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Kenmotsu type representation formula for surfaces with prescribed mean curvature in the hyperbolic 3-space

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Abstract. Our primary object of this paper is to give a representation formula for surfaces with prescribed mean curvature in the hyperbolic 3-space of curvature -1 in terms of their normal Gauss maps. For CMC (constant mean curvature) surfaces, we derive another representation formula in terms of their adjusted Gauss maps. These formulas are hyperbolic versions of the Kenmotsu representation formula for surfaces in the Euclidean 3-space. As an application, we give a construction of complete simply connected CMC H ($|H| < 1$) surfaces embedded in the hyperbolic 3-space.

Introduction.

Let $\boldsymbol{H}^3(-c^2)$ be the hyperbolic 3-space of constant curvature $-c^2$ ($c > 0$). In this paper, we give a representation formula for surfaces with prescribed (not necessarily constant) mean curvature in $\boldsymbol{H}^3(-c^2)$, as a hyperbolic version of the Kenmotsu representation formula [**Ke**] for surfaces in $\boldsymbol{E}^3$. Hence we call it the *Kenmotsu type representation formula*. This is given via an integrable differential equation of first order in terms of the prescribed mean curvature and the *normal Gauss map*. The normal Gauss map was defined by Kokubu [**Ko1**], and he gave a representation formula for minimal surfaces in $\boldsymbol{H}^3(-c^2)$ by means of the normal Gauss map. (The Kenmotsu type representation formula for minimal surfaces in $\boldsymbol{H}^3(-c^2)$ coincides with Kokubu's representation formula.)

In Section 1, we review the definition of the normal Gauss maps for surfaces in $\boldsymbol{H}^3(-c^2)$ in terms of complex 2×2 matrices. In Section 2, we show that the normal Gauss map G of a surface with mean curvature H in $\boldsymbol{H}^3(-c^2)$ satisfies a nonlinear partial differential equation of second order. If we put $c = 0$ in it, we can obtain the generalized harmonic map equation (abbreviated to GH equation) for the Gauss map of a surface with mean curvature H in $\boldsymbol{E}^3$. Then we call it the hyperbolic GH equation. The hyperbolic GH equation is the integrability condition for a surface with mean curvature H in $\boldsymbol{H}^3(-c^2)$, from which we obtain the Kenmotsu type representation formula in $\boldsymbol{H}^3(-c^2)$. The idea to obtain the Kenmotsu type representation formula in

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$\boldsymbol{H}^3(-c^2)$ is based on a new viewpoint for the mechanism of obtaining the Kenmotsu representation formula in $\boldsymbol{E}^3$ (see Appendix I).

In Section 3, we concentrate on constant mean curvature (abbreviated to CMC) surfaces in $\boldsymbol{H}^3(-c^2)$. The Gauss map for a CMC surface in $\boldsymbol{E}^3$ is a harmonic map to the unit 2-sphere $\boldsymbol{S}^2$ with the standard metric g_0. In contrast, the normal Gauss map for a CMC H surface in $\boldsymbol{H}^3(-c^2)$ is not so. However, it is a harmonic map to $\boldsymbol{S}^2$ with a different metric $h_{c,H}$ from g_0, where $h_{c,H}$ has singularities in case $|H| \leqq c$. When $H = c$ (resp. $|H| > c$), using the notion of the Lawson correspondence [**L**] at 'adapted frame level' (cf. Bobenko [**Bo**]), we can adjust the normal Gauss map to a holomorphic map (resp. a harmonic map) to $(\boldsymbol{S}^2, g_0)$, and obtain the Bryant representation formula [**Br**] (reformulated by Umehara-Yamada [**UY**]) (resp. the Kenmotsu-Bryant type representation formula [**AA1**]). Our method here of 'adjusting' gives a new viewpoint to the proof of these formulas. When $|H| < c$, using the Lawson type correspondence (cf. Fujioka [**F**]) at adapted frame level, we can adjust the normal Gauss map of a CMC H surface in $\boldsymbol{H}^3(-c^2)$ to the one of a minimal surface in $\boldsymbol{H}^3(-c_0^2)$ ($c_0 = \sqrt{c^2 - H^2}$), and obtain another representation formula for CMC H surfaces in $\boldsymbol{H}^3(-c^2)$. (We call it the Kokubu-Bryant type representation formula.)

In Section 4, we construct complete simply connected CMC H ($|H| < 1$) surfaces embedded in $\boldsymbol{H}^3(-1)$ by applying the result [**AA4**] of the Dirichlet problem for harmonic maps from the unit disk to $(\boldsymbol{S}^2, h_{1,H})$.

In Appendix II, we remark that for a CMC H ($|H| < c$) (not totally umbilic) surface in $\boldsymbol{H}^3(-c^2)$, there exists a dual CMC H spacelike surface in the de Sitter 3-space of constant curvature c^2 arising from a parallel translation.

In Appendix III, we give spin versions of the above representation formulas in $\boldsymbol{H}^3(-c^2)$.

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1. Normal Gauss map of surfaces in $\boldsymbol{H}^3(-c^2)$.

The hyperbolic 3-space $\boldsymbol{H}^3(-c^2)$ is defined as the following hyperquadric in the Minkowski 4-space $\boldsymbol{L}^4 = (\boldsymbol{R}^4, \langle\,,\rangle)$:

$$\boldsymbol{H}^3(-c^2) = \left\{ \mathbf{x} = (x_0, x_1, x_2, x_3) \in \boldsymbol{L}^4 \,\middle|\, \langle \mathbf{x}, \mathbf{x} \rangle := -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -\frac{1}{c^2}, x_0 > 0 \right\}.$$

We will consider $\boldsymbol{L}^4$ as the space Herm(2) of 2×2 Hermitian matrices by

$$\mathbf{x} = (x_0, x_1, x_2, x_3) \in \boldsymbol{L}^4 \mapsto \underline{\mathbf{x}} = \begin{pmatrix} x_0 + x_3 & x_1 + \sqrt{-1}x_2 \\ x_1 - \sqrt{-1}x_2 & x_0 - x_3 \end{pmatrix}. \tag{1.1}$$

Here $\langle \mathbf{x}, \mathbf{x} \rangle = -\det \underline{\mathbf{x}}$ and the canonical basis of $\boldsymbol{L}^4$ is given by

$$\underline{\mathbf{e}_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \underline{\mathbf{e}_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underline{\mathbf{e}_2} = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \quad \underline{\mathbf{e}_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The complex special linear group $SL(2;\boldsymbol{C})$ acts isometrically on $\boldsymbol{L}^4$ by

$$\mathrm{g}\cdot\mathbf{x} = \mathrm{g}\underline{\mathbf{x}}\mathrm{g}^* \quad (\mathrm{g}\in SL(2;\boldsymbol{C}), \mathbf{x}\in\boldsymbol{L}^4),$$

where $\mathrm{g}^* = {}^t\bar{\mathrm{g}}$. $SL(2;\boldsymbol{C})$ also acts on $\boldsymbol{H}^3(-c^2)$ isometrically and transitively. Indeed, $SL(2;\boldsymbol{C})$ is the double cover of the identity component $SO_+(1,3)$ of the isometry group of $\boldsymbol{H}^3(-c^2)$. The isotropy group at a point $(1/c)\underline{\mathbf{e}_0}\in\boldsymbol{H}^3(-c^2)$ is the special unitary group $SU(2)$, and hence

$$\boldsymbol{H}^3(-c^2) = SL(2;\boldsymbol{C})/SU(2) = \left\{\frac{1}{c}e_0(\mathrm{g}) := \frac{1}{c}\mathrm{g}\mathrm{g}^* \,\middle|\, \mathrm{g}\in SL(2;\boldsymbol{C})\right\}.$$

$SU(2)$ is the double cover of the rotational group $SO(3)$. Indeed, the subgroup $SU(2)$ in $SL(2;\boldsymbol{C})$ acts isometrically on the Euclidean 3-space $\boldsymbol{E}^3 = \{\mathbf{x}\in\boldsymbol{L}^4 \mid x_0 = 0\} \cong \sqrt{-1}\cdot\mathfrak{su}(2)(\subset\mathrm{Herm}(2))$, and acts on the unit 2-sphere $\boldsymbol{S}^2$ (with the standard metric g_0) in $\boldsymbol{E}^3$ transitively (and isometrically). Then

$$\boldsymbol{S}^2 = SU(2)/U(1) = \{[\mathrm{h}] := \mathrm{h}\underline{\mathbf{e}_3}\mathrm{h}^* \mid \mathrm{h}\in SU(2)\},$$

where $U(1) = \{(\cos\theta)\underline{\mathbf{e}_0} + \sqrt{-1}(\sin\theta)\underline{\mathbf{e}_3} \mid \theta\in[0,2\pi)\} \cong \boldsymbol{S}^1 \subset \boldsymbol{C}$.

The Gram-Schmidt procedure for complex row-vectors of each matrix $\mathrm{g}\in SL(2;\boldsymbol{C})$ gives the (Iwasawa) decomposition

$$SL(2;\boldsymbol{C}) = S\cdot SU(2) \ni \mathrm{g} = \mathrm{sh}, \tag{1.2}$$

$$\mathrm{s}\in S = \left\{\begin{pmatrix} a & w \\ 0 & 1/a \end{pmatrix} \,\middle|\, a>0, w\in\boldsymbol{C}\right\}, \quad \mathrm{h}\in SU(2). \tag{1.3}$$

$\boldsymbol{H}^3(-c^2)$ can be identified with the Lie group S:

$$\boldsymbol{H}^3(-c^2) = \left\{\frac{1}{c}e_0(\mathrm{s}) = \frac{1}{c}\mathrm{s}\mathrm{s}^* \in\boldsymbol{L}^4 \,\middle|\, \mathrm{s}\in S\right\} \cong S. \tag{1.4}$$

Under this identification, the action of S on $\boldsymbol{H}^3(-c^2)$ can be regarded as the left translation L of the Lie group S on itself. Let ϖ denote the projection from $SL(2;\boldsymbol{C})$ to S.

Take a coordinate (w,t) on S as the following map Ψ, then we obtain the upper half-space model $\boldsymbol{H}^3_+(-c^2)$ of $\boldsymbol{H}^3(-c^2)$:

$$\Psi : \boldsymbol{R}^3_+ = \boldsymbol{C}\times\boldsymbol{R}_+ \ni (w,t) \to \begin{pmatrix} \sqrt{ct} & \sqrt{c}w/\sqrt{t} \\ 0 & 1/\sqrt{ct} \end{pmatrix} \in S(\subset SL(2;\boldsymbol{C})),$$

$$\boldsymbol{H}^3_+(-c^2) = (\boldsymbol{R}^3_+, g_c), \quad g_c = \frac{|dw|^2 + dt^2}{c^2t^2}.$$

LEMMA 1.1. *For a tangent vector* $X = (X_1 + \sqrt{-1}X_2, X_3)$ *at* $(w,t)\in\boldsymbol{H}^3_+(-c^2)$, *put* $\tilde{X} = d\Psi(X)\in T_{\mathrm{s}}S$ (*where* $\mathrm{s} = \Psi(w,t)$) *and* $\tilde{\tilde{X}} = (1/c)d(e_0\circ\Psi)(X)\in T_x\boldsymbol{H}^3(-c^2)\subset$ *Herm*(2) (*where* $x = (1/c)\mathrm{s}\mathrm{s}^*$). *Then, under the identification* $T_{\underline{\mathbf{e}_0}}S \cong T_{(1/c)\underline{\mathbf{e}_0}}\boldsymbol{H}^3_+(-c^2) = \sqrt{-1}\cdot\mathfrak{su}(2)$ via $(1/c)de_0$,

$$dL_{\mathrm{s}^{-1}}\tilde{X} = \mathrm{s}^{-1}\tilde{\tilde{X}}(\mathrm{s}^{-1})^* = \frac{1}{ct}(X_1\underline{\mathbf{e}_1} + X_2\underline{\mathbf{e}_2} + X_3\underline{\mathbf{e}_3}) = \frac{1}{ct}\underline{X}\in\sqrt{-1}\cdot\mathfrak{su}(2).$$

Now let M be a Riemann surface and $f : M \to \boldsymbol{H}^3(-c^2)$ a conformal immersion. Let $\mathscr{E} : U \to SL(2;\boldsymbol{C})$ be an *adapted framing* of f on every contractible open set U of M, that is, $\mathscr{E} : U \to SL(2;\boldsymbol{C})$ is a smooth map such that $f|_U = (1/c)e_0 \circ \mathscr{E} = (1/c)\mathscr{E}\cdot\underline{\mathbf{e}_0}$, $\mathscr{E}\cdot\underline{\mathbf{e}_3}$ is a unit normal vector field of f and $\mathscr{E}\cdot(\underline{\mathbf{e}_1} - \sqrt{-1}\underline{\mathbf{e}_2})$ is a vector field of type $(1,0)$. Here we remark that the adapted framing on U is determined uniquely up to the right action of a $U(1)$-valued function. Corresponding to the decomposition $SL(2;\boldsymbol{C}) = S\cdot SU(2)$, we have a decomposition $\mathscr{E} = \mathscr{S}h$, where $\mathscr{S} = \varpi \circ \mathscr{E} : U \to S$ and $h : U \to SU(2)$. It should be pointed out that $\mathscr{S}$ is a *framing* of f defined globally on M, that is, $f = (1/c)e_0 \circ \mathscr{S}$. Put $G = [h]$, and then $G = G_f$ is a map from M to $\boldsymbol{S}^2$. The following lemma implies that G coincides with the *normal Gauss map* of f defined by Kokubu [**Ko1**].

