$. By the upper semicontinuity of F, if x is close to o then F_x is close to $y = F_o$. Thus, for any point x in $C(y)$ close enough to o we obtain $F_x \subset B(y, \varepsilon\rho(x, y))$, whence $F_x = y$, and the claim is proved.

Therefore, F_y^{-1} is a subtree of $C(y)$ and moreover, all points of $C(y)$ close enough to o belong to F_y^{-1}. So, if the ramification points of $C(y)$ are close to each other then they all belong to F_y^{-1}. □

Remark. By a somewhat similar – yet, since it settles by direct induction a variant of Lemma 5, quite lengthy – argument, one can prove that for any tree T there exists a convex pyramid P of apex y with total angle $\theta_y = \pi$ such that $F_y^{-1} \sim C(y) \sim T$. The constructed surface can be smoothened everywhere except at y, while keeping the desired properties.

Example. The set M_y^{-1} may be a local tree but not necessarily a tree.

To see this, consider a flat Riemannian surface $F \in \mathscr{A}$ and a point $z \in F$. The radius of injectivity $\mathrm{inj}(z)$ at z is positive, hence we may cut off from F a disk D around z of radius smaller than $\mathrm{inj}(z)$, and smoothly glue instead a right circular cone of apex y whose total angle is $\lambda T_y = \pi$, such that the curvature of the glued piece is nonpositive everywhere except at y. Lemmas 3 and 6 now show that, on the new surface, M_y^{-1} contains all points in, possibly excepting some extremities (if any), of $C(y)$. □

Remark. If $\lambda T_y < \pi$ then $M_y^{-1} = S \setminus \{y\}$, directly from Lemma 3. Conversely, if M_y^{-1} has nonempty interior in S then $\lambda T_y \leq \pi$, by $M_y^{-1} \setminus C(y) \neq \emptyset$ and Lemma 3 again.

If $\lambda T_y = \pi$ then M_y^{-1} may be a local tree (as shown in Theorem 3 or by the previous example), or it may have interior points. The last situation is illustrated by the special case of a Tannery surface with parameters $p = 2$ and $q = 1$ (see [**2**], p.95 and p.102 for the precise definitions), as it follows from Theorem 11 in [**14**].

Concluding, if $\lambda T_y < \pi$ then $M_y^{-1} = S \setminus \{y\}$ and there is nothing more to say, and if $\lambda T_y = \pi$ then one cannot generally characterize M_y^{-1}. The main part of this section will be devoted to describe the structure of M_y^{-1} in the case $\lambda T_y > \pi$.

We continue with a result treating – slightly more generally – the sets Q_y^{-1} instead of M_y^{-1}. Before, notice that $\lambda T_y \leq \pi$ directly implies, by the definition of the critical points, $Q_y^{-1} = S \setminus \{y\}$. The case $\lambda T_y = 2\pi$ is treated in [**1**].

Theorem 4. *If the surface S is orientable and $y \in S$ such that $\lambda T_y > \pi$ then Q_y^{-1} is contained in a local tree of $C(y)$ with less than $\frac{\lambda T_y}{\lambda T_y - \pi}$ extremities outside the cycles of $C(y)$.*

Proof. Each point of Q_y^{-1} is joined to y by at least two segments, because $\lambda T_y > \pi$. If $Q_y^{-1} \subset C^{cp}(y)$ there is nothing to prove, since $C^{cp}(y)$ is itself a local tree of $C(y)$ without extremities. So we can assume that Q_y^{-1} is included in some connected local tree T' of $C(y)$ with m extremities outside $C^{cp}(y)$, say $x_1, \ldots, x_m$. Consider a minimal with respect to inclusion, connected local tree T of T' which contains $x_1, \ldots, x_m$; in particular, T has no other extremity.