Let N be the unit normal vector field of $f : M \to \boldsymbol{H}^3_+(-c^2)(\cong \boldsymbol{H}^3(-c^2))$, and put $\tilde{N} = d\Psi(N)$, $\tilde{\tilde{N}} = (1/c)d(e_0 \circ \Psi)(N)$.

Lemma 1.2. $G = dL_{\mathscr{S}^{-1}}\tilde{N}$.

Proof. $G = h\underline{\mathbf{e}_3}h^* = \mathscr{S}^{-1}(\mathscr{E}\underline{\mathbf{e}_3}\mathscr{E}^*)(\mathscr{S}^{-1})^* = \mathscr{S}^{-1}\tilde{\tilde{N}}(\mathscr{S}^{-1})^* = dL_{\mathscr{S}^{-1}}\tilde{N}$. □

Lemma 1.2 combined with Lemma 1.1 gives the following geometric interpretation of the normal Gauss map for a surface in $\boldsymbol{H}^3_+(-c^2) = (\boldsymbol{R}^3_+, g_c)$. Regarding the underlying space $\boldsymbol{R}^3_+(\subset\boldsymbol{R}^3)$ of $\boldsymbol{H}^3_+(-c^2)$ as the half-space of the Euclidean 3-space $\boldsymbol{E}^3$, parallel translate the unit normal vector $N(z)$ at a point $f(z) = (w,t)$ on the immersed surface in $\boldsymbol{H}^3_+(-c^2)$ to the origin of $\boldsymbol{E}^3$, then we obtain the vector $\underline{N(z)}$ in $\sqrt{-1}\cdot\mathfrak{su}(2) \cong \boldsymbol{E}^3$. Next normalize $\underline{N(z)}$ with respect to the Euclidean norm, then we obtain the normal Gauss map $G = \overline{(1/ct)}\underline{N} : M \to \boldsymbol{S}^2$.

Example 1. The hyperbolic cylinder in $\boldsymbol{H}^3_+(-c^2)$ is defined by

$$f : \boldsymbol{R}^2 \to \boldsymbol{H}^3_+(-c^2), \quad f(x,y) = \frac{1}{c(c_2\cosh c_1x - c_1\sin c_2y)}\left(c_2\sinh c_1x,\ c_1\cos c_2y,\ \frac{c_1c_2}{c}\right)$$

for some positive constants c_1, c_2 satisfying $-c_1^{-2} + c_2^{-2} = -c^{-2}$. This is a flat surface with CMC $H = c(c_1/2c_2 + c_2/2c_1)$ $(|H| > c)$. Its normal Gauss map G is given by

$$G(x,y) = \left(\frac{-c_1c_2\sinh c_1x\sin c_2y}{c^2(c_2\cosh c_1x - c_1\sin c_2y)}, \frac{-c_1c_2\cosh c_1x\cos c_2y}{c^2(c_2\cosh c_1x - c_1\sin c_2y)}, \frac{c_1\cosh c_1x - c_2\sin c_2y}{c(c_2\cosh c_1x - c_1\sin c_2y)}\right).$$

Then the image of G is the set $\{(x_1,x_2,x_3) \in \boldsymbol{S}^2 \mid x_3 \neq c_1/c_2\} \cup \{(0, \pm c_1/c, c_1/c_2)\}$. See Figure 1 for the hyperbolic cylinder and its image under G.

We remark that the 'isometry group' of $\boldsymbol{E}^3$ is the semi-direct product $\boldsymbol{E}^3 \rtimes SU(2)$ (see Appendix I), in contrast, $SL(2;\boldsymbol{C})$ can not be decomposed into a semi-direct product of $SU(2)$ and a complementary part. Then the transformation law $G_{\mathrm{g}\cdot f} = \mathrm{h}\cdot G_f$ for normal Gauss maps holds only for each element $\mathrm{g} = \mathrm{sh}$ in the normalizer $\mathscr{N}(S)$ of S:

$$\mathscr{N}(S) = \left\{\begin{pmatrix} z & w \\ 0 & 1/z\end{pmatrix}\,\middle|\, z, w \in \boldsymbol{C}, z \neq 0\right\}.$$

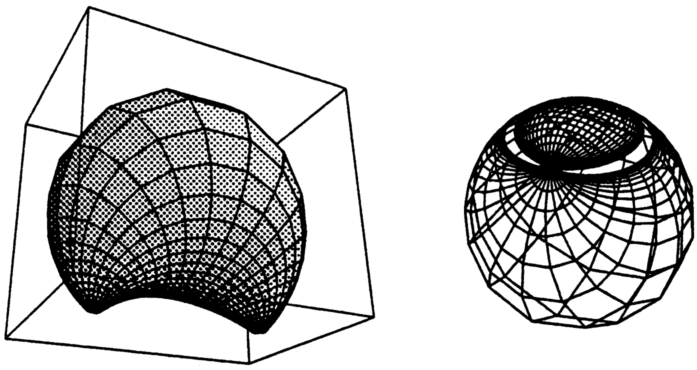

Figure 1.

Regarding $\boldsymbol{S}^2$ as the extended complex plane $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ by the stereographic projection $P_1 : \boldsymbol{S}^2 \backslash \{\underline{\mathbf{e}}_3\} \to \boldsymbol{C}$ (resp. $P_2 : \boldsymbol{S}^2 \backslash \{-\underline{\mathbf{e}}_3\} \to \boldsymbol{C}$), we can identify the normal Gauss map G with the map to $\hat{\boldsymbol{C}}$: On every contractible open set U,

$$
\begin{aligned}
&G = P_1 \circ G = \frac{q}{p} = \frac{\overline{\mathscr{E}_{22}}}{\mathscr{E}_{21}} \quad \left(\text{resp. } G = P_2 \circ G = \frac{p}{q} = \frac{\mathscr{E}_{21}}{\overline{\mathscr{E}_{22}}}\right), \\
&\text{where} \quad \mathscr{E} = \begin{pmatrix} \mathscr{E}_{11} & \mathscr{E}_{12} \\ \mathscr{E}_{21} & \mathscr{E}_{22} \end{pmatrix} = \mathscr{S}h, \quad h = \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix}.
\end{aligned} \tag{1.5}
$$

Remark 1.3. We here review the generalized Gauss map for a conformal immersion $f : M \to \boldsymbol{H}^3(-c^2)$. The *generalized Gauss map* $\mathscr{G} : M \to \mathrm{Gr}_2(\boldsymbol{L}^4)$ of f carries a point on M to the oriented spacelike 2-space in $\boldsymbol{L}^4$, which is given by the parallel translation of the tangent space of M at the point to the origin of $\boldsymbol{L}^4$. The Grassmann manifold $\mathrm{Gr}_2(\boldsymbol{L}^4)$ consisting of oriented spacelike 2-planes in $\boldsymbol{L}^4$ has the following structure of a homogeneous space:

$$\mathrm{Gr}_2(\boldsymbol{L}^4) = SL(2;\boldsymbol{C})/(\boldsymbol{R}_+ \cdot U(1)) = \boldsymbol{Q}_1^2 := \{[\mathrm{g}(\underline{\mathbf{e}}_1 - \sqrt{-1}\underline{\mathbf{e}}_2)\mathrm{g}^*] \mid \mathrm{g} \in SL(2;\boldsymbol{C})\},$$

where $[\mathbf{w}]$ is the complex line through 0 and $\mathbf{w} \in \boldsymbol{C}_1^4 = \boldsymbol{L}^4 \otimes_{\boldsymbol{R}} \boldsymbol{C}$. Take a complex coordinate $(\mathrm{g}_{11}/\mathrm{g}_{21}, \mathrm{g}_{12}/\mathrm{g}_{22}) \in \hat{\boldsymbol{C}} \times \hat{\boldsymbol{C}}$ $(\mathrm{g} = (\mathrm{g}_{ij}) \in SL(2;\boldsymbol{C}))$ on $\mathrm{Gr}_2(\boldsymbol{L}^4)$. Then $\mathscr{G}$ is considered as the pair $(\mathscr{G}_1, \mathscr{G}_2)$ of the following two *hyperbolic Gauss maps* (cf. [**Br**], [**AA1**]):

$$\mathscr{G}_1 = \frac{\mathscr{E}_{11}}{\mathscr{E}_{21}}, \quad \mathscr{G}_2 = \frac{\mathscr{E}_{12}}{\mathscr{E}_{22}} : M \to \hat{\boldsymbol{C}}.$$

($\mathscr{G}_1$ and $\mathscr{G}_2$ are independent of the choice of adapted framings, and then can be defined globally on M.) However, this decomposition of $\mathscr{G}$ is not invariant by the action of $SL(2;\boldsymbol{C})$ on $\boldsymbol{H}^3(-c^2)$. $\mathscr{G}_1, \mathscr{G}_2$ have the following relation to the $G : M \to \boldsymbol{S}^2 \cong \hat{\boldsymbol{C}}$ (via P_1):

$$\mathscr{G}_1 = \mathscr{S}[G], \quad \mathscr{G}_2 = \mathscr{S}[-1/\overline{G}],$$

where $\mathrm{g}[\zeta] = (\mathrm{g}_{11}\zeta + \mathrm{g}_{12})/(\mathrm{g}_{21}\zeta + \mathrm{g}_{22})$ $(\zeta \in \hat{\boldsymbol{C}})$, that is, the conformal action on $\hat{\boldsymbol{C}}$ by $\mathrm{g} = (\mathrm{g}_{ij}) \in SL(2;\boldsymbol{C})$.

Let ϕ be the dual $(1,0)$-form to $\mathscr{E}\cdot(\underline{\mathbf{e}_1} - \sqrt{-1}\underline{\mathbf{e}_2})$ on U. Then the induced metric is given by $f^*ds^2 = \phi \cdot \bar{\phi}$ and $d\phi = -\rho \wedge \phi$, where ρ stands for the connection form of f^*ds^2. We denote by H the mean curvature of f and by $\Phi = Q\phi \cdot \phi$ its Hopf differential. The pullback $\mathscr{E}^{-1}d\mathscr{E}$ of the Maurer-Cartan form τ on $SL(2;\boldsymbol{C})$ by the adapted framing $\mathscr{E}$ on U is represented as follows (cf. [**Br**], [**AA1**]):

$$\mathscr{E}^{-1}d\mathscr{E} = \mathscr{E}^*\tau_{\mathfrak{h}} + \mathscr{E}^*\tau_{\mathfrak{m}},$$

$$\mathscr{E}^*\tau_{\mathfrak{h}} = \frac{1}{2}\begin{pmatrix} \sqrt{-1}\rho & H\phi + \overline{Q\phi} \\ -H\bar{\phi} - Q\phi & -\sqrt{-1}\rho \end{pmatrix}, \quad \mathscr{E}^*\tau_{\mathfrak{m}} = \frac{c}{2}\begin{pmatrix} 0 & \phi \\ \bar{\phi} & 0 \end{pmatrix}, \tag{1.6}$$

where $\tau = \tau_{\mathfrak{h}} + \tau_{\mathfrak{m}}$ is the decomposition corresponding to the reductive orthogonal decomposition $\mathfrak{sl}(2;\boldsymbol{C}) = \mathfrak{su}(2) \oplus \sqrt{-1}\mathfrak{su}(2) =: \mathfrak{h} \oplus \mathfrak{m}$.

2. Kenmotsu type representation formula for surfaces in $\boldsymbol{H}^3(-c^2)$.

Let $f : M \to \boldsymbol{H}^3(-c^2)$ be a conformal immersion and $\mathscr{E} = \mathscr{S}h : U \to SL(2;\boldsymbol{C}) = S\cdot SU(2)$ the adapted framing of f on a contractible open set U of M. The framing $\mathscr{S} = \varpi \circ \mathscr{E} : M \to S \subset SL(2;\boldsymbol{C})$ satisfies

$$\mathscr{S}^{-1}d\mathscr{S} = h(\mathscr{E}^{-1}d\mathscr{E} - h^{-1}dh)h^{-1} = h\big(\mathscr{E}^*\tau_{\mathfrak{m}} + (\mathscr{E}^*\tau_{\mathfrak{h}} - h^{-1}dh)\big)h^{-1}$$

on every U, from which we obtain

$$\mathscr{S}^{-1}d\mathscr{S} + (\mathscr{S}^{-1}d\mathscr{S})^* = 2h(\mathscr{E}^*\tau_{\mathfrak{m}})h^{-1} = c(\alpha + \alpha^*),$$

where α is an $\mathfrak{sl}(2;\boldsymbol{C})$-valued $(1,0)$-form defined on U by

$$\alpha = hE_{12}h^*\phi, \quad \text{where} \quad E_{12} = \frac{1}{2}(\underline{\mathbf{e}_1} - \sqrt{-1}\underline{\mathbf{e}_2}).$$

We note that α is a global section of $T^{*(1,0)}M \otimes G^{-1}T^{(1,0)}\boldsymbol{S}^2$, where $G^{-1}T^{(1,0)}\boldsymbol{S}^2$ is the pullback bundle of $T^{(1,0)}\boldsymbol{S}^2$ via the normal Gauss map G of f. By means of each composition $P_i \circ G : M \to \hat{\boldsymbol{C}}$ $(i = 1,2)$ in (1.5), α can be represented by

$$\alpha = \begin{cases} \begin{pmatrix} -P_1 \circ G & (P_1 \circ G)^2 \\ -1 & P_1 \circ G \end{pmatrix}\omega_1 & \text{on} \quad G^{-1}(\boldsymbol{S}^2)\backslash\{\underline{\mathbf{e}_3}\} \subset M, \\ \begin{pmatrix} -P_2 \circ G & 1 \\ -(P_2 \circ G)^2 & P_2 \circ G \end{pmatrix}\omega_2 & \text{on} \quad G^{-1}(\boldsymbol{S}^2)\backslash\{\underline{-\mathbf{e}_3}\} \subset M, \end{cases}$$

where $\omega_1 = p^2\phi$ and $\omega_2 = q^2\phi$. We often denote this α briefly by

$$\alpha = \begin{pmatrix} -G & G^2 \\ -1 & G \end{pmatrix}\omega, \tag{2.1}$$

regarding G as $P_1 \circ G$ and $\omega = \omega_1$ as the nowhere-vanishing $(1,0)$-form defined everywhere on M.

The Lie algebra $\mathfrak{s}$ of S is the space of upper triangular matrices $\{t\underline{\mathbf{e}_3} + zE_{12} \mid t \in \boldsymbol{R}, z \in \boldsymbol{C}\}(\subset \mathfrak{sl}(2;\boldsymbol{C}))$. Then the $\mathfrak{s}$-valued 1-form $\mathscr{S}^{-1}d\mathscr{S}$ is represented by

$$\mathscr{S}^{-1}d\mathscr{S} = \frac{c}{2}(\alpha + \alpha^*) + \frac{c}{4}[\underline{\mathbf{e}}_3, \alpha + \alpha^*].$$

The $\mathfrak{h}$-part $\tau_{\mathfrak{h}}$ and $\mathfrak{m}$-part $\tau_{\mathfrak{m}}$ of $\mathscr{S}^{-1}d\mathscr{S}$ are given respectively by

$$\tau'_{\mathfrak{h}} = \frac{c}{4}[\underline{\mathbf{e}}_3, \alpha], \quad \tau''_{\mathfrak{h}} = \frac{c}{4}[\underline{\mathbf{e}}_3, \alpha^*], \quad \tau'_{\mathfrak{m}} = \frac{c}{2}\alpha, \quad \tau''_{\mathfrak{m}} = \frac{c}{2}\alpha^*.$$

From the equation (2.11) in [**AA1**], α and H satisfy the equation

$$d\alpha + \frac{c}{4}[[\underline{\mathbf{e}}_3, \alpha^*] \wedge \alpha] = -\frac{1}{2}H[\alpha^* \wedge \alpha]. \tag{2.2}$$

From this equation (2.2) combined with (2.1), we obtain that

$$\omega(= \omega_1) = \frac{2(\bar{G})_z}{\{c(1 - |G|^2) + H(1 + |G|^2)\}(1 + |G|^2)}dz. \tag{2.3}$$

From Proposition 2.1 and Remark 2.2 in [**AA1**], we have the following:

Proposition 2.1. *The induced metric f^*ds^2 on M and the Hopf differential Φ of f are given by*

$$f^*ds^2 = (1 + |G|^2)^2\omega \cdot \bar{\omega}, \tag{2.4}$$

$$\Phi = 2G_z\omega \cdot dz = \frac{4G_z(\bar{G})_z}{\{c(1 - |G|^2) + H(1 + |G|^2)\}(1 + |G|^2)}dz \cdot dz. \tag{2.5}$$

Remark 2.2. It follows from (2.3) and (2.4) that the above 1-form ω is smooth everywhere and G is nowhere-holomorphic on M. From (2.5), $G_z(w) = 0$ at $w \in M$ if and only if w is an umbilic point of f.

From the equations (2.1), (2.2) and (2.3), we also obtain the following nonlinear partial differential equation of second order for G.

Theorem 2.3. *The normal Gauss map $G(= P_1 \circ G) : M \to \boldsymbol{S}^2 \cong \hat{\boldsymbol{C}}$ of f satisfies*

$$\frac{c(1 - |G|^2) + H(1 + |G|^2)}{1 + |G|^2}G_{z\bar{z}} + \frac{2\{c|G|^2 - H(1 + |G|^2)\}\bar{G}}{(1 + |G|^2)^2}G_zG_{\bar{z}} = H_zG_{\bar{z}}. \tag{2.6}$$

Remark 2.4. If we use P_2 instead of P_1, $\omega = \omega_2$ and $G(= P_2 \circ G) : M \to \boldsymbol{S}^2 \cong \hat{\boldsymbol{C}}$ satisfy the equations (2.3) and (2.6) in which H replaced by $-H$.

Put $c = 0$ in the equation (2.6), it is the GH equation for the Gauss map of a surface in $\boldsymbol{E}^3$ with mean curvature H (see Appendix I). Moreover, when H is constant, the equation (2.6) implies the following harmonicity for the normal Gauss map. Hence we call it the *hyperbolic GH equation.*

Theorem 2.5. *For a CMC H conformal immersion $f : M \to \boldsymbol{H}^3(-c^2)$, the normal Gauss map $G(= P_1 \circ G)$ of f is a non-holomorphic harmonic map from M into $\hat{\boldsymbol{C}}$ equipped with the following metric $h_{c,H}$:*

$$h_{c,H} = \frac{4|d\zeta|^2}{|(1 + |\zeta|^2)\{c(1 - |\zeta|^2) + H(1 + |\zeta|^2)\}|}.$$

This metric $h_{c,H}$ is regular if and only if $|H| > c$. When $H = c$ (resp. $H = -c$), $h_{c,H}$ has a singular point only at the point $\zeta = \infty \in \hat{\boldsymbol{C}}$ (resp. $\zeta = 0 \in \hat{\boldsymbol{C}}$). When $|H| < c$, the singular set of $h_{c,H}$ is the round circle $|\zeta| = \sqrt{(c+H)/(c-H)}$ in $\hat{\boldsymbol{C}}$.

Remark 2.6. Especially the singular metric $h_0 := (1/c)h_{c,0} = 4|d\zeta|^2/|(1-|\zeta|^2)(1+|\zeta|^2)|$ (for $H = 0$) we call *Kokubu's metric* ([**Ko1**]). The isometry group of h_0 is biggest among those of the singular metrics $h_{c,H}(|H| < c)$. Namely, when $0 < |H| < c$ the orientation preserving isometry group of $(\boldsymbol{S}^2, h_{c,H})$ is given by $U(1)/\{\pm \underline{\mathbf{e}}_0\}$, in contrast, when $H = 0$ it is given by the subgroup in $SL(2;\boldsymbol{C})/\{\pm \underline{\mathbf{e}}_0\}$ generated by $U(1)$ and $\underline{\mathbf{e}}_1$.

Now we can state the following:

Theorem 2.7 (Kenmotsu type representation formula in $\boldsymbol{H}^3(-c^2)$). *Let M be a simply connected Riemann surface with a reference point z_0. Given a real-valued smooth function H on M, let $v : M \to \hat{\boldsymbol{C}}$ be a non-holomorphic smooth map satisfying the following equation*:

$$\frac{c(1-|v|^2) + H(1+|v|^2)}{1+|v|^2} v_{z\bar{z}} + \frac{2\{c|v|^2 - H(1+|v|^2)\}\bar{v}}{(1+|v|^2)^2} v_z v_{\bar{z}} = H_z v_{\bar{z}}. \tag{2.6$'$}$$

Define a 1*-form ω on M as follows and assume that it is smooth on M*:

$$\omega = \frac{2(\bar{v})_z}{\{c(1-|v|^2) + H(1+|v|^2)\}(1+|v|^2)} dz.$$

Put an $\mathfrak{sl}(2;\boldsymbol{C})$-valued 1*-form α and an $\mathfrak{s}$-valued* 1*-form μ on M by*

$$\alpha = \begin{pmatrix} -v & v^2 \\ -1 & v \end{pmatrix} \omega, \quad \mu = \frac{c}{2}(\alpha + \alpha^*) + \frac{c}{4}[\underline{\mathbf{e}}_3, \alpha + \alpha^*].$$

Then there exists uniquely a smooth map $\mathscr{S} : M \to S$ such that $\mathscr{S}(z_0) = \underline{\mathbf{e}}_0$ and $\mathscr{S}^{-1} d\mathscr{S} = \mu$. Put $f = (1/c)\mathscr{S}\mathscr{S}^*$, then $f : M \to \boldsymbol{H}^3(-c^2)$ is a conformal immersion outside $\{w \in M | \omega(w) = 0\}$ with prescribed mean curvature H and the normal Gauss map $G = P_1^{-1} \circ v$. The induced metric $f^*ds^2 = (1+|v|^2)^2 \omega \cdot \bar{\omega}$ and the Hopf differential $\Phi = 2v_z\omega \cdot dz$.

Proof. One needs only to check the hyperbolic GH equation (2.6′) implies the integrability condition $d\mu + \mu \wedge \mu = 0$. □

Remark 2.8. When $H \equiv 0$, the Kenmotsu type representation formula coincides with Kokubu's representation formula [**Ko1**] for minimal surfaces in $\boldsymbol{H}^3(-c^2)$.

Remark 2.9. The Kenmotsu type representation formula for surfaces in $\boldsymbol{H}^3(-c^2)$ can be deformed to the Kenmotsu representation formula for surfaces in $\boldsymbol{E}^3$, as the Lie group $SL(2;\boldsymbol{C})$ collapses into the Abelian group $\boldsymbol{C}^3$ (cf. [**UY**], [**AA1**]).

3. Adjusted representation formulas for CMC surfaces in $\boldsymbol{H}^3(-c^2)$.

In this section, we concentrate on CMC surfaces in $\boldsymbol{H}^3(-c^2)$. We introduce the notion of 'adjusting' the normal Gauss maps to more suitable maps. The Bryant formula ([**Br**], [**UY**]) represents CMC c surfaces in $\boldsymbol{H}^3(-c^2)$ by means of holomorphic data, and the Kenmotsu-Bryant type formula ([**AA1**]) represents CMC H ($|H|>c$) surfaces in $\boldsymbol{H}^3(-c^2)$ by means of harmonic maps to $(\boldsymbol{S}^2, g_0)$. The Gauss data in these formulas can be regarded as an adjustment of the normal Gauss map through the Lawson correspondence at adapted frame level as below. With these understood, we can also obtain the Kokubu-Bryant type representation formula for CMC H ($|H|<c$) surfaces in $\boldsymbol{H}^3(-c^2)$ by means of harmonic maps to $\boldsymbol{S}^2$ equipped with Kokubu's metric.