Denote by Δ_i the maximal domain of S bounded by segments from x_i to y such that $T \subset S \setminus \Delta_i$, and by α_i the angle of Δ_i at y, hence $\lambda T_y - \alpha_i \leq \pi$. Since $\bigcup_{i=1}^m \Delta_i$ is strictly included in S, $\Sigma_{i=1}^m \alpha_i < \lambda T_y$ and we get

$$m\pi \geq \Sigma_{i=1}^m (\lambda T_y - \alpha_i) = m\lambda T_y - \Sigma_{i=1}^m \alpha_i > (m-1)\lambda T_y,$$

whence $\lambda T_y < \frac{m}{m-1}\pi$ or, equivalently, $m < \frac{\lambda T_y}{\lambda T_y - \pi}$. □

THEOREM 5. *Suppose the surface S is orientable of genus g and $y \in S$ such that $\lambda T_y > \pi$. Then M_y^{-1} is a local tree with at most* $\max\{1, 10g-5\}$ *components, it has at most $2g$ generating cycles and less than $\frac{\lambda T_y}{\lambda T_y - \pi}$ extremities outside the cycles of $C(y)$.*

PROOF. Last part of Lemma 3 together with $\lambda T_y > \pi$ directly imply $M_y^{-1} \subset C(y)$. Theorem 1 and the properties of $C(y)$ show that M_y^{-1} is a local tree, the tree case being obtained from Corollary 1.

The number of components of M_y^{-1} follows from Theorem 2 and $M_y^{-1} \subset C(y)$.

Since $C(y)$ has $2g$ generating cycles, this gives an upper bound for the number of the generating cycles of $M_y^{-1} \subset C(y)$.

Finally if M_y^{-1} has only one extremity outside $C^{cp}(y)$ then clearly $1 < \frac{\lambda T_y}{\lambda T_y - \pi}$. Otherwise, for each extremity x_i $(i = 1, \ldots, m)$ of M_y^{-1} outside $C^{cp}(y)$, there is a tree component T_i of x_i in $M_y^{-1} \cap C(y) \setminus C^{cp}(y)$ such that $M_y^{-1} \supset \mathrm{cl} T_i \cap C^{cp}(y) \neq \emptyset$, by *b*) of Theorem 1, and Theorem 4 ends the proof. $\square$

The next result is obtained by adding to the upper bound in Theorem 5 the maximal number of vertices of $C^{cp}(y)$, as well as twice the corresponding number of edges of $C^{cp}(y)$ (see the final part in the proof of Theorem 2).

COROLLARY 2. *If the surface S is orientable and $y \in S$ such that $\lambda T_y > \pi$ then M_y^{-1} is a local tree with less than $\frac{\lambda T_y}{\lambda T_y - \pi} + 16g - 8$ extremities.*

REMARK. Theorems 4 and 5 may be compared to results of J. Itoh [**7**] and T. Zamfirescu [**17**], valid for Riemannian surfaces. They showed that, eventhough Q_x may be totally disconnected and may have uncountably many points, and $C(x)$ may be non-triangulable, Q_x must belong to a single handsome tree in $C(x)$, the number of endpoints of which is bounded by above by a constant depending only on the positive curvature of S.

The statement of Theorem 5 can also be seen as a restriction on λT_y put by the set M_y^{-1}, in which case we get the following theorem and corollary.

THEOREM 6. *Let S be an orientable surface and y a point in S. If M_y^{-1} is included in $C(y)$ and has $m > 1$ extremities outside the cycles of $C(y)$, or if a tree component of M_y^{-1} contains no vertex of $C^{cp}(y)$ and has $m > 1$ extremities, then $\pi \leq \lambda T_y < \frac{m}{m-1}\pi$.*

PROOF. The inequality $\lambda T_y \geq \pi$ follows from Lemma 3, and the first case is covered by Theorem 5.

The second case needs a little more care. Denote by x_i $(i = 1, \ldots, m)$ the extremities of a tree component C of M_y^{-1} without vertices of $C^{cp}(y)$. Theorem 1 together with int$C = \emptyset$ imply C is included in the (arcwise connected) union of $C(y)$ with some subsegments of segments from y. If there is an extremity x_i outside $C(y)$ then $\lambda T_y = \pi$ (by Lemma 3) and the inequality is satisfied. So we may assume that all extremities x_i are on $C(y)$; therefore, $C \subset C(y)$ and the ramification points of C are ramifications of $C(y)$. Then, because the set of ramification points of the tree $C(y)$ is at most countable, there are points $x'_i \in C \cap C(y)$ which are joined to y by precisely two segments, say γ_i and γ'_i.