The *Lawson correspondence* [**L**] is a bijective correspondence between the space of isometric immersions with CMC $H(\geqq c)$ into $\boldsymbol{H}^3(-c^2)$ and the space of isometric immersions with CMC $H_0 := \sqrt{H^2-c^2}$ into $\boldsymbol{E}^3$. This correspondence is local or for simply connected surfaces. Let M be a contractible Riemann 2-manifold with $(1,0)$-type coframe ϕ, and the metric is $ds_0^2 = \phi\cdot\bar{\phi}$. We denote by ρ the connection form on M. Let $f: M\to \boldsymbol{H}^3(-c^2)$ be an isometric immersion with CMC $H=H_c$ and the Hopf differential $\Phi_0 = Q\phi\cdot\phi$. Let $\mathscr{E}=\mathscr{E}_c : M\to SL(2;\boldsymbol{C})$ be the adapted framing of f, and hence it satisfies the following equation (3.1) in which $t=c$:

$$\mathscr{E}_t^{-1}d\mathscr{E}_t = \tau_t := \frac{1}{2}\begin{pmatrix} \sqrt{-1}\rho & (t+H_t)\phi+\overline{Q\phi} \\ (t-H_t)\bar{\phi}-Q\phi & -\sqrt{-1}\rho \end{pmatrix}. \tag{3.1}$$

Put $H_t = \sqrt{H_0^2+t^2}$, then this equation (3.1) for each $t\geqq 0$ is also integrable (cf. [**Bo**, Theorem 14.3]). Namely, we can obtain a smooth 1-parameter family $\{\mathscr{E}_t : M\to SL(2;\boldsymbol{C})\,|\,t\geqq 0\}$ of the unique solution of each equation $\mathscr{E}_t^{-1}d\mathscr{E}_t=\tau_t$ (up to left translation by a constant matrix in $SL(2;\boldsymbol{C})$). When $t>0$, put $f_t := (1/t)\mathscr{E}_t\mathscr{E}_t^*$. Here we note that $f_c = f$. $f_t: M\to \boldsymbol{H}^3(-t^2)$ $(t>0)$ is an isometric CMC H_t immersion with the Hopf differential Φ_0, and $\mathscr{E}_t$ is the adapted framing of f_t. When $t=0$, the solution $\mathscr{E}_0 : M\to SU(2)$ gives a harmonic map $G_0 = [\mathscr{E}_0] : M\to(\boldsymbol{S}^2, g_0)$, and there exists an isometric CMC H_0 immersion $f_0 : M\to\boldsymbol{E}^3$ with the Hopf differential Φ_0 and the Gauss map G_0. The family $\{f_t|t\geqq 0\}$ of the above immersions is called the *canonical* 1-*parameter family* associated with $f=f_c : M\to\boldsymbol{H}^3(-c^2)$ (cf. [**UY**]). Hence $\{\mathscr{E}_t|t\geqq 0\}$ is the *canonical* 1-*parameter family of adapted framings* associated with $f=f_c$. (Here we remark that the above $\mathscr{E}_0$ is not exactly an adapted framing of f_0 in the sense of Appendix I.)

Example 2. Let $f_c(=f)$ be the hyperbolic cylinder defined in Example 1. Put $a=\cosh^{-1}(c/c_1)(=\sinh^{-1}(c/c_2))$. The canonical 1-parameter family $\{f_t|t\geqq 0\}$ of f_c is given as follows:

$$f_t(t>0): \boldsymbol{R}^2\to\boldsymbol{H}^3_+(-t^2),$$

$$f_t(x,y) = \frac{1}{t(k_2\cosh k_1x - k_1\sin k_2y)}\left(k_2\sinh k_1x,\ k_1\cos k_2y,\ \frac{k_1k_2}{t}\right)$$

$$f_0 : \boldsymbol{R}^2\to\boldsymbol{E}^3,\quad f_0(x,y) = \left(x,\ \frac{1}{k}\cos ky,\ \frac{1}{k}\sin ky\right),$$

where

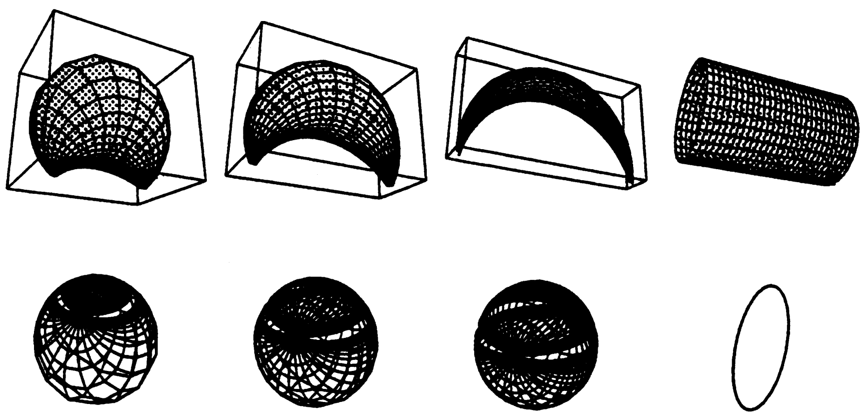

Figure 2.

$$k = \frac{2c}{\sinh 2a} (= 2H_0), \quad k_1 = \frac{t}{\cosh a(t)}, \quad k_2 = \frac{t}{\sinh a(t)},$$

$$a(t) = \frac{1}{2} \log\left\{ \frac{t}{c} \left(\sinh 2a + \sqrt{\sinh^2 2a + \frac{c^2}{t^2}} \right) \right\}.$$

See Figure 2 for this deformation $\{f_t\}_{t \in [0,c]}$, together with the images under the normal Gauss maps G_t $(0 < t \leqq c)$ and the Gauss map G_0.

Now we decompose the adapted framing $\mathscr{E}_t : M \to SL(2;\boldsymbol{C})$ $(t \geqq 0)$ into $\mathscr{E}_t = \mathscr{S}_t h_t$ corresponding to the decomposition $SL(2;\boldsymbol{C}) = S \cdot SU(2)$, where $\mathscr{S}_t : M \to S$ and $h_t : M \to SU(2)$. (When $t = 0$, $\mathscr{S}_0 = \underline{\mathbf{e}_0}$ and $h_0 = \mathscr{E}_0$.) When $t > 0$, $G_t := [h_t]$ is the normal Gauss map of f_t. Recall that the Kenmotsu type representation formula for $f = f_c$ was given via the integrable equation of first order for the framing $\mathscr{S}_c = \mathscr{E}_c h_c^{-1}$. It is described by only G_c and some constants, in which both ρ and Φ_0 never appear. Since $\mathscr{S}_c^{-1} d\mathscr{S}_c = h_c(\mathscr{E}_c^{-1} d\mathscr{E}_c - h_c^{-1} dh_c)h_c^*$, ρ and Φ_0 in $\mathscr{E}_c^{-1} d\mathscr{E}_c$ appear only in the term $h_c^{-1} dh_c$. ρ and Φ_0 are independent of $t(\geqq 0)$, and moreover, the manner which they appear in the equation (3.1) is also independent of $t(\geqq 0)$. Then the framing $\mathscr{F}_t := \mathscr{E}_c h_t^{-1}$ of $f = f_c$ also satisfies an equation described by only $G_t : M \to \boldsymbol{S}^2 \cong \hat{\boldsymbol{C}}$ (via P_1):

$$\begin{aligned} \mathscr{F}_t^{-1} d\mathscr{F}_t &= h_t(\mathscr{E}_c^{-1} d\mathscr{E}_c - \mathscr{E}_t^{-1} d\mathscr{E}_t)h_t^{-1} + \mathscr{S}_t^{-1} d\mathscr{S}_t \\ &= \frac{1}{2}(c + H_c - H_t)\alpha_t + \frac{1}{2}(c - H_c + H_t)\alpha_t^* + \frac{t}{4}[\underline{\mathbf{e}_3}, \alpha_t + \alpha_t^*], \end{aligned} \tag{3.2}$$

$$\text{where} \quad \alpha_t = h_t E_{12} h_t^* \phi = \begin{pmatrix} -G_t & G_t^2 \\ -1 & G_t \end{pmatrix} \omega_t,$$

$$\omega_t = \frac{2(\overline{G_t})_z}{\{t(1 - |G_t|^2) + H_t(1 + |G_t|^2)\}(1 + |G_t|^2)} dz.$$

In this way, we can adjust the normal Gauss map G_c of f_c to each G_t and deform the framing $\mathscr{S}_c$ of f_c to the framing $\mathscr{F}_c$ satisfying the equation (3.2). If we will adjust G_c to G_0, we can obtain the framing $\mathscr{F} := \mathscr{F}_0 = \mathscr{E}_c h_0^{-1}$ of $f = f_c$ satisfying the integrable differential equation described by only the harmonic map $G_0 : M \to (\boldsymbol{S}^2, g_0)$. When

$H_0 \neq 0$, this integrable equation gives the *Kenmotsu-Bryant type representation formula* [**AA1**, Theorem 3.2] for a CMC $H_c(>c)$ immersion f_c. When $H_0 = 0$, we can obtain the *Bryant representation formula* [**Br**, Theorem A] (reformulated in [**UY**]) for a CMC c immersion. Hence we will call the map G_0 the *adjusted Gauss map* of f. For example, the adjusted Gauss map of the hyperbolic cylinder f_c in $\boldsymbol{H}^3_+(-c^2)$ is given by the Gauss map G_0 of the right circular cylinder f_0 in $\boldsymbol{E}^3$, as in Example 2. We remark that, for a non-simply connected surface M immersed in $\boldsymbol{H}^3(-c^2)$, the adjusted Gauss map is only defined on the universal cover $\tilde{M}$.

We can apply the above mechanism to leading the more suitable representation formula in the case of CMC H ($|H| < c$) surfaces in $\boldsymbol{H}^3(-c^2)$.

There exists a Lawson type bijective correspondence between the space of isometric immersions with CMC H ($0 < H < c$) in $\boldsymbol{H}^3(-c^2)$ and the space of isometric minimal immersions into $\boldsymbol{H}^3(-c_0^2)$ of $c_0 = \sqrt{c^2 - H^2}$ (cf. [**F**]). It follows from this Lawson type correspondence that there exist the *canonical* 1*-parameter family* $\{f_t : M \to \boldsymbol{H}^3(-t^2) \mid t \geqq c_0\}$ of isometric CMC $H_t = \sqrt{t^2 - c_0^2}$ immersions with the Hopf differential Φ_0, and the *canonical* 1*-parameter family of the adapted framings* $\mathscr{E}_t : M \to SL(2;\boldsymbol{C})$ ($t \geqq c_0$) of f_t. For every $t \geqq c_0$, we decompose $\mathscr{E}_t = \mathscr{S}_t h_t$, where $\mathscr{S}_t : M \to S$ and $h_t : M \to SU(2)$. (In this case, even if $t \searrow c_0$, S-component $\mathscr{S}_t$ never collapse to $\underline{\mathbf{e}_0}$ as in the previous case of $H \geqq c$, that is, $\mathscr{S}_{c_0} \not\equiv \underline{\mathbf{e}_0}$.) For a conformal CMC $H = H_c$ ($0 \leqq H < c$) immersion $f = f_c : M \to \boldsymbol{H}^3(-c^2)$, choose the framing $\mathscr{F}_t := \mathscr{E}_c h_t^{-1}$ ($t \geqq c_0$). As in the previous argument, this framing $\mathscr{F}_t$ gives another representation of $f = f_c$ by means of the normal Gauss map $G_t = [h_t]$ of f_t ($t \geqq c_0$). Especially we will choose the Gauss data G_{c_0}, which is a harmonic map to $\boldsymbol{S}^2 \cong \hat{\boldsymbol{C}}$ equipped with Kokubu's metric. Recall that it has the biggest isometry group among the singular metrics $\{h_{t,H_t} | t \geqq c_0\}$. We will also call the map $G_{c_0} = [h_{c_0}] : M \to \boldsymbol{S}^2$ the *adjusted Gauss map* of f. Now we can obtain the following Kokubu-Bryant type representation formula.

Theorem 3.1 (Kokubu-Bryant type representation formula). *Let M be a simply connected Riemann surface with a reference point z_0. Let $v : M \to (\boldsymbol{S}^2 \cong \hat{\boldsymbol{C}}, h_0 = 4|d\zeta|^2/|1 - |\zeta|^4|)$ be a non-holomorphic harmonic map. For a nonnegative constant H such that $H < c$, put $c_0 = \sqrt{c^2 - H^2}$. Define a* 1*-form ω on M as follows and assume that it is smooth on M*:

$$\omega = \frac{2(\bar{v})_z}{c_0(1 - |v|^4)} dz.$$

Put an $\mathfrak{sl}(2;\boldsymbol{C})$-valued 1*-form α and an $\mathfrak{sl}(2;\boldsymbol{C})$-valued* 1*-form τ on M by*

$$\alpha = \begin{pmatrix} -v & v^2 \\ -1 & v \end{pmatrix} \omega, \quad \tau = \frac{1}{2}\{(c + H)\alpha + (c - H)\alpha^*\} + \frac{c_0}{4}[\underline{\mathbf{e}_3}, \alpha + \alpha^*].$$

Then there exists uniquely a smooth map $\mathscr{F} : M \to SL(2;\boldsymbol{C})$ such that $\mathscr{F}(z_0) = \underline{\mathbf{e}_0}$ and $\mathscr{F}^{-1}d\mathscr{F} = \tau$. Put $f = (1/c)\mathscr{F}\mathscr{F}^*$, then $f : M \to \boldsymbol{H}^3(-c^2)$ is a conformal CMC H immersion outside isolated degenerate points $\{z \in M \mid \omega(z) = 0\}$ with the adjusted Gauss map v. The induced metric f^*ds^2 and the Gauss curvature K of f are given by

$$f^*ds^2 = (1 + |v|^2)^2 \omega \cdot \bar{\omega}, \quad K = -c_0^2 \left[1 + \frac{|v_z|^2(1 - |v|^2)^2}{|v_{\bar{z}}|^2(1 + |v|^2)^2}\right].$$

Conversely, every conformal CMC H $(|H| < c)$ immersion $f : M \to \boldsymbol{H}^3(-c^2)$ is congruent with the one constructed by a nowhere-holomorphic harmonic map $g : M \to (\hat{\boldsymbol{C}}, h_0)$ as above.

Remark 3.2. For minimal surfaces in $\boldsymbol{H}^3(-c^2)$, the Kokubu-Bryant type representation formula coincides with Kokubu's representation formula [**Ko1**], and the adjusted Gauss map is the normal Gauss map.

4. Construction of complete simply connected CMC surfaces in $\boldsymbol{H}^3(-1)$.

In this section, we give a construction of complete simply connected CMC H $(|H| < 1)$ surfaces embedded in $\boldsymbol{H}^3 = \boldsymbol{H}^3(-1)$ by applying the Kenmotsu type representation formula. We note that our construction is different from those of Anderson [**An**], Polthier [**P**] and Tonegawa [**T**]. It is rather similar to that in [**Ak**].

First, we rewrite the formula by using the upper half-space model $\boldsymbol{H}^3_+$ of $\boldsymbol{H}^3$.

Theorem 4.1 (Kenmotsu type representation formula for CMC surfaces in $\boldsymbol{H}^3_+$). *Let M be a simply connected Riemann surface with a reference point z_0. For a non-negative constant H satisfying $H < 1$, put $k = \sqrt{(1-H)/(1+H)}$ $(0 < k \leqq 1)$. Let v be a non-holomorphic harmonic map from M to $\hat{\boldsymbol{C}} \cong \boldsymbol{S}^2$ (via P_2^{-1}) equipped with the singular metric*

$$h_k = \frac{4|d\zeta|^2}{|(k^2 - |\zeta|^2)(1 + |\zeta|^2)|}.$$

Define a 1-form ω on M as follows and assume that it is smooth on M:

$$\omega = \ell \cdot \frac{(\bar{v})_z}{(k^2 - |v|^2)(1 + |v|^2)} dz, \quad \ell := \frac{2}{1+H}. \tag{4.1}$$

Then the path integrals

$$t(z) = \exp\left[-2\,\mathrm{Re}\int_{z_0}^{z} v\omega\right], \quad w(z) = \int_{z_0}^{z} t(\omega - \bar{v}^2\bar{\omega}) \tag{4.2}$$

do not depend on any path from z_0 to z in M, and

$$f(z) = (w(z), t(z)) : M \to \boldsymbol{H}^3_+$$

is a conformal CMC H immersion outside $\{z \in M | \omega(z) = 0\}$ with the normal Gauss map $G = P_2^{-1} \circ v : M \to \boldsymbol{S}^2$. The induced metric is given by

$$f^* ds^2 = \ell^2 \cdot \frac{|v_{\bar{z}}|^2 |dz|^2}{(k^2 - |v|^2)^2}. \tag{4.3}$$

The singular set of the above metric h_k is the round circle $|\zeta| = k$ in $\hat{\boldsymbol{C}}$. Let $\boldsymbol{D}_k$ denote the open disk $D_k = \{\zeta \in \boldsymbol{C} \,|\, |\zeta| < k\}$ equipped with the metric h_k.

Remark 4.2. The diameter of $\boldsymbol{D}_k$ is finite, which implies that $\boldsymbol{D}_k$ is not complete. The Gauss curvature at each point ζ of $\boldsymbol{D}_k$ is negative (resp. nonpositive) if $0 < H < 1$ (resp. if $H = 0$) and decreases uniformly to $-\infty$ as ζ goes to the boundary $\partial\overline{D_k}$.

Let v be a nowhere-holomorphic harmonic map from the unit open disk $D = \{z \in \boldsymbol{C} \mid |z| < 1\}$ to $\boldsymbol{D}_k$. Applying the above formula to the map v, we can construct a simply connected CMC H surface in $\boldsymbol{H}^3_+$. The following result guarantees the existence of such a harmonic map and the completeness of the CMC surface constructed from it.

Theorem 4.3 ([**AA4**]). *Given $0 < \beta \leqq 1$, let φ be a $C^{1,\beta}$ diffeomorphism from $\partial\bar{D}$ to $\partial\overline{D_k}$ such that $\deg(\varphi) = -1$. Then there exists uniquely a harmonic diffeomorphism $v \in C^\infty(D, \boldsymbol{D}_k)$ satisfying the following properties*:

(1) $v \in C^{1,\gamma}(\bar{D}, \overline{D_k})$ *for* $0 < \gamma < \beta$.

(2) $v|_{\partial\bar{D}} = \varphi$.

(3) *There exists a positive constant C such that $C^{-1}(1 - |z|^2) \leqq k^2 - |v(z)|^2 \leqq C(1 - |z|^2)$ for all $z \in \bar{D}$.*

(4) *There exists a positive constant δ such that $0 < \delta \leqq |v_{\bar{z}}| \leqq \delta^{-1}$ on $\bar{D}$.*

Moreover, we can prove the following:

Theorem 4.4. *Give a constant H such that $0 \leqq H < 1$, and put $k = \sqrt{(1-H)/(1+H)}$ $(0 < k \leqq 1)$. For $0 < \beta \leqq 1$, let φ be a $C^{1,\beta}$ diffeomorphism from $\partial\bar{D}$ to $\partial\overline{D_k}$ such that $\deg(\varphi) = -1$. Then there exists a complete conformal CMC H embedding $f : D \to \boldsymbol{H}^3_+$ whose normal Gauss map G is the composition $P_2^{-1} \circ v$ of P_2^{-1} and a harmonic diffeomorphism $v \in C^\infty(D, \boldsymbol{D}_k)$ satisfying $v \in C^{1,\gamma}(\bar{D}, \overline{D_k})$ for $0 < \gamma < \beta$ and $v|_{\partial\bar{D}} = \varphi$. Moreover, the embedding f can be extended uniquely to a $C^{1,\gamma}$ embedding $\hat{f} : \bar{D} \to \overline{\boldsymbol{H}^3_+} := \{(w,t) \in \boldsymbol{C} \times \boldsymbol{R} \mid w \in \boldsymbol{C}, t \geqq 0\}$ satisfying $\hat{f}(\partial\bar{D}) \subset \partial\overline{\boldsymbol{H}^3_+} = \{(w,0) \in \boldsymbol{C} \times \boldsymbol{R} \mid w \in \boldsymbol{C}\}$.*

Proof. Let $v \in C^{1,\gamma}(\bar{D}, \overline{D_k}) \cap C^\infty(D, \boldsymbol{D}_k)$ be the nowhere-holomorphic harmonic diffeomorphism with $v|_{\partial\bar{D}} = \varphi$, which is constructed in Theorem 4.3. By Theorem 4.1, we can construct a conformal CMC H immersion $f : D \to \boldsymbol{H}^3_+$ whose normal Gauss map coincides with $P_2^{-1} \circ v$. First, we prove that $f(D)$ is complete in $\boldsymbol{H}^3_+$. It follows from (4.3) combined with (3) and (4) in Theorem 4.3 that

$$f^*ds^2 \geqq \frac{\ell^2\delta^2}{4C^2} \cdot \frac{4|dz|^2}{(1-|z|^2)^2}.$$

Here $4|dz|^2/(1-|z|^2)^2$ is the Poincaré metric on D. Then $f(D)$ is complete in $\boldsymbol{H}^3_+$.

Second, we prove that f can be extended uniquely up to a $C^{1,\gamma}$ map $\hat{f} : \bar{D} \to \overline{\boldsymbol{H}^3_+}$ satisfying $f(\partial\bar{D}) \subset \partial\overline{\boldsymbol{H}^3_+}$, for all $0 < \gamma < \beta$. The uniqueness is obvious if such an extension exists. (4.2) implies the following:

$$d(\log t(z)) = -(v\omega + \overline{v\omega}), \quad dw(z) = t(\omega - \bar{v}^2\bar{\omega}). \tag{4.4}$$

Represent $z \in \bar{D}$ and $v(z) \in \overline{D_k}$ as $z = \rho e^{\sqrt{-1}\eta}$ and $v(z) = r(z)e^{\sqrt{-1}\theta(z)}$ in terms of polar coordinates. Then a similar argument to [**LT**, Lemma 1.3] asserts that on $\partial\bar{D}$

$$\left(\frac{\partial r}{\partial\rho}\right)(e^{\sqrt{-1}\eta}) = k\,|d\varphi(e^{\sqrt{-1}\eta})|, \quad \left(\frac{\partial\theta}{\partial\rho}\right)(e^{\sqrt{-1}\eta}) = 0 \tag{4.5}$$

(see [**AA4**]). By (4.1), (4.4) and (4.5), then there exist $a_1(z), a_2(z) \in C^{0,\gamma}(\bar{D})$ and

$b_1(z), b_2(z), c(z) \in C^{0,\gamma}(\bar{D}, \boldsymbol{C})$ such that

$$\frac{\partial(\log t(z))}{\partial \rho} = -\frac{1 + a_1(z)(1-\rho)^\gamma}{1-\rho}, \quad \frac{\partial(\log t(z))}{\partial \eta} = a_2(z)(1-\rho)^\gamma, \tag{4.6}$$

$$\frac{\partial w(z)}{\partial \rho} = \frac{t(z)\{(1-k)b_1(z) + c(z)(1-\rho)^\gamma\}}{1-\rho}, \quad \frac{\partial w(z)}{\partial \eta} = \frac{t(z)b_2(z)}{1-\rho}, \tag{4.7}$$

where $b_1(z)$ and $b_2(z)$ never vanish around $\partial \bar{D}$. (4.6) implies that $t(z)$ can be extended to a $C^{1,\gamma}$ function $\hat{t}(z)$ on $\bar{D}$ satisfying

$$C_1^{-1}(1-\rho) \leqq \hat{t}(z) \leqq C_1(1-\rho), \tag{4.8}$$

where C_1 is a positive constant depending only on γ, δ and C. From (4.7) and (4.8), there exist an extension $\hat{w} \in C^{1,\gamma}(\bar{D}, \boldsymbol{C})$ of $w(z)$ and $\tilde{b}_1(z), \tilde{b}_2(z), \tilde{c}(z) \in C^{0,\gamma}(\bar{D}, \boldsymbol{C})$ such that

$$\frac{\partial \hat{w}(z)}{\partial \rho} = (1-k)\tilde{b}_1(z) + \tilde{c}(z)(1-\rho)^\gamma, \quad \frac{\partial \hat{w}(z)}{\partial \eta} = \tilde{b}_2(z) \tag{4.9}$$

where $\tilde{b}_1(z)$ and $\tilde{b}_2(z)$ also never vanish around $\partial \bar{D}$. Then we obtain a $C^{1,\gamma}$ extension $\hat{f} = (\hat{w}, \hat{t}) : \bar{D} \to \overline{\boldsymbol{H}_+^3}$ of f satisfying $\hat{f}(\partial \bar{D}) \subset \partial \overline{\boldsymbol{H}_+^3}$.