Since $C \subset C(y)$ contains no vertex of $C^{cp}(y)$, there exists a subset S_C of S homeomorphic to a cylinder such that all segments from y to the points of C are included in S_C. Thus, if x'_i is close to x_i then $\gamma_i \cup \gamma'_i$ separates in S_C x_i from all extremities of $C \setminus \{x_i\}$. The rest of the proof runs similarly to that of Theorem 5 and will not be repeated. □

Let T_m denote any tree with m extremities. An interesting consequence of Theorem 6 is the following.

Corollary 3. *A convex surface S contains at most 7 points y such that $M_y^{-1} \sim T_{\geq 3}$, or at most 5 points y such that $M_y^{-1} \sim T_{4 \leq m \leq 6}$, or at most 4 points y such that $M_y^{-1} \sim T_{\geq 7}$. Moreover, S contains at most 3 points y with* int$M_y^{-1} \neq \emptyset$.

Proof. Let $S \in \mathscr{A}$ be convex and $m \geq 3$; the total curvature at y can be expressed, by Theorem 6, as

$$\omega_y = 2\pi - \lambda T_y > 2\pi - \frac{m}{m-1}\pi = \pi - \frac{1}{m-1}\pi \geq \pi/2.$$

Since the total curvature of S is equal to 4π, there are at most 7 points $y \in S$ such that $M_y^{-1} \sim T_m$ with $m \geq 3$. The other estimations follow similarly.

Assume now int$M_y^{-1} \neq \emptyset$, hence $\lambda T_y \leq \pi$ by Lemma 3. Then the total curvature at y is $\omega_y = 2\pi - \lambda T_y \geq \pi$, and there are at most $k \leq 4$ points $y \in S$ with int$M_y^{-1} \neq \emptyset$. Suppose $k = 4$, so S is flat everywhere excepting its vertices y (where $\omega_y = \lambda T_y = \pi$), hence S is either a doubly covered rectangle or a tetrahedron with curvature π at each of its vertices. On doubly covered rectangles M_y^{-1} is an arc for each vertex y, while in the case of tetrahedra all vertices y have $M_y^{-1} \sim T_3$, because $C(y) \sim T_3$ and $M_y^{-1} \subset C(y)$, $M_y^{-1} \sim C(y)$, just as in the proof of Theorem 3. Therefore, $k < 4$. □

Example. In the following we provide a convex surface having a countable set of points x_n with F_{x_n} an arc, and a countable set of points y_n with $F_{y_n}^{-1}$ an arc, so proving that the last inequality in Theorem 6 – and thus that in Theorem 5 – is sharp. It also shows that the assumption $m \geq 3$ in $M_y^{-1} \sim T_m$ within Corollary 3 is necessary.

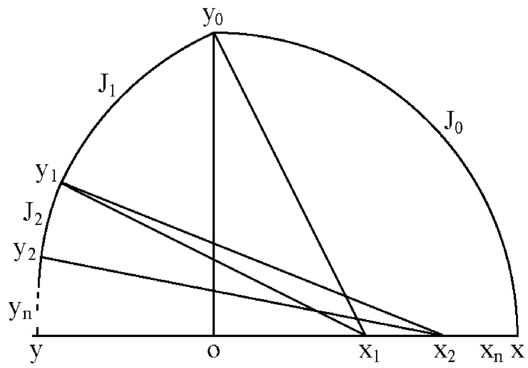

Figure 3.

Take in a plane a quarter of circle with the centre at o, bounded by the radii $[ox]$ and $[oy_0]$, and denote it by J_0 (see Figure 3).