Finally, we prove that $\hat{f}$ is an embedding. Let $\pi : \overline{\boldsymbol{H}_+^3} \ni (w, t) \mapsto w = (w, 0) \in \partial \overline{\boldsymbol{H}_+^3}$ be the projection, and put $\mathscr{D} := \pi(f(D))$. From the fact that f is a C^∞ immersion and $(P_2^{-1} \circ \nu)(D) \subset \boldsymbol{S}_+^2 := \{(x_1, x_2, x_3) \in \boldsymbol{S}^2 \mid x_3 > 0\}$, $\pi \circ f : D \to \mathscr{D}$ is a local diffeomorphism, and then $\mathscr{D}$ is open in $\boldsymbol{H}_+^3$. We also note that $\overline{\mathscr{D}} = \pi(\hat{f}(\bar{D}))$. Suppose that $f(D)$ is a graph over $\mathscr{D}$. It should be pointed out that the unit normal vector field along the $C^{1,\gamma}$ curve $\hat{f}|_{\partial \bar{D}} : \partial \bar{D} \to \partial \overline{\boldsymbol{H}_+^3}$ coincides with $\nu/|\nu| : \partial \bar{D} \to \boldsymbol{C}(\cong \partial \overline{\boldsymbol{H}_+^3})$. From the fact $|\nu| = k$ on $\partial \bar{D}$ and (2) in Theorem 4.3, it is not hard to show that $\hat{f}|_{\partial \bar{D}} : \partial \bar{D} \to \partial \overline{\boldsymbol{H}_+^3}$ is a simple closed $C^{1,\gamma}$ curve. This implies that $\hat{f}(\bar{D})$ is also a graph over $\overline{\mathscr{D}}$. In this case, it is obvious that $\hat{f}$ is an embedding. Hence, to complete the proof, it is enough to show that $f(D)$ is a graph over $\mathscr{D}$.

Give any R $(0 < R < k)$ and fix it. Let $\Omega_R := \nu^{-1}(D_R)$ $(\subset D)$ be the inverse image of $D_R := \{\zeta \in \boldsymbol{C} \mid |\zeta| < R\}$ by ν. Note that $\min\{|z| \mid z \in \partial \overline{\Omega_R}\} \nearrow 1$ if $R \nearrow k$. By (4.6), (4.8) and (4.9), we have

$$\frac{\partial t(z)}{\partial \eta} \Big/ \frac{\partial w(z)}{\partial \eta} = O((k-R)^\gamma) \quad (R \nearrow k). \tag{4.10}$$

It then follows from (4.10) that the unit normal vector filed N_R along the C^∞ curve $c_R := \pi \circ (f|_{\partial \overline{\Omega_R}}) : \partial \overline{\Omega_R} \to \partial \overline{\boldsymbol{H}_+^3}$ almost coincides with $\nu/|\nu| = \nu/R : \partial \overline{\Omega_R} \to \boldsymbol{C}(\cong \partial \overline{\boldsymbol{H}_+^3})$ for $R(<k)$ sufficiently close to k. Indeed,

$$\left| N_R(z) - \frac{\nu}{R}(z) \right| = O((k-R)^\gamma) \quad (R \nearrow k). \tag{4.11}$$

By (4.11) combined with (2) in Theorem 4.3, it is also not hard to show that, for $R(<k)$ sufficiently close to k, $c_R : \partial \overline{\Omega_R} \to \partial \overline{\boldsymbol{H}_+^3}$ is a simple closed C^∞ curve. Then the Jordan-Brouwer Separation Theorem implies that there exists a relatively compact simply

connected domain $\mathscr{D}_R$ in $\partial \overline{\boldsymbol{H}^3_+}(\cong \boldsymbol{C})$ such that

$$\partial \overline{\mathscr{D}_R} = c_R(\partial \overline{\Omega_R}), \quad \partial \overline{\boldsymbol{H}^3_+} = c_R(\partial \overline{\Omega_R}) \amalg \mathscr{D}_R \amalg (\partial \overline{\boldsymbol{H}^3_+} \backslash \overline{\mathscr{D}_R}).$$

Since $\pi \circ (f|_{\overline{\Omega_R}}) : \overline{\Omega_R} \to \partial \overline{\boldsymbol{H}^3_+}$ is a local diffeomorphism, we have $\overline{\mathscr{D}_R} = \pi(f(\overline{\Omega_R}))$. For any point $w \in \overline{\mathscr{D}_R}$ the set $(\pi \circ (f|_{\overline{\Omega_R}}))^{-1}(w)$ is finite, and then $\pi \circ (f|_{\overline{\Omega_R}}) : \overline{\Omega_R} \to \overline{\mathscr{D}_R}$ is a covering map. Hence $\pi \circ (f|_{\overline{\Omega_R}})$ must be a diffeomorphism because $\overline{\mathscr{D}_R}$ is a simply connected. This implies that $f(\overline{\Omega_R})$ is a graph over $\overline{\mathscr{D}_R}$ for each $0 < R < k$. Letting $R \nearrow k$, and then $\overline{\Omega_R} \to D$, $\overline{\mathscr{D}_R} \to \mathscr{D}$. Therefore, $f(D)$ is a graph over $\mathscr{D}$. This completes the proof of Theorem 4.4.

Appendix I. The Kenmotsu representation formula for surfaces in $\boldsymbol{E}^3$.

We give another proof of the Kenmotsu representation formula [**Ke**] for surfaces in $\boldsymbol{E}^3$, which method could be extended in the case of non-flat Riemannian 3-space forms as in Section 2 and [**AA2**].

Under the identification $\boldsymbol{E}^3 \cong \mathrm{span}\{\underline{\mathrm{e}_1}, \underline{\mathrm{e}_2}, \underline{\mathrm{e}_3}\} = \sqrt{-1}\mathfrak{su}(2)$, the identity component of the isometry group of $\boldsymbol{E}^3$ is given by $(\boldsymbol{E}^3 \rtimes SU(2))/\{\pm 1\}$, where $\boldsymbol{E}^3 \rtimes SU(2)$ acts $\boldsymbol{E}^3$ by

$$(\mathbf{v}, \mathrm{h}) \cdot \mathbf{x} = \mathrm{h}\underline{\mathbf{x}}\mathrm{h}^* + \underline{\mathbf{v}}, \quad (\mathbf{x}, \mathbf{v} \in \boldsymbol{E}^3, \mathrm{h} \in SU(2)).$$

Let f be a conformal immersion from a contractible Riemann surface M into $\boldsymbol{E}^3$ with the Gauss map g. A smooth map $\mathscr{E} = (f, h) : M \to \boldsymbol{E}^3 \rtimes SU(2)$ is called an *adapted framing* of f if $\mathscr{E} \cdot \underline{\mathrm{e}_3} = [h] = g$, which exists uniquely up to the right multiplication of a $U(1)$-valued function. Let ϕ be the dual $(1,0)$-form to $h \cdot E_{12}$ on M, then the induced metric is given by $f^*ds^2 = \phi \cdot \bar{\phi}$. We denote by ρ the connection form on M. Let H be the mean curvature of f and $\Phi = Q\phi \cdot \phi$ its Hopf differential. The pullback $\mathscr{E}^{-1}d\mathscr{E}$ of the Maurer-Cartan form on $\boldsymbol{E}^3 \rtimes SU(2)$ by the adapted framing $\mathscr{E} = (f, h)$ is given by

$$\text{(AI.1)} \qquad \mathscr{E}^{-1}d\mathscr{E} = \begin{pmatrix} 0 & \phi \\ \bar{\phi} & 0 \end{pmatrix} \oplus h^{-1}dh, \quad h^{-1}dh = \frac{1}{2}\begin{pmatrix} \sqrt{-1}\rho & H\phi + \overline{Q\phi} \\ -H\bar{\phi} - Q\phi & -\sqrt{-1}\rho \end{pmatrix}.$$

In order to get an equation for a framing of f which is independent of Q and ρ, it is enough to remove merely the rotational part from the adapted framing $\mathscr{E}$ of f. Put $\underline{h} = (\mathbf{0}, h) : M \to \boldsymbol{E}^3 \rtimes SU(2)$ and $\mathscr{F} = \mathscr{E}\underline{h}^{-1}$. Then $\mathscr{F}$ is the framing of f given by

$$\mathscr{F} = (f, h)(\mathbf{0}, h^{-1}) = (f, \underline{\mathrm{e}_0}) : M \to \boldsymbol{E}^3 \rtimes SU(2).$$

Since $\underline{h}^{-1}d\underline{h} = \mathbf{0} \oplus h^{-1}dh$, we can remove the part containing Q and ρ from the framing equation (AI.1) as follows:

$$df \oplus \mathbf{0} = \mathscr{F}^{-1}d\mathscr{F} = \underline{h}(\mathscr{E}^{-1}d\mathscr{E} - \underline{h}^{-1}d\underline{h})\underline{h}^{-1} = h\begin{pmatrix} 0 & \phi \\ \bar{\phi} & 0 \end{pmatrix}h^* \oplus \mathbf{0}.$$

Then we obtain that

$$\text{(AI.2)} \qquad df = \alpha + \alpha^*,$$

where

$$\alpha := hE_{12}h^*\phi = \begin{pmatrix} -g & g^2 \\ -1 & g \end{pmatrix}\omega, \quad g = P_1 \circ g : M \to \hat{\boldsymbol{C}}.$$

Here the induced metric f^*ds^2 is given by $f^*ds^2 = \phi \cdot \bar{\phi} = \mathrm{tr}(\alpha\alpha^*) = (1+|g|^2)^2\omega \cdot \bar{\omega}$.

Take an isothermal coordinate z on M such that $\phi = e^u dz$. Put $\omega = wdz$ and $\alpha = Adz$. Since the mean curvature H of f is given by

$$H = 2e^{-2u}\langle f_{z\bar{z}}, g\rangle = e^{-2u}\,\mathrm{tr}(A_{\bar{z}}g) = \frac{2g_{\bar{z}}}{(1+|g|^2)^2\overline{w}},$$

the nowhere-vanishing $(1,0)$-form ω is given by

$$\omega = wdz = \frac{2(\bar{g})_z}{H(1+|g|^2)^2}dz$$

unless H vanishes identically on an open subset of M. In this case, the integrability condition for the equation (AI.2) is the following equation for the Gauss map g of the immersion f:

$$\text{(AI.3)} \qquad H\left(g_{z\bar{z}} - \frac{2\bar{g}}{1+|g|^2}g_z g_{\bar{z}}\right) = H_z g_{\bar{z}}.$$

In [**Ke**], this equation (AI.3) is called the *generalized harmonic (GH) equation.*

Remark. Indeed, when H is nonzero constant, it follows from the GH equation (AI.3) and the fact $\omega \neq 0$ that g is a nowhere-holomorphic harmonic map of M to $(\boldsymbol{S}^2, g_0)$ (cf. [**RV**]). When $H \equiv 0$, g is a holomorphic map from M to $\boldsymbol{S}^2$ as well known and $\omega = wdz$ is a holomorphic 1-form on M.

The Kenmotsu representation formula guarantees a converse of the above argument.

Theorem (Kenmotsu representation formula [**Ke**]). *Let M be a simply connected Riemann surface with a reference point z_0. Let $v : M \to \hat{\boldsymbol{C}}$ be a non-holomorphic smooth map satisfying the GH equation* (AI.3) *for a given smooth function H on M*:

$$H\left(v_{z\bar{z}} - \frac{2\bar{v}}{1+|v|^2}v_z v_{\bar{z}}\right) = H_z v_{\bar{z}}.$$

Define a 1*-form ω on M as follows and assume that it is smooth on M*:

$$\omega = \frac{2(\bar{v})_z}{H(1+|v|^2)^2}dz.$$

Put an $\mathfrak{sl}(2;\boldsymbol{C})$-valued 1*-form α on M by*

$$\alpha = \begin{pmatrix} -v & v^2 \\ -1 & v \end{pmatrix}\omega.$$

Then there exists uniquely a smooth map $f : M \to \sqrt{-1}\cdot\mathfrak{su}(2) = \boldsymbol{E}^3$ such that

$$df = \alpha + \alpha^* \quad \text{and} \quad f(z_0) = \boldsymbol{0}.$$

f is a conformal immersion outside $\{z \in M \mid \omega(z) = 0\}$ with prescribed mean curvature H and the Gauss map $g = P_1^{-1} \circ \nu : M \to \boldsymbol{S}^2$. Moreover, the induced metric $f^* ds^2 = (1+|\nu|^2)^2 \omega \cdot \bar{\omega}$, the Hopf differential $\Phi = 2\nu_z \omega \cdot dz$, and the Gauss curvature $K = H^2(1 - (|\nu_z|/|\nu_{\bar{z}}|)^2)$.

Appendix II. Bonnet pairs of CMC surfaces in $\boldsymbol{H}^3(-c^2)$ (and $\boldsymbol{S}_1^3(c^2)$).

We give a duality for CMC surfaces in $\boldsymbol{H}^3(-c^2)$ (and $\boldsymbol{S}_1^3(c^2)$).

The classical Bonnet theorem implies a duality for CMC surfaces in $\boldsymbol{E}^3$: Let $f : M \to \boldsymbol{E}^3$ be an immersed CMC $H(\neq 0)$ surface oriented by a given unit normal vector field N. If f has no umbilic points, then the parallel set defined by $\tilde{f} = f + (1/H)N$ is an immersed CMC H surface with the reversed orientation.

For a CMC H surface in $\boldsymbol{H}^3(-c^2)$, when $|H| > c$, one can obtain a 'parallel' CMC H surface in $\boldsymbol{H}^3(-c^2)$ (see [PT]). When $|H| < c$, we can obtain a CMC H spacelike surface in the de Sitter 3-space $\boldsymbol{S}_1^3(c^2)$ of constant curvature c^2. $\boldsymbol{S}_1^3(c^2)$ is the Lorentzian 3-space form defined as the pseudo-sphere in $\boldsymbol{L}^4$ of radius $1/c$:

$$\boldsymbol{S}_1^3(c^2) = \left\{ \mathbf{x} \in \boldsymbol{L}^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = \frac{1}{c^2} \right\}.$$

Let $f : M \to \boldsymbol{H}^3(-c^2)$ be a conformal CMC H immersion. Take the adapted framing $\mathscr{E} = (\mathscr{E}_{ij}) : U \to SL(2; \boldsymbol{C})$ of f on every contractible open set U of M. Put $\mathscr{E}_\theta = \mathscr{E} h_\theta$ for $h_\theta := \cosh(\theta/2)\underline{\mathbf{e}}_0 + \sinh(\theta/2)\underline{\mathbf{e}}_3$. Then the following maps are defined globally on M:

$$f_{0,\theta} = \frac{1}{c}\mathscr{E}_\theta \cdot \underline{\mathbf{e}}_0 = (\cosh\theta) f + \frac{1}{c}(\sinh\theta) N = \exp_f^H(\theta N/c) : M \to \boldsymbol{H}^3(-c^2),$$

$$f_{3,\theta} = \frac{1}{c}\mathscr{E}_\theta \cdot \underline{\mathbf{e}}_3 = (\sinh\theta) f + \frac{1}{c}(\cosh\theta) N = \exp_{(N/c)}^S(\theta f) : M \to \boldsymbol{S}_1^3(c^2),$$

where $N = \mathscr{E} \cdot \underline{\mathbf{e}}_3$ is the unit normal vector of f, $\exp^H$ and $\exp^S$ are the exponential maps of $\boldsymbol{H}^3(-c^2)$ and $\boldsymbol{S}_1^3(c^2)$ respectively. Assume that $f : M \to \boldsymbol{H}^3(-c^2)$ is not totally umbilic, that is, the Hopf differential Φ is not identically zero. Let $\underline{M}$ denote the Riemannian 2-manifold M with the reversed orientation.

Proposition AII.1. (1) *When $|H| > c$, put $\theta = \tanh^{-1}(c/H)$. Then $\tilde{f} = f_{0,\theta}$ is a conformal CMC H immersion from $\underline{M}$ into $\boldsymbol{H}^3(-c^2)$ (with isolated degenerate points). Here the induced metric $\tilde{f}^* ds^2 = [|\Phi|^2/(H^2 - c^2)] f^* ds^2$ and the Hopf differential $\Phi_{\tilde{f}} = \bar{\Phi}$.*

(2) *When $|H| < c$, put $\theta = \tanh^{-1}(H/c)$. Then $\tilde{f} = f_{3,\theta}$ is a conformal CMC H immersion from $\underline{M}$ into $\boldsymbol{S}_1^3(c^2)$ (with isolated degenerate points). Here the induced metric $\tilde{f}^* ds^2 = [|\Phi|^2/(c^2 - H^2)] f^* ds^2$ and the Hopf differential $\Phi_{\tilde{f}} = \bar{\Phi}$.*

Remark. For a conformal CMC H ($|H| < c$) immersion $f' : M \to \boldsymbol{S}_1^3(c^2)$, we can also obtain a conformal CMC H immersion $\widetilde{f'} : \underline{M} \to \boldsymbol{H}^3(-c^2)$ defined by $\widetilde{f'} = \sinh\theta\, f' + (1/c)\cosh\theta\, N'$ $(\theta = \tanh^{-1}(H/c))$, where $N' : M \to \boldsymbol{H}^3(-1)$ is the unit normal vector field of f'. Then f and $\tilde{\tilde{f}}$ are identical.

Let $G(= P_2 \circ G)$ be the normal Gauss map of $f : M \to \boldsymbol{H}^3(-c^2)$. As in the case of a surface in $\boldsymbol{H}^3(-c^2)$, we can define the normal Gauss map for a spacelike surface in $\boldsymbol{S}^3_1(c^2)$ (see [**AA3**]). In each case of Proposition AII.1, the map $\tilde{\mathscr{E}} = (\tilde{\mathscr{E}}_{ij}) := \mathscr{E}_\theta$ is an adapted framing of the conformal immersion $\tilde{f}$ from $\underline{M}$. The normal Gauss map $\tilde{G} : \underline{M} \to \hat{\boldsymbol{C}}$ of the parallel surface $\tilde{f}$ is given by

$$\tilde{G} = \overline{\left(\frac{\tilde{\mathscr{E}}_{21}}{\overline{\tilde{\mathscr{E}}_{22}}}\right)} = e^{\theta}\overline{\left(\frac{\mathscr{E}_{21}}{\overline{\mathscr{E}_{22}}}\right)} = \sqrt{\frac{|H+c|}{|H-c|}}\,\overline{G}.$$

From Theorem 2.5, G is a non-holomorphic harmonic map to $(\hat{\boldsymbol{C}}, \check{h}_{c,H})$, where

$$\check{h}_{c,H} = \frac{4|d\zeta|^2}{|(1+|\zeta|^2)\{c(1-|\zeta|^2) - H(1+|\zeta|^2)\}|}, \quad \text{and} \quad P_1^* h_{c,H} = P_2^* \check{h}_{c,H} \text{ on } \boldsymbol{S}^2.$$

When $|H| > c$, $\tilde{G} = \sqrt{(H+c)/(H-c)}\,\overline{G}$ is also a non-holomorphic harmonic map from $\underline{M}$ to $(\hat{\boldsymbol{C}}, h_{c,H})$. The metric $h_{c,H}$ is the regular metric on $\boldsymbol{S}^2$, which deforms to the standard metric $4|d\zeta|^2/\{|H|(1+|\zeta|^2)^2\}$ on $\boldsymbol{S}^2$ as c goes to 0 for a fixed nonzero H. When $|H| < c$, $\tilde{G} = \sqrt{(c+H)/(c-H)}\,\overline{G}$ is a non-holomorphic harmonic map from $\underline{M}$ to $\hat{\boldsymbol{C}}$ equipped with the following metric $h'_{c,H}$:

$$h'_{c,H} = \frac{4|d\zeta|^2}{|(1-|\zeta|^2)\{c(1+|\zeta|^2) + H(1-|\zeta|^2)\}|}.$$

The metric $h'_{c,H}$ restricted on the unit open disk deforms to the hyperbolic metric $4|d\zeta|^2/\{|H|(1-|\zeta|^2)^2\}$ as c goes to 0 for a fixed nonzero H.

Appendix III. Spin version of representation formulas.

Finally, we give spin versions of the Kenmotsu type, the Kenmotsu-Bryant type and the Kokubu-Bryant type representation formulas for surfaces in $\boldsymbol{H}^3(-c^2)$. We treat a spin structure on a Riemann surface M as a complex line bundle whose square is the holomorphic tangent bundle $T^{(1,0)}M$ of M, namely, a minus spin bundle of M. Kusner and Schmitt [**KS**] gave a spin version of the Kenmotsu representation formula for conformal immersions from M into $\boldsymbol{E}^3$ (cf. [**KT**]), by choosing a spin structure $\mathrm{Spin}(M)$ on M canonically induced from the pullback of the unique spin structure $\mathrm{Spin}(\boldsymbol{S}^2)$ on $\boldsymbol{S}^2$ via the Gauss map, and by means of the lift $\psi : \mathrm{Spin}(M) \to \mathrm{Spin}(\boldsymbol{S}^2)$. Using the framing method, we can modify the approach by Kusner and Schmitt (see [**AA2**] for details) in order to apply it to the representation formulas for a conformal immersion $f : M \to \boldsymbol{H}^3(-c^2)$. Namely, we choose a spin structure $\mathrm{Spin}(M)$ on M induced from $\mathrm{Spin}(\boldsymbol{S}^2)$ via the normal Gauss map or the adjusted Gauss map, and give the condition that the lift $\psi : \mathrm{Spin}(M) \to \mathrm{Spin}(\boldsymbol{S}^2)$ induces the integrable differential equation for f.

First, we recall that $SU(2) \ni \mathrm{h} \to [\mathrm{h}] \in \boldsymbol{S}^2 = SU(2)/U(1)$ is the unique principal $\mathrm{Spin}(2)$-bundle $\tilde{P}$ on $\boldsymbol{S}^2$, and the spin structure $\mathrm{Spin}(\boldsymbol{S}^2)$ on $\boldsymbol{S}^2$ (i.e. the minus spin bundle associated to $\tilde{P}$) can be regarded as $\boldsymbol{R}^* \cdot SU(2)$ excepting the image $\boldsymbol{0}(\boldsymbol{S}^2)(\cong \boldsymbol{S}^2)$ of the zero-section.