Let x_1 be the mid-point of $[ox]$ and y_1 the point on the bisector of the angle $\angle ox_1y_0$ determined by $\|x_1 - y_0\| = \|x_1 - y_1\|$. Let J_1 be the smallest arc of circle centered at x_1 between y_0 and y_1. Inductively, let x_n be the mid-point of the segment $[xx_{n-1}]$ and y_n the point on the bisector of the angle $\angle ox_ny_{n-1}$ such that $\|x_n - y_{n-1}\| = \|x_n - y_n\|$. Denote by J_n the smallest arc of circle centered at x_n between y_{n-1} and y_n. The sequence $\{y_n\}_{n\geq 0}$ converges to a point y on the line ox and $\lim_{n\to\infty} x_n = x$.

Let S be the doubly-covered compact planar region bounded by $\cup_{n\geq 0} J_n \cup [xy]$. One can easily check on S that $F_x = y$, and for all integers $n \geq 1$ we have $F_{x_n} = J_n$ and $F_{y_n}^{-1} = [x_nx_{n+1}]$. $\square$

Acknowledgements. Thanks are due to the referee for his remarks. The comments of Professor T. Zamfirescu on a preliminary version of the manuscript helped to improve the clarity of the presentation, especially of the proof of Theorem 1. The work was finalized during the author's stay at Kumamoto University, supported by JSPS.

References

[1] I. Bárány, J. Itoh, C. Vîlcu and T. Zamfirescu, Every point is critical, to appear.
[2] A. L. Besse, Manifolds all of whose Geodesics are Closed, Springer-Verlag, New York, 1978.
[3] Y. Burago, M. Gromov and G. Perelman, A. D. Alexandrov spaces with curvature bounded below, Russian Math. Surveys, **47** (1992), 1–58.
[4] H. T. Croft, K. J. Falconer and R. K. Guy, Unsolved Problems in Geometry, Springer-Verlag, New York, 1991.

[5] K. Grove, Critical point theory for distance functions, Amer. Math. Soc., Proc. Sympos. Pure Math., **54** (1993), 357–385.
[6] K. Grove and P. Petersen, A radius sphere theorem, Invent. Math., **112** (1993), 577–583.
[7] J. Itoh, Essential cut locus on a surface, Proc. 5th Pacific Rim Geometry Conference, Tohoku Math. Publ., **20**, Tohoku Univ., Sendai, 2001, pp. 53–59.
[8] J. Itoh and C. Vîlcu, Farthest points and cut loci on some degenerate convex surfaces, J. Geom., **80** (2004), 106–120.
[9] J. Itoh and T. Zamfirescu, On the length of the cut locus on surfaces, Rend. Circ. Mat. Palermo, Serie II, Suppl., **70** (2002), 53–58.
[10] Y. Otsu and T. Shioya, The Riemannian structure of Alexandrov spaces, J. Differential Geom., **39** (1994), 629–658.
[11] K. Shiohama and M. Tanaka, Cut loci and distance spheres on Alexandrov surfaces, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sém. Congr., Soc. Math. France, **1**, (1996), 531–559.
[12] C. Vîlcu, On two conjectures of Steinhaus, Geom. Dedicata, **79** (2000), 267–275.
[13] C. Vîlcu, Properties of the farthest point mapping on convex surfaces, Rev. Roumaine Math. Pures Appl., **51** (2006), 125–134.
[14] C. Vîlcu and T. Zamfirescu, Multiple farthest points on Alexandrov surfaces, Adv. Geom., **7** (2007), 83–100.
[15] T. Zamfirescu, Farthest points on convex surfaces, Math. Z., **226** (1997), 623–630.
[16] T. Zamfirescu, On the cut locus in Alexandrov spaces and applications to convex surfaces, Pacific J. Math., **217** (2004), 375–386.
[17] T. Zamfirescu, On the critical points of a Riemannian surface, Adv. Geom., **6** (2006), 493–500.

Costin Vîlcu
Institute of Mathematics Simion Stoilow
of the Romanian Academy
P.O. Box 1-764, Bucharest 014700
Romania
E-mail: Costin.Vilcu@imar.ro