Let $M = (M, ds^2)$ be an oriented connected Riemannian 2-manifold. Take a local isothermal coordinate z on M with $ds^2 = e^{2u}|dz|^2$, and put $\phi = e^u dz$. Let v be a smooth map from M to $\boldsymbol{S}^2$, and (α, v) the (fiber metric preserving) bundle map of $T^{(1,0)}M$ to $T^{(1,0)}\boldsymbol{S}^2$. Take a local lift $h : M \to SU(2)$ of v, that is, $v = [h]$, $\alpha \in \Gamma(T^{*(1,0)}M \otimes v^{-1}T^{(1,0)}\boldsymbol{S}^2)$ is locally described as follows:

$$\alpha = hE_{12}h^*\phi = \begin{pmatrix} -v & v^2 \\ -1 & v \end{pmatrix}\omega, \quad \text{where} \quad h = \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix}, \quad v(= P_1 \circ v) = \frac{q}{p}, \quad \omega = p^2\phi.$$

Let $\mathrm{Spin}(M) = S^-$ be the (unique) pullback bundle of $\mathrm{Spin}(\boldsymbol{S}^2)$ under α. Then $\mathrm{Spin}(M)$ defines a spin structure on M, that is, the minus spin bundle associated to the principal spin bundle $\tilde{P}_M$ on M defined uniquely from $\tilde{P}$. The lift $\psi : \mathrm{Spin}(M) \to \mathrm{Spin}(\boldsymbol{S}^2)$ of α is described by a pair $\big(\psi_1(z,\bar{z})\sqrt{dz}, \psi_2(z,\bar{z})\sqrt{dz}\big)$ of smooth sections of the plus spin bundle S^+ associated to $\tilde{P}_M$, where we consider ψ merely as the map from $\mathrm{Spin}(M)\backslash\mathbf{0}(M)$ into $\boldsymbol{R}^*{\cdot}SU(2)$. Then the following diagram is commutative. (We remark that ψ maps the zero spinor of $\mathrm{Spin}(M)$ to the zero spinor of $\mathrm{Spin}(\boldsymbol{S}^2)$.) We call $\psi = (\psi_1\sqrt{dz}, \psi_2\sqrt{dz})$ the *spinor representation* of the bundle map $(\alpha, v) : T^{(1,0)}M \to T^{(1,0)}\boldsymbol{S}^2$.

$$\begin{array}{ccccl}
\zeta\sqrt{\phi^*}|_z \in \mathrm{Spin}(M) & \xrightarrow{\psi=(\psi_1\sqrt{dz},\psi_2\sqrt{dz})} & \mathrm{Spin}(\boldsymbol{S}^2) & \ni h(z)\begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix} = \begin{pmatrix} q\zeta & -\overline{p\zeta} \\ p\zeta & \overline{q\zeta} \end{pmatrix} \\
\downarrow & & \downarrow & \qquad\qquad\qquad \leftrightarrow (p(z)\zeta, q(z)\zeta) \\
\zeta^2\phi^*|_z \in T^{(1,0)}M & \xrightarrow{\alpha=h\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)h^*\phi} & T^{(1,0)}\boldsymbol{S}^2 & \ni \zeta^2 h(z)E_{12}h(z)^* \\
\downarrow & & \downarrow & \\
z \in M & \xrightarrow{v=h\underline{\mathbf{e}}_3h^*} & \boldsymbol{S}^2(\cong\hat{\boldsymbol{C}}) & \ni h(z)\underline{\mathbf{e}}_3h(z)^* \ (= q(z)/p(z)),
\end{array}$$

where $\zeta \in \boldsymbol{C}$ and $\phi^* = e^{-u}\partial/\partial z$ denotes the dual to ϕ. Then $\psi_1 = e^{u/2}q, \psi_2 = e^{u/2}p$, and

$$v(= P_1 \circ v) = \frac{\psi_1}{\psi_2}, \quad \omega = \psi_2^2 dz, \tag{AIII.1}$$

$$\alpha = \begin{pmatrix} -\psi_1\psi_2 & \psi_2^2 \\ -\psi_1^2 & \psi_1\psi_2 \end{pmatrix} dz. \tag{AIII.2}$$

The *Dirac operator* $\not{D}$ for the spinor representation $\psi = \big(\psi_1(z,\bar{z})\sqrt{dz}, \psi_2(z,\bar{z})\sqrt{dz}\big)$, which is a smooth section of $S^+ \oplus S^+$, is defined by

$$\not{D}\psi = \not{D}\begin{pmatrix} \psi_1\sqrt{dz} \\ \psi_2\sqrt{dz} \end{pmatrix} = 2\begin{pmatrix} \partial\overline{\psi_2}\sqrt{dz} \\ -\partial\overline{\psi_1}\sqrt{dz} \end{pmatrix}.$$

Now we review the following proposition, which is already obtained as a part of the proof of Theorem 2.7.

Proposition AIII.1. *Let M be a Riemann surface and $\tilde{M}$ the universal cover of M. For a bundle map (α, v) of $T^{(1,0)}M$ into $T^{(1,0)}\boldsymbol{S}^2$, define an $\mathfrak{su}(2)$-valued 1-form μ as follows*:

$$\mu = \frac{c}{2}(\alpha + \alpha^*) + \frac{c}{4}[\underline{\mathbf{e}}_3, \alpha + \alpha^*].$$

If μ satisfies the integrability condition $d\mu + \mu \wedge \mu = 0$, there exists a smooth map $\mathscr{S} : \tilde{M} \to S(\subset SL(2; \boldsymbol{C}))$ satisfying $\mathscr{S}^{-1} d\mathscr{S} = \mu$. $f := (1/c)\mathscr{S}\mathscr{S}^* : \tilde{M} \to \boldsymbol{H}^3(-c^2)$ is a conformal immersion with the normal Gauss map $G = \nu$.

From (AIII.2), we can rewrite the integrability condition $d\mu + \mu \wedge \mu = 0$ as follows, in terms of the spinor representation $\psi = (\psi_1\sqrt{dz}, \psi_2\sqrt{dz})$ of (α, ν):

$$\begin{cases} \operatorname{Im}(\overline{\psi_1}\partial\overline{\psi_2} + \overline{\psi_2}\partial\overline{\psi_1}) = 0, \\ \psi_1\left(\bar{\partial}\psi_1 - \dfrac{c}{2}\overline{\psi_2}|\psi_1|^2\right) + \overline{\psi_2}\left(\partial\overline{\psi_2} - \dfrac{c}{2}\psi_1|\psi_2|^2\right) = 0. \end{cases}$$

These equations are equivalent to the following:

$$\text{(AIII.3)} \qquad \not{D}\psi = 2\begin{pmatrix} \partial\overline{\psi_2} \\ -\partial\overline{\psi_1} \end{pmatrix} = \begin{pmatrix} (c|\psi_2|^2 - r)\psi_1 \\ -(c|\psi_1|^2 + r)\psi_2 \end{pmatrix},$$

where r is a real-valued function on M. Let H denote the mean curvature of f. It follows from (2.3) combined with (AIII.1) that

$$\text{(AIII.4)} \qquad \bar{\psi} \cdot \not{D}\psi = -\{c(|\psi_2|^2 - |\psi_1|^2) + H|\psi|^2\}|\psi|^2,$$

where $\cdot$ stands for the complex bilinear inner product on $\boldsymbol{C}^2$. Then, from (AIII.3) and (AIII.4), we obtain that $r = c(|\psi_2|^2 - |\psi_1|^2) + H|\psi|^2$ and the following:

Proposition AIII.2. *The integrability condition $d\mu + \mu \wedge \mu = 0$ is equivalent to the following non-linear Dirac equation for $\psi = (\psi_1\sqrt{dz}, \psi_2\sqrt{dz})$:*

$$\not{D}\psi = 2\begin{pmatrix} \partial\overline{\psi_2} \\ -\partial\overline{\psi_1} \end{pmatrix} = \begin{pmatrix} (c|\psi_1|^2 - H|\psi|^2)\psi_1 \\ -(c|\psi_2|^2 + H|\psi|^2)\psi_2 \end{pmatrix}.$$

From Propositions AIII.1 and AIII.2, we obtain the following theorem.

Theorem AIII.3 (Spin version of Kenmotsu type representation formula). *Let M be a simply connected Riemann surface with a reference point z_0 and H a smooth function on M. Let $\psi = (\psi_1, \psi_2) : M \to \boldsymbol{C}^2$ be a nowhere-vanishing $\boldsymbol{C}^2$-valued smooth function satisfying the equation*

$$\not{D}\psi = 2\begin{pmatrix} \partial\overline{\psi_2} \\ -\partial\overline{\psi_1} \end{pmatrix} = \begin{pmatrix} (c|\psi_1|^2 - H|\psi|^2)\psi_1 \\ -(c|\psi_2|^2 + H|\psi|^2)\psi_2 \end{pmatrix}.$$

Put $\mathfrak{sl}(2; \boldsymbol{C})$-valued 1-forms α and μ by

$$\alpha = \begin{pmatrix} -\psi_1\psi_2 & \psi_1^2 \\ -\psi_2^2 & \psi_1\psi_2 \end{pmatrix} dz, \qquad \mu = \frac{c}{2}(\alpha + \alpha^*) + \frac{c}{4}[\underline{\mathbf{e}}_3, \alpha + \alpha^*].$$

Then there exists uniquely a smooth map $\mathscr{S} : M \to S(\subset SL(2; \boldsymbol{C}))$ such that $\mathscr{S}(z_0) = \underline{\mathbf{e}}_0$ and $\mathscr{S}^{-1} d\mathscr{S} = \mu$. Put $f = (1/c)\mathscr{S}\mathscr{S}^*$, then $f : M \to \boldsymbol{H}^3(-c^2)$ is a conformal im-

mersion with the mean curvature H and the normal Gauss map $G = P_1^{-1} \circ (\psi_1/\psi_2)$. Here the induced metric f^*ds^2 and the Hopf differential Φ are given by

$$f^*ds^2 = |\psi|^4|dz|^2, \quad \Phi = -(\psi \cdot \not{D}\bar{\psi})dz \cdot dz.$$

REMARK AIII.4. The above formula coincides with the representation formula obtained independently by Kokubu [**Ko2**].

Similarly, we can represent CMC surfaces in $\boldsymbol{H}^3(-c^2)$ with the adjusted Gauss maps $g = \nu$ using the spinor representations of (α, ν).

THEOREM AIII.5 (Spin version of Kenmotsu-Bryant type representation formula). *Let M be a simply connected Riemann surface with a reference point z_0. Give a positive constant $H(\geqq c)$, and put $H_0 = \sqrt{H^2 - c^2}$. Let $\psi = (\psi_1, \psi_2) : M \to \boldsymbol{C}^2$ be a nowhere-vanishing $\boldsymbol{C}^2$-valued smooth function satisfying the equation*

$$\not{D}\psi = -H_0|\psi|^2\psi.$$

Put $\mathfrak{sl}(2;\boldsymbol{C})$-valued 1-forms α and τ by

$$\alpha = \begin{pmatrix} -\psi_1\psi_2 & \psi_1^2 \\ -\psi_2^2 & \psi_1\psi_2 \end{pmatrix} dz, \quad \tau = \frac{1}{2}\{(c - H_0 + H)\alpha + (c + H_0 - H)\alpha^*\}.$$

Then there exists uniquely a smooth map $\mathscr{F} : M \to SL(2;\boldsymbol{C})$ such that $\mathscr{F}(z_0) = \underline{\mathbf{e}}_0$ and $\mathscr{F}^{-1}d\mathscr{F} = \tau$. Put $f = (1/c)\mathscr{F}\mathscr{F}^*$, then $f : M \to \boldsymbol{H}^3(-c^2)$ is a conformal CMC $H(\geqq c)$ immersion with the adjusted Gauss map $g = P_1^{-1} \circ (\psi_1/\psi_2)$.

THEOREM AIII.6 (Spin version of Kokubu-Bryant type representation formula). *Let M be a simply connected Riemann surface with a reference point z_0. Give a non-negative constant $H(\leqq c)$, and put $c_0 = \sqrt{c^2 - H^2}$. Let $\psi = (\psi_1, \psi_2) : M \to \boldsymbol{C}^2$ be a nowhere-vanishing $\boldsymbol{C}^2$-valued smooth function satisfying the equation*

$$\not{D}\psi = c_0 \begin{pmatrix} |\psi_1|^2\psi_1 \\ -|\psi_2|^2\psi_2 \end{pmatrix}.$$

Put $\mathfrak{sl}(2;\boldsymbol{C})$-valued 1-forms α and τ by

$$\alpha = \begin{pmatrix} -\psi_1\psi_2 & \psi_1^2 \\ -\psi_2^2 & \psi_1\psi_2 \end{pmatrix} dz, \quad \tau = \frac{1}{2}\{(c + H)\alpha + (c - H)\alpha^*\} + \frac{c_0}{4}[\underline{\mathbf{e}}_3, \alpha + \alpha^*].$$

Then there exists uniquely a smooth map $\mathscr{F} : M \to SL(2;\boldsymbol{C})$ such that $\mathscr{F}(z_0) = \underline{\mathbf{e}}_0$ and $\mathscr{F}^{-1}d\mathscr{F} = \tau$. Put $f = (1/c)\mathscr{F}\mathscr{F}^*$, then $f : M \to \boldsymbol{H}^3(-c^2)$ is a conformal CMC $H(\leqq c)$ immersion with the adjusted Gauss map $g = P_1^{-1} \circ (\psi_1/\psi_2)$.

REMARK AIII.7. When $H = c$ in the above Theorems AIII.5 and AIII.6, we obtain a spin version of the Bryant representation formula ([**Br**], [**UY**]) for CMC c surfaces in $\boldsymbol{H}^3(-c^2)$. The integrability condition is given by the linear Dirac equation $\not{D}\psi = 0$, which is the same as the one in the spin version of the Weierstrass representation formula for minimal surfaces in $\boldsymbol{E}^3$ (cf. [**KT**], [**KS**]).

